FEDERER-ČECH COUPLES

MICHEAL DYER

1. Introduction. In (**5**), I considered two-term conditions in π-exact couples, of which the exact couple $\mathfrak{C}(X, Y, v)$ of Federer (**7**) is an example. Let $M(X, Y)$ be the space of all maps from X to Y with the compact-open topology. Our aim in this paper is to construct a π-exact couple $\check{\mathfrak{C}}^2(X, Y)$, where X is a finite-dimensional (in the sense of Lebesgue) metric space and $Y \in \mathfrak{W}$, a certain (rather large) class of spaces. Specifically, $\mathfrak{W}$ is the class of all topological spaces X which possess the following property (P).

(P) Let Y be a (possibly infinite) simplicial complex. There exists $x_0 \in X$ and $y_0 \in Y$ such that $[X, x_0] \simeq [Y, y_0]$.

In § 5 it will be seen that $\mathfrak{W}$ contains all CW complexes and all metric absolute neighbourhood retracts (ANR)s.

In § 4, it is shown that, in the exact couple $\check{\mathfrak{C}}^2(X, Y)$, $\check{E}_{p,q}{}^2(X, Y) \approx \check{H}^q(X; \pi_{p+q}(Y))$, where $\check{H}^q(X; \pi_{p+q}(Y))$ represents the qth Čech cohomology with coefficients in $\pi_{p+q}(Y)$, and that $\check{E}_\infty(X, Y)$ is associated with the homotopy groups of $M(X, Y)$. The method used is similar to one used by Barratt (**1**), and, in fact, relies heavily on Theorem 12.21 of that same paper.

The existence of a π-exact couple with the above properties allows the extension of the results of (**5**) to mapping spaces $M(X, Y)$, where $Y \in \mathfrak{W}$ is arc-connected and simple, and X is an arc-connected k-dimensional metric space ($k < \infty$). For instance, we will prove the following two theorems.

THEOREM A. *Let $k = 2$, $Y = U$, the infinite unitary group, and let v: $X \to U$ be constant. Then if $\bar{\pi}_j = \pi_j(M(X, U), v)$,*

$$\bar{\pi}_{2i} \approx \check{H}^1(X), \qquad \bar{\pi}_{2i-1} \approx \check{H}^2(X) \oplus Z, \qquad i = 1, 2, \ldots .$$

THEOREM B. *Let $k = 4$, $Y = O_+$, a component of the infinite orthogonal group, and let v: $X \to Y$ be constant. Then, if $\bar{\pi}_j = \pi_j(M(X, O_+), v)$, we have:*

$$\begin{aligned} \bar{\pi}_{1+8i} &\approx \check{H}^2(X) + Z_2, & i &= 0, 1, 2, 3, 4, \ldots, \\ \bar{\pi}_{2+8i} &\approx \check{H}^1(X), & i &= 0, 1, 2, \ldots, \\ \bar{\pi}_{3+8i} &\approx \check{H}^4(X) + Z, & i &= 0, 1, 2, \ldots . \end{aligned}$$

The sequence

$$\check{H}^2(X) \xrightarrow{d^2} \check{H}^4(X; Z_2) \xrightarrow{\phi} \bar{\pi}_{4+8i} \xrightarrow{\psi} \check{H}^3(X) \to 0, \qquad i = 0, 1, 2, \ldots,$$

is exact.

These theorems are part of two larger theorems, the statements of which are precisely given in (**5**, Propositions 10.1 and 10.2), except that X is a metric

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space of dimension k instead of a CW complex and the cohomology is Čech instead of singular.

The organization of the paper is as follows. In § 2 I give a short exposition of directed systems of exact couples and show that the direct limit of the derived couple is naturally isomorphic to the derived couple of the direct limit. The definition of the Federer exact couple is reviewed in § 3. In § 4 the construction of $\check{\mathfrak{C}}^2(X, Y)$ is given and in § 5 a trivial extension of the theorem of Spanier-Barratt (12.21) mentioned above is given. In § 6, certain pertinent results of (**5**, § 6) are carried over into $\check{\mathfrak{C}}^2(X, Y)$. In § 7 some results on the homotopy groups of joins, which are useful in the examples of § 8, are given. Finally, in § 9 I give the computation of the homotopy groups of $M(\mathrm{CP}^k, U)$, $M(V_{n+1,2}, U)$ and some of the stable homotopy groups of $M(L(p, q), S^n)$, p odd, where $L(p, q)$ is a 3-dimensional Lens space. This last computation is an interesting example of a pair of spaces which do not satisfy Gap Theorem I or II (see **5**, §§ 8 and 9) and still have a two-term condition in $\mathfrak{C}^2(L(p, q), S^n)$.

I wish to acknowledge my indebtedness to M. Barratt [in (**1**)] for several crucial ideas and to Professor Sze-Tsen Hu for warm encouragement and many stimulating conversations.

2. Directed systems of exact couples. Let M be a set with a partial ordering "$<$" (partial ordering is a relation which is reflexive and transitive). Such a pair $(M, <)$ is said to be *directed* if and only if given any $\alpha, \beta \in M$ there is a $\gamma \in M$ such that $\gamma > \alpha$ and $\gamma > \beta$. A subset M' of M (M' partially ordered by $<$ also) is said to be *cofinal* if and only if for any $\alpha \in M$ there is a $\beta \in M'$ such that $\beta > \alpha$.

Let $\mathfrak{C} = \{D, E, i, j, k\}$ be an exact couple in the sense of Federer; i.e., D is a (not necessarily abelian) group, E is an abelian group, and i, j, and k are homomorphisms such that the following triangle is exact (see **7**):

$$\begin{array}{ccc} D & \xrightarrow{\;i\;} & D \\ & {\scriptstyle k}\nwarrow \quad \swarrow{\scriptstyle j} & \\ & E & \end{array}$$

Definition 2.1. Given exact couples $\mathfrak{C}_\alpha$ and $\mathfrak{C}_\beta$. A *mapping* $\xi_{\alpha\beta}$: $\mathfrak{C}_\alpha \to \mathfrak{C}_\beta$ is a pair of homomorphisms $(\phi_{\alpha\beta}, \psi_{\alpha\beta})$ such that

$$\phi_{\alpha\beta}: D(\alpha) \to D(\beta), \qquad \psi_{\alpha\beta}: E(\alpha) \to E(\beta)$$

and

$$\phi_{\alpha\beta} \circ i_\alpha = i_\beta \circ \phi_{\alpha\beta}, \quad \psi_{\alpha\beta} \circ j_\alpha = j_\beta \circ \phi_{\alpha\beta}, \quad \phi_{\alpha\beta} \circ k_\alpha = k_\beta \circ \psi_{\alpha\beta}.$$

Definition 2.2. A directed system of exact couples $\mathfrak{C}^*$ over a directed set M is a set $\{\mathfrak{C}_\alpha | \mathfrak{C}_\alpha$ is an exact couple, $\alpha \in M\}$ such that for each relation $\alpha < \beta$ there is a mapping of couples $\xi_{\alpha\beta}$: $\mathfrak{C}_\alpha \to \mathfrak{C}_\beta$ such that

(a) $\alpha = \beta \Rightarrow \xi_{\alpha\alpha} =$ identity on $\mathfrak{C}_\alpha$, and
(b) $\alpha < \beta < \gamma \Rightarrow \xi_{\beta\gamma} \circ \xi_{\alpha\beta} = \xi_{\alpha\gamma} (\phi_{\beta\gamma} \circ \phi_{\alpha\beta} = \phi_{\alpha\gamma}$ and $\psi_{\beta\gamma} \circ \psi_{\alpha\beta} = \psi_{\alpha\gamma})$.

In this case, define the direct limit couple $\bar{\mathfrak{C}} = \text{dir lim}_{\alpha \in M} \mathfrak{C}_\alpha$ by

$$\bar{\mathfrak{C}} = \{\bar{D}, \bar{E}, \bar{\imath}, \bar{\jmath}, \bar{k}\},$$

where

$$\bar{D} = \operatorname*{dir\,lim}_{\alpha \in M} D(\alpha), \qquad \bar{E} = \operatorname*{dir\,lim}_{\alpha \in M} E(\alpha),$$

$$\bar{\imath} = \operatorname*{dir\,lim}_{\alpha \in M} i_\alpha, \qquad \bar{\jmath} = \operatorname*{dir\,lim}_{\alpha \in M} j_\alpha, \qquad \bar{k} = \operatorname*{dir\,lim}_{\alpha \in M} k_\alpha;$$

thus, $\bar{\imath}(x) = \overline{i_\alpha(x_\alpha)}$, where $\alpha \in M$, $x_\alpha \in D(\alpha)$, and $x_\alpha \in x \in \bar{D}$, i.e., x_α is a representative of the class $x \in \bar{D}$, and $\overline{i_\alpha(x_\alpha)}$ means the class in $\bar{D}$ of which $i_\alpha(x_\alpha)$ is a member.

Since, in general, $D(\alpha)$ is non-abelian, the alternative definition given in (**6**, p. 222) is applicable and extends to non-abelian groups. Since the direct limit of an exact sequence is exact (see **6**, p. 225), the following is true.

THEOREM 2.3. *The direct limit $\bar{\mathfrak{C}}$ of a directed system $\mathfrak{C}^*$ of exact couples is an exact couple.*

Let $\{\mathfrak{C}_\alpha, \xi_{\alpha\beta} | \alpha, \beta \in M\}$ be a directed system of exact couples. Let $\alpha < \beta$ and consider the map $\xi_{\alpha\beta}$: $\mathfrak{C}_\alpha \rightarrow \mathfrak{C}_\beta$. Let $\mathfrak{C}_\alpha'$ and $\mathfrak{C}_\beta'$ be the derived couples of $\mathfrak{C}_\alpha$ and $\mathfrak{C}_\beta$, respectively. Then the mapping $\xi_{\alpha\beta}$ induces a mapping $\xi_{\alpha\beta}'$: $\mathfrak{C}_\alpha' \rightarrow \mathfrak{C}_\beta'$ (see **8**, p. 242) defined by

$$\phi_{\alpha\beta}'(y) = \phi_{\alpha\beta}(y) \quad \text{for } y \in D'(\alpha) = i_\alpha(D(\alpha))$$

and

$\psi_{\alpha\beta}'$ (homology class of x) = (homology class of $\psi_{\alpha\beta}(x)$) for $x \in \ker d_\alpha \subset E(\alpha)$.

The following lemmas can be easily proved.

LEMMA 2.4. (a) *$\xi_{\alpha\alpha}'$ is the identity on $\mathfrak{C}_\alpha'$, and*
(b) *$\xi_{\beta\gamma}' \circ \xi_{\alpha\beta}' = \xi_{\alpha\gamma}'$ for $\alpha < \beta < \gamma$.*

LEMMA 2.5. (a) $\phi_{\alpha\beta}' \circ i_\alpha' = i_\beta' \circ \phi_{\alpha\beta}'$,
(b) $\psi_{\alpha\beta}' \circ j_\alpha' = j_\beta' \circ \phi_{\alpha\beta}'$, *and*
(c) $\phi_{\alpha\beta}' \circ k_\alpha' = k_\beta' \circ \psi_{\alpha\beta}'$.

Thus, the set $\{\mathfrak{C}_\alpha', \xi_{\alpha\beta}' | \alpha, \beta \in M\}$ is again a directed system of exact couples. Let $\mathfrak{C}_\alpha' \equiv \mathfrak{C}_\alpha^2$, $\xi_{\alpha\beta}' \equiv \xi_{\alpha\beta}^2$. Iterate the process for $n \geqq 1$.

THEOREM 2.6. *The set $\mathfrak{C}^{*n} = \{\mathfrak{C}_\alpha^n, \xi_{\alpha\beta}^n\}$ is a directed system of exact couples.*

For each $n \geqq 1$, define the limit exact couple of $\mathfrak{C}^{*n}$ as:

$$\hat{\mathfrak{C}}^n = \operatorname*{dir\,lim}_{\alpha \in M} \mathfrak{C}_\alpha^n \quad [\text{note that } \hat{\mathfrak{C}}^1 \equiv \bar{\mathfrak{C}}].$$

Definition 2.7. A mapping $\xi_{\alpha\beta}$: $\mathfrak{C}_\alpha \rightarrow \mathfrak{C}_\beta$ is an isomorphism if and only if $\phi_{\alpha\beta}$ and $\psi_{\alpha\beta}$ are isomorphisms.

The following theorem answers the natural question.

THEOREM 2.8. *$\hat{\mathfrak{C}}^n \approx (\bar{\mathfrak{C}})^n$; i.e.,* $\text{dir lim}_{\alpha \in M}(\mathfrak{C}_\alpha{}^n) \approx (\text{dir lim}_{\alpha \in M} \mathfrak{C}_\alpha)^n$, *where $(\bar{\mathfrak{C}})^n$ is the $(n-1)$st derived couple of $\bar{\mathfrak{C}}$.*

Theorem 2.8 follows directly from the following two lemmas.

LEMMA 2.9. *Let Φ: $\{G_\alpha, \eta_{\alpha\beta}\} \to \{H_\alpha, \theta_{\alpha\beta}\}$ be a mapping of the directed system $\{G_\alpha\}$ over M into the directed system $\{H_\alpha\}$ over M. (Thus, $\Phi = \{\Phi_\alpha | \alpha \in M\}$, where Φ_α: $F_\alpha \to H_\alpha$ is a homomorphism such that if $\alpha < \beta$, then $\Phi_\beta \circ \eta_{\alpha\beta} = \theta_{\alpha\beta} \circ \Phi_\alpha$.) Let F_α be a subgroup of G_α for each $\alpha \in M$ such that if $\alpha < \beta$, then $\eta_{\alpha\beta}(F_\alpha) \subset F_\beta$. Then*

$$\operatorname*{dir\,lim}_{\alpha\in M}(\Phi_\alpha | F_\alpha) \approx \left(\operatorname*{dir\,lim}_{\alpha\in M} \Phi_\alpha\right)\Big|\left(\operatorname*{dir\,lim}_{\alpha\in M} F_\alpha\right) = \Phi_\infty | F_\infty.$$

LEMMA 2.10. *Let $\{G_\alpha, \psi_{\alpha\beta}\}$ be a directed system of groups over M. Let*

$$\Phi: \{G_\alpha, \psi_{\alpha\beta}\} \to \{G_\alpha, \psi_{\alpha\beta}\}$$

be a map of systems. Then

$$\operatorname*{dir\,lim}_{\alpha\in M}(\ker \Phi_\alpha) \approx \ker \Phi_\infty \subset G_\infty \quad \text{and} \quad \operatorname*{dir\,lim}_{\alpha\in M}(\operatorname{im} \Phi_\alpha) \approx \operatorname{im} \Phi_\infty \subset G_\infty$$

in the natural way.

In addition, suppose that G_α is abelian for all $\alpha \in M$ and that $\Phi^2 = \Phi \circ \Phi = 0$. Then

$$\operatorname*{dir\,lim}_{\alpha\in M}\left(\frac{\ker \Phi_\alpha}{\operatorname{im} \Phi_\alpha}\right) \approx \frac{\ker \Phi_\infty}{\operatorname{im} \Phi_\infty}$$

in the natural way.

3. The Federer exact couple $\mathfrak{C}(X, Y, v)$. In this section a brief description of $\mathfrak{C}(X, Y, v)$ is given. Let X be a CW complex and let Y be any path-connected space. Let X^n be the n-dimensional skeleton of X and U_j the arc-component of $M(X^j, Y)$ containing $v_j = v|X^j$. Define the map r: $U_j \to U_{j-1}$ by $r(f) = f|X^{j-1}$ ($f \in M(X^j, Y)$). Since X is a CW complex, r is a fibering in the sense of Serre; see (**8**). Let

$$F_j = r^{-1}(v_{j-1}) = \{f \in U_j | \; f|X^{j-1} = v_{j-1}\}.$$

F_j is a fibre of r.

Define

$$D = \sum_{p,q} D_{p,q},$$

where $D_{p,q} = \pi_p(U_q, v_q)$ if $p, q \geqq 0$, $D_{p,q} = 0$ otherwise, and

$$E = \sum_{p,q} E_{p,q},$$

where $E_{p,q} = \pi_p(F_q, v_q)$ if $p, q \geqq 0$ and $E_{p,q} = 0$ otherwise. Then the homotopy sequence of the fibering above becomes

$$\ldots \to E_{i,j} \xrightarrow{k} D_{i,j} \xrightarrow{i} D_{i,j-1} \xrightarrow{j} E_{i-1,j} \to \ldots,$$

where k is inclusion induced, $i = r_*$, and $j = \partial$. This makes $\{D, E, i, j, k\}$ an exact couple, denoted by $\mathfrak{C}(X, Y, v)$.

The following theorem is stated for future reference. The proof may be found in (**7**, p. 351).

THEOREM 3.1. *If X is a* CW *complex of dimension $k < \infty$ and if Y is arc-connected and simple ($=$ n-simple for all $n > 0$), then*

(a) $E_{p,q} \approx C^q(X, \pi_{p+q}(Y))$, *the group of q-dimensional cochains on X with coefficients in $\pi_{p+q}(Y)$, for $p \geqq 1$. If $p = 0$, then $E_{0,q} \approx$ subgroup of $C^q(X, \pi_q(Y))$;*

(b) $E_{p,q}{}^2 \approx H^q(X, \pi_{p+q}(Y))$ *for $p \geqq 1$ and if $p = 0$, $E_{0,q}{}^2 \approx$ subgroup of $H^q(X, \pi_q(Y))$.*

Filter $\pi_p(M(X, Y), v)$ as follows:

$$\pi_p(M(X, Y), v) \supset \pi_{p,0} \supset \pi_{p,1} \supset \ldots \supset \pi_{p,k-1} \supset 0, \tag{3.1}$$

where

$$\pi_{p,q} = \ker\{i^{(k-q)}: D_{p,k} \to D_{p,q}\},$$

$q < k = \dim X$ and $i^{(j)} = i \circ i \circ i \circ \ldots \circ i$ (j times). Then the usual proposition is true; see (**7**).

PROPOSITION 3.2. $\pi_{p,q-1}/\pi_{p,q} \approx E_{p,q}{}^\infty$.

4. The Federer-Čech π-exact couple $\check{\mathfrak{C}}^2(X, Y)$. This section contains the construction of the π-couple $\check{\mathfrak{C}}^2(X, Y)$ and the proof that $\check{E}^2(X, Y) \approx \check{H}^q(X; \pi_{p+q}(Y))$.

Let X be a paracompact topological space; i.e., X is Hausdorff and the set S of all locally finite open coverings of X is cofinal in the set M of all open coverings. Here the set M is directed by refinement; that is to say, $\beta > \alpha$ if and only if β is a refinement of α ($\alpha, \beta \in M$).

If $\alpha \in S$, then the nerve of α, N_α, is a simplicial complex. For each $\alpha \in S$, consider the exact couple $\mathfrak{C}(N_\alpha, Y, v_\alpha)$ defined in the last section, where Y is any arc-connected, simple space.

Let $\alpha, \beta \in S$ be such that $\alpha < \beta$. Thus, β is a refinement of α. Vertex inclusion defines simplicial maps $f_{\beta\alpha}: N_\beta \to N_\alpha$. Any two maps so defined are homotopic; see (**6**, p. 235).

PROPOSITION 4.1. *Let $\beta > \alpha$, $\alpha, \beta \in S$, and assume that $v_\gamma: N_\gamma \to Y$ is the constant map to some point $y_0 \in Y$ for $\gamma = \alpha, \beta$. Let $f_{\beta,\alpha}, f_{\beta\alpha}': N_\beta \to N_\alpha$ be any two simplicial maps defined by vertex inclusion. Then the maps*

$$\phi_{\alpha\beta}, \phi_{\alpha\beta}': D_{p,q}{}^2(N_\alpha, Y, v_\alpha) \to D_{p,q}{}^2(N_\beta, Y, v_\beta),$$

induced by $f_{\beta\alpha}, f_{\beta\alpha}'$, respectively, are identical; i.e., vertex inclusion induces a unique homomorphism $\phi_{\alpha\beta}: D_{p,q}{}^2(\alpha) \to D_{p,q}{}^2(\beta)$ provided v_α and v_β are constant.

Proof. Let $f_{\beta,\alpha}$ and $f_{\beta,\alpha}'$ be any two maps defined above. There exists a homotopy

$$\phi_t: f_{\beta,\alpha} \simeq f_{\beta,\alpha}' \qquad (t \in I).$$

By (**7**, p. 354), the following triangle commutes,

$$\begin{array}{ccc} & D_{p,q}{}^2(N_\alpha, Y, v_\alpha) & \\ {\scriptstyle \phi_{\alpha\beta}}\swarrow & & \searrow{\scriptstyle \phi_{\alpha\beta}'} \\ D_{p,q}{}^2(N_\beta, Y, v_\beta) & \xrightarrow{\eta} & D_{p,q}{}^2(N_\beta, Y, v_\beta) \end{array}$$

where $\phi_{\alpha\beta}$ is induced by $f_{\beta\alpha}$, $\phi_{\alpha\beta}'$ by $f_{\beta\alpha}'$, and η is the isomorphism induced by the curve $V: I \to M(N_\beta, Y)$, $[V(t)](x) = (v_\alpha \circ \phi_t)(x)$ $(t \in I, x \in N_\beta)$. However, $v_\alpha(N_\alpha) = \{y_0\} \Rightarrow [V(t)](N_\beta) = v_\alpha(\phi_t(N_\beta)) = \{y_0\}$. Thus, $V(I) = \{v_\beta\}$ and η is the identity. Therefore, $\phi_{\alpha\beta} = \phi_{\alpha\beta}'$: $D_{p,q}{}^2(\alpha) \to D_{p,q}{}^2(\beta)$. Hence, no matter what vertex map is chosen, the induced map on $D^2(N_\alpha, Y, v_\alpha)$ is uniquely defined. This proves Proposition 4.1.

If $\alpha, \beta \in S$, $\alpha < \beta$, we have the following diagram ($p \geqq 0$, $q \geqq 0$):

$$(*)\qquad \begin{array}{ccc} E_{p,2}{}^2(\alpha) & \xrightarrow{\psi_{\alpha\beta}'} & E_{p,q}{}^2(\beta) \\ \approx\downarrow\gamma & & \approx\downarrow\gamma \\ H^q(N_\alpha, \pi_{p+q}(Y)) & \xrightarrow[f_{\beta\alpha}^*]{} & H^q(N_\beta, \pi_{p+q}(Y)) \end{array}$$

where γ is the injection of Theorem 3.1(b) (surjective if $p \geqq 1$) (for a definition of γ, see (**7**, p. 345)), $\psi_{\alpha\beta}'$ is the homomorphism induced by the homomorphism $\psi_{\alpha\beta}$: $\pi_p(F_q{}^\alpha, v_q{}^\alpha) \to \pi_p(F_q{}^\beta, v_q{}^\beta)$ which is in turn induced by $f_{\beta\alpha}$: $N_\beta \to N_\alpha$, and $f_{\beta\alpha}^*$ is the homomorphism induced by $f_{\beta\alpha}$. This diagram commutes since both maps are induced by $f_{\beta\alpha}$. Then $f_{\beta\alpha}^*$ is uniquely defined implies that $\psi_{\alpha\beta}'$ is also uniquely defined. Thus, the groups $\{D^2, \phi_{\alpha\beta}\}$ and $\{E^2, \psi_{\alpha\beta}'\}$ form two directed systems of groups over $(S, <)$. This proves the following theorem; see (**7**, p. 353).

THEOREM 4.2. *Let X be paracompact and Y arc-connected and simple. Then* $\mathfrak{C}^* = \{\mathfrak{C}^2(N_\alpha, Y, v_\alpha), (\phi_{\alpha\beta}, \psi_{\alpha\beta}') \mid \alpha \in S, v_\alpha \text{ constant}\}$ *forms a directed system of exact couples over S, the set of locally finite open coverings of X.*

Definition. The exact couple

$$\check{\mathfrak{C}}^2(X, Y) = \operatorname*{dir\,lim}_{\alpha \in S} (\mathfrak{C}^2(N_\alpha, Y, v_\alpha))$$

is called the *Federer-Čech couple for the pair* (X, Y).

Note that $\check{\mathfrak{C}}^2(X, Y)$ is an exact couple in the sense of Federer such that degree $\bar{j}^2 = (-1, 2)$, deg $\bar{\imath}^2 = (0, -1)$, deg $\bar{k}^2 = (0, 0)$, where $\check{\mathfrak{C}}^2(X, Y)$ is the exact couple below:

$$\begin{array}{ccc} \bar{D}^2(X, Y) & \xrightarrow{\bar{\imath}^2} & \bar{D}^2(X, Y) \\ {\scriptstyle \bar{k}^2}\nwarrow & & \swarrow{\scriptstyle \bar{j}^2} \\ & \bar{E}^2(X, Y) & \end{array}$$

The following proposition is crucial.

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PROPOSITION 4.3. *In* $\mathfrak{C}^2(X, Y)$, *and for* $p > 0$, $\bar{E}_{q,q}{}^2(X, Y) \approx \check{H}^q(X, \pi_{p+q}(Y))$, *where* $\check{H}^q(X, \pi_{p+q}(Y))$ *denotes the q-dimensional Čech cohomology group of* X *with coefficients in* $\pi_{p+q}(Y)$, *based on all open coverings of* X.

Proof. Since X is paracompact, the set S is cofinal in M, the set of all open coverings of X. Thus, by Theorem 3.1(b),

$$\gamma\colon E_{p,q}{}^2(\alpha) \approx H^q(N_\alpha, \pi_{p+q}(Y))$$

for $p \geqq 1$, $q \geqq 0$, and each $\alpha \in S$. Since $(*)$ is commutative for each $\alpha < \beta$ we have:

$$\check{H}^q(X, \pi_{p+q}(Y)) = \operatorname{dir}\lim_{\alpha \in M} H^q(N_\alpha, \pi_{p+q}(Y)) = \operatorname{dir}\lim_{\alpha \in S} H^q(N_\alpha, \pi_{p+q}(Y))$$

$$\approx \operatorname{dir}\lim_{\alpha \in S} E_{p,q}{}^2(\alpha) \approx \bar{E}_{p,q}{}^2 \quad \text{(by Theorem 2.8)}.$$

It is also clear that if $p = 0$, then $\bar{E}_{0,q}{}^2(X, Y) \approx$ subgroup of $\check{H}^q(X, \pi_q(Y))$. This proves Proposition 4.3.

Definition 4.4. $\dim_N X \leqq k$, where $N \subset M$, the set of all open coverings of X, if and only if any covering $\alpha \in N$ has a refinement $\beta \in N$ such that the dimension of the nerve of β is at most k. We say that $\dim_N X = k$ if and only if k is the least integer such that $\dim_N X \leqq k$. Dowker has shown (**2**) that, if X is normal, then $\dim_F X = \dim_{SF} X = \dim_S X$, where F, SF, and S are the classes of finite, star-finite, and locally finite covers of X, respectively.

Definition 4.5. If X is normal, then: X has Lebesgue dimension k means $\dim_S X = k$.

THEOREM 4.6. *Suppose that* X *is a paracompact space of Lebesgue dimension* k. *Then* $\check{\mathfrak{C}}^2(X, Y)$ *satisfies* (**5**, *Definition* 2.1, *properties* (1), (2), *and* (3)).

Proof. (**5**, Definition 2.1 (1) and (2)) are true since they are true for each $\alpha \in S$. $k = \dim_S X$ implies that the set T of all open covers α of X such that N_α has dimension k and $\alpha \in S$ is cofinal in S (and hence in M). Therefore,

$$\bar{E}_{p,q}{}^2 = \operatorname{dir}\lim_{\alpha \in S} E_{p,q}{}^2(\alpha) = \operatorname{dir}\lim_{\alpha \in T} E_{p,q}{}^2(\alpha) = 0 \quad \text{for } q > k.$$

Thus (**5**, Definition 2.1 (3)) holds.

$\mathfrak{C}^2(X, Y)$ could almost be called a π-exact couple, except that $\deg \bar{j}^2 = (-1, 2)$. However, the definition of a π-exact couple may be broadened to read as follows.

Definition 4.7. $\mathfrak{C}$ is a π-exact couple if and only if $\mathfrak{C}$ is an exact couple in the sense of Federer, $\deg i = (0, -1)$, $\deg j = (-1, n)$, $n \geqq 1$, and $\deg k = (0, 0)$ such that $\mathfrak{C}$ satisfies (**5**, Definition 2.1 (1), (2), (3)).

In this case, not only is $\mathfrak{C}^2(X, Y)$ a π-exact couple, but also all the derived couples $\mathfrak{C}^2(X, Y, v)$ of § 3. This seems much more natural. Thus, $\mathfrak{C}^2(X, Y)$ is a π-exact couple.

5. A theorem of Spanier and Barratt. $\mathfrak{W}$ is the class of all spaces X for which the pair $(X, x_0) \simeq (Y, y_0)$ for some $x_0 \in X$, $y_0 \in Y$, where Y is a simplicial complex. Thus, if $X \in \mathfrak{W}$, the following is true.

PROPOSITION 5.1. *Let Z be any space.*

$$(X, x_0) \simeq (Y, y_0) \Rightarrow (M(Z, X), v_{x_0}) \simeq (M(Z, Y), v_{y_0}),$$

where $v_{x_0}(Z) = \{x_0\}$ and $v_{y_0}(Z) = \{y_0\}$.

Proof. Let f: $(X, x_0) \to (Y, y_0)$ and g: $(Y, y_0) \to (X, x_0)$ be homotopy inverses and let H_t: $(X, x_0) \to (X, x_0)$ be a homotopy of $g \circ f$ and $1_{(X,x_0)}$, and G_t: $(Y, y_0) \to (Y, y_0)$ a homotopy of $f \circ g$ and $1_{(Y,y_0)}$. Then, *composition* induces homotopy inverses

$$f_*: (M(Z, X), v_{x_0}) \to (M(Z, Y), v_{y_0}), \quad g_*: (M(Z, Y), v_{y_0}) \to (M(Z, X), v_{x_0}),$$

and H_{t*}: $g_* \circ f_* \simeq 1_{(M(Z, X), v_{x_0})}$, $\quad G_{t*}$: $f_* \circ g_* \simeq 1_{(M(Z,Y), v_{y_0})}$.

COROLLARY 5.2. $(X, x_0) \simeq (Y, y_0) \Rightarrow \pi_q(M(Z, X), v_{x_0}) \simeq \pi_q(M(Z, Y), v_{y_0})$ *for all q.*

In (**1**, Chapter 6), Barratt proved the following extension of a theorem of Spanier (**14**).

THEOREM 5.3 (*Spanier-Barratt*). *Let P be normal, paracompact, and locally compact or first countable. If X is any simplicial complex, then there is an isomorphism*

$$\Phi_*: \operatorname{dir}\lim_{\alpha \in L} \pi_q(M(N_\alpha, X), v_\alpha) \approx \pi_q(M(P, X), v) \qquad (q > 0),$$

where L is the class of locally finite open coverings of P, N_α is the nerve of α, $v_\alpha(N_\alpha) = \{x_0\} = v(P)$.

COROLLARY 5.4. *If P is as above, and $X \in \mathfrak{W}$, then*

$$\operatorname{dir}\lim_{\alpha \in L} \pi_q(M(N_\alpha, X), v_\alpha) \approx \pi_q(M(P, X), v).$$

The proof of Theorem 5.3 is essentially given in (**1**, Chapter 6), although there it is given for track groups instead of homotopy groups. A rather elegant proof of this fact in a more general setting appears in (**11**).

In order to identify some of the spaces in $\mathfrak{W}$, we need to discuss the *homotopy extension property* (HEP).

Definition 5.6. Let A be a *closed* subspace of a space X, and let Y be any space. The triple $(X, A; Y)$ is said to have HEP if and only if given any map f: $X \to Y$ and any homotopy h_t: $A \to Y$ $(t \in I)$ of $f|A$, there exists a homotopy g_t: $X \to Y$ $(t \in I)$ of f such that $g_t|A = h_t$ $(t \in I)$.

The following triples $(X, A; Y)$, A closed in X, have HEP:

HEP 1. Y metric ANR, X metric space (see **10**);

HEP 2. Y compact ANR, X is normal and paracompact (see **2**);

HEP 3. Y is separable and topologically complete ANR, X is normal and countably paracompact (see **3**);

HEP 4. Y metric ANR, X countably paracompact, collectionwise normal and A is a G_δ set (see **4**).

For the remaining, consider the triples $(X, A; Y)$, A closed in X, and Y an arbitrary space.

HEP 5. X, A are compact metric ANR (see **18**);

HEP 6. (X, A) is a CW pair (see **19**);

HEP 7. X, A are metric ANR or perfectly normal ANR;

HEP 8. X, A are paracompact ANR, X is normal, A is G_δ;

HEP 9. X, A are normal ANR or collectionwise normal ANR, X is countably paracompact, A is G_δ (for HEP 7, 8, 9, see **4**).

Let $A = x_0$ and consider triples $(X, \{x_0\}; Y)$ which have HEP, where Y is a simplicial complex. Note that if X is Hausdorff and first countable, then $\{x_0\}$ is a G_δ set, and the G_δ condition is satisfied in HEP 8 and HEP 9.

THEOREM 5.7. *Let $(X, \{x_0\}; Y)$ have HEP. In addition, let $X \simeq Y$, where Y is a simplicial complex. Then there is a $y_0 \in Y$ such that $(X, x_0) \simeq (Y, y_0)$.*

For the proof, see (**1**, p. 311).

Thus, by HEP 1–HEP 9, $\mathfrak{W}$ contains the following classes:

$\mathfrak{W}_{\mathrm{CW}}$ = class of all CW complexes;

$\mathfrak{W}_{\mathrm{ANR}}$ = class of all metric ANRs;

$\mathfrak{W}_{\mathrm{M}}$ = class of all metric spaces which have the homotopy type of a locally finite simplicial complex;

$\mathfrak{W}_{\mathrm{NP}}$ = class of all normal, paracompact spaces which have the homotopy type of a finite simplicial complex;

$\mathfrak{W}_{\mathrm{NCP}}$ = class of all normal, countably paracompact spaces which have the homotopy type of a separable, topologically complete, locally finite simplicial complex;

$\mathfrak{W}_{\mathrm{CPCN}}$ = class of all countably paracompact, collectionwise normal, Hausdorff, first countable spaces which have the homotopy type of a locally finite simplicial complex;

$\mathfrak{W}_{\mathrm{PNANR}}$ = class of all perfectly normal ANRs which have the homotopy type of a simplicial complex.

$\mathfrak{W}_{\mathrm{CW}}$ is a subclass of $\mathfrak{W}$ since any CW complex has the homotopy type of a simplicial complex; see (**12**). $\mathfrak{W}_{\mathrm{ANR}}$ is a subclass of W since X being a metric ANR implies that X is dominated by a simplicial complex (**10**, p. 138) which in turn implies by (**12**) that X has the homotopy type of a simplicial complex with the weak topology. Thus $\mathfrak{W}$ is a rather large class.

6. Two-term conditions in $\mathfrak{C}^2(X, Y)$.

THEOREM 6.1. *Let X be paracompact, of Lebesgue dimension k, and either first*

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countable or locally compact. Let Y be simple and arc-connected such that $Y \in \mathfrak{W}$. Then there is a filtration

$$\pi_n(M(X, Y), v) \supset \bar{\pi}_{n,0} \supset \bar{\pi}_{n,1} \supset \bar{\pi}_{n,2} \supset \ldots \supset \bar{\pi}_{n,k-1} \supset 0,$$

of $\pi_n(M(X, Y), v)$ such that

$$\frac{\pi_n(M(X, Y), v)}{\bar{\pi}_{n,0}} \approx \bar{E}_{n,0}{}^{\infty} \quad \text{and} \quad \frac{\bar{\pi}_{n,q-1}}{\bar{\pi}_{n,q}} \approx \bar{E}_{n,q}{}^{\infty} \qquad (1 \leqq q \leqq k).$$

Proof. X is of Lebesgue dimension k. Filter $\pi_n(M(X, Y), v)$ as follows (v constant). Consider the set T of open covers α such that

$$\alpha \in T \Rightarrow \dim N_\alpha \leqq k.$$

T is cofinal in M. If $\alpha \in T$, then (3.1) defines a filter of subgroups:

$$\pi_n(M(N_\alpha, Y), v_\alpha) \supset \pi_{n,0}(\alpha) \supset \pi_{n,1}(\alpha) \supset \ldots \supset \pi_{n,k-1}(\alpha) \supset 0.$$

Since $\alpha < \beta$, $\alpha, \beta \in T \Rightarrow \phi_{\alpha\beta}(\pi_{n,i}(\alpha)) \subset \pi_{n,i}(\beta)$, then the set $\{\pi_{n,i}(\alpha), \phi_{\alpha\beta}\}$ forms a directed system of groups.

Define

$$\bar{\pi}_{n,i} = \operatorname*{dir\,lim}_{\alpha \in T}(\pi_{n,i}(\alpha)) = \operatorname*{dir\,lim}_{\alpha \in M}(\pi_{n,i}(\alpha)) \qquad (0 \leqq i \leqq k-1).$$

This filters $\pi_n(M(X, Y), v)$ (Theorem 5.3).

For each $\alpha \in T$, $\pi_{n,i-1}(\alpha)/\pi_{n,i}(\alpha) \approx E_{n,i}{}^{\infty}(\alpha)$. By Theorem 4.2, the set $\{E_{n,i}{}^{\infty}(\alpha), \psi_{\alpha\beta}\}$ forms a directed system of groups over S and by Theorem 2.8,

$$\begin{aligned} \bar{E}_{n,i}{}^{\infty} &\approx \operatorname*{dir\,lim}_{\alpha \in S} E_{n,i}{}^{\infty}(\alpha) \\ &\approx \operatorname*{dir\,lim}_{\alpha \in S}\left[\frac{\pi_{n,i-1}(\alpha)}{\pi_{n,i}(\alpha)}\right] \\ &\approx \operatorname*{dir\,lim}_{\alpha \in S \cap T}\left[\frac{\pi_{n,i-1}(\alpha)}{\pi_{n,i}(\alpha)}\right] \qquad (S \cap T \text{ cofinal in } S) \\ &\approx \frac{\bar{\pi}_{n,i-1}}{\bar{\pi}_{n,i}}; \end{aligned}$$

see (**6**, p. 228), where $\bar{E}_{n,i}{}^{\infty}$ is a term derived from $\check{\mathfrak{C}}^2(X, Y)$. This completes the proof of Theorem 6.1.

COROLLARY 6.2. *Let X be a k-dimenisonal metric space. Then Theorem* 6.1 *is true.*

Theorem 6.1 and the fact that $\check{\mathfrak{C}}^2(X, Y)$ is a π-exact couple in the sense of Definition 4.7 implies the following theorem.

THEOREM 6.3. *Let Y be simple, arc-connected, and an element of $\mathfrak{W}$. Let X be paracompact, of Lebesgue dimension k, and either first countable or locally*

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compact. Then, if a two-term condition $\{\lambda, \mu; 2\}$ *holds on* $\bar{E}^2$ *of* $\check{\mathfrak{S}}^2(X, Y)$, *the following sequence is exact:*

$$\check{H}^{b_\mu}(X, \pi_{\mu+b_\mu}(Y)) \xrightarrow{\phi_\mu} \pi_\mu(M(X, Y), v) \xrightarrow{\psi_\mu} \check{H}^{a_\mu}(X, \pi_{\mu+a_\mu}(Y)) \xrightarrow{\theta_\mu} \ldots$$
$$\xrightarrow{\psi_\lambda} \check{H}^{a_\lambda}(X, \pi_{\lambda+a_\lambda}(Y));$$

see (**5**, Definition 3.1 and Theorem 3.2).

In the next two theorems, let X be a k-dimensional, paracompact, *arc-connected*, and first countable or locally compact space. Let $Y \in \mathfrak{W}$ be such that Y is simple and arc-connected.

THEOREM 6.4. *In* $\check{\mathfrak{S}}^2(X, Y)$, *the differential operator* $\bar{d}^i\colon \bar{E}_{n,0}{}^i \to \bar{E}_{n-1,i}{}^i$ *is zero for any* n *and* $i \geqq 2$.

Proof. Let T be the set of all locally finite open covers of X such that $\dim N_\alpha \leqq k$. T is cofinal in M. For each $\alpha \in T$, $d^i\colon E_{n,0}{}^i(\alpha) \to E_{n-1,i}{}^i(\alpha)$ is zero by (**5**, Theorem 6.1). By Theorem 2.8,

$$\bar{d}^i = \operatorname*{dir\,lim}_{\alpha \in T} d^i = 0 \quad \text{for all } n \text{ and } i \geqq 2.$$

The following theorem will show that certain exact sequences split.

THEOREM 6.5. *Suppose that, for some fixed* p *and each locally finite cover* α *of* X *such that* $\dim N_\alpha \leqq k$, $\pi_{p,i}(\alpha) = 0$ *for* $i > 0$ *and* $E_{p,0}{}^\infty(\alpha) = E_{p,0}{}^2(\alpha)$. *Then the following sequence*

$$0 \to \bar{\pi}_{p,0} \to \pi_p(M(X, Y), v) \xrightarrow{\bar{k}} \pi_p(Y) \to 0,$$

is split exact, where $\bar{k}$ *is induced by the evaluation map.*

Proof. For each $\alpha \in T$, $\pi_{p,i}(\alpha) = 0$ for $i > 0$ and $E_{p,0}{}^\infty(\alpha) = E_{p,0}{}^2(\alpha)$ $(\approx H^0(N_\alpha; \pi_p(Y)) \approx \pi_p(Y))$ implies

$$\bar{\pi}_{p,i} = 0 \quad \text{and} \quad \bar{E}_{p,0}{}^2 = \bar{E}_{p,0}{}^\infty \quad (\approx \check{H}^0(X; \pi_p(Y)) \approx \pi_p(Y)).$$

Thus,

$$0 \to \pi_{p,0}(\alpha) \to \pi_p(M(N_\alpha, Y), v_\alpha) \xrightarrow{k(\alpha)} \pi_p(Y) \to 0,$$

where $k(\alpha)$ is induced by the evaluation map, is exact for each α, and, in the limit,

$$0 \to \bar{\pi}_{p,0}(\alpha) \to \pi_p(M(X, Y), v_\alpha) \xrightarrow{k} \pi_p(Y) \to 0.$$

Define $j\colon Y \subset M(X, Y)$ to be the injection of Y into constant maps and $j_\alpha \subset M(N_\alpha, Y)$ similarly for each $\alpha \in T$. By (**5**, Theorem 6.4), $k(\alpha) \circ j_{\alpha *} = 1_{\pi_p(Y)}$ for all $\alpha \in T$. It follows from Corollary 5.4 that

$$j_* = \operatorname*{dir\,lim}_{\alpha \in T} j_{\alpha *} \quad \text{and} \quad \bar{k} = \operatorname*{dir\,lim}_{\alpha \in T} k(\alpha).$$

This implies that

$$\bar{k} \circ j_* = \operatorname*{dir\,lim}_{\alpha \in T} k(\alpha) \circ \operatorname*{dir\,lim}_{\alpha \in T} j_{\alpha *} = \operatorname*{dir\,lim}_{\alpha \in T} (k(\alpha) \circ j_{\alpha *}) = 1_{\pi_p(Y)}.$$

This proves Theorem 6.5.

It is clear that any pair of spaces (X, Y) which satisfy Gap Theorem I (i.e., $\dim X = k$ is small and there are certain gaps in the homotopy groups of Y) will give rise to a two-term condition in $\mathfrak{C}^2(X, Y)$. It then becomes clear that (**5**, Propositions 10.1 and 10.2) hold for X metric, where Čech cohomology replaces singular cohomology everywhere. In particular, Theorems A and B are true.

7. Some results on the homotopy groups of joins. Prior to several examples of the homotopy groups of mapping spaces, we compute in this section some of the homotopy groups of the join $X * Y$ of spaces X and Y. The general reference for this section is (**17**). In (**17**), Whitehead defined the exact couple $\mathfrak{C}(X * Y)$ for $X * Y$, where Y is a CW complex, X is $(m-1)$-connected, and Y is $(n-1)$-connected $(m, n \geqq 2)$; thus, $X * Y$ is $(m+n)$-connected. This couple is a regular ∂-couple, and thus many of the results of (**8**, Chapter VIII) apply. The E^2 term of $\mathfrak{C}(X * Y)$ has been partially identified as

$$E_{p,q}{}^2(X * Y) \approx H_{n+p}(Y; \pi_{m+q}(X)) \qquad (q \leqq m-2)$$

and $\pi_{m+n+1+p}(X * Y)$ can be identified with $D_{p+1,-1}$ in $\mathfrak{C}(X * Y)$. Using two-term conditions in $\mathfrak{C}(X * Y)$ we obtain the following results.

THEOREM 7.1. *Let $L(\pi, n)$ be a Moore space for an abelian group π, $n \geqq 2$. Let X be an $(m-1)$-connected space $(m \geqq 2)$. Then the following sequence*

$$0 \to \pi \otimes \pi_{m+q}(X) \xrightarrow{\Phi} \pi_{m+n+1+q}(X * L) \to \operatorname{Tor}(\pi, \pi_{m+q-1}(X)) \to 0,$$

where Φ is induced by the join operation (see **17**, p. 59), *is exact for $q \leqq m-2$ and split exact for $q = 1$.*

This generalizes (**17**, Theorem 3.1).

Proof. $\tilde{H}_q(L) = 0$ for $q \neq n$ and $H_n(L) \approx \pi \approx \pi_n(L)$. Thus, if $q \leqq m-2$,

$$E_{p,q}{}^2 \approx \begin{cases} \pi \otimes \pi_{m+q}(X) & \text{if } p = 0, \\ \operatorname{Tor}(\pi, \pi_{m+q}(X)) & \text{if } p = 1, \\ 0 & \text{if } p > 1. \end{cases}$$

(**8**, Chapter VIII, Theorem 8.2) implies that the sequence is exact. To see the splitting, consider the ladder:

$$\begin{array}{ccccccccc} 0 \to & \pi \otimes \tilde{H}_{m+1}(X) & \to & H_{m+n+2}(X * L) & \underset{s}{\overset{r}{\rightleftarrows}} & \operatorname{Tor}(\pi, \tilde{H}_m(X)) & \to 0 \\ & \downarrow h_n \otimes h_{m+1} & & \downarrow h_{m+n+2} & & \downarrow \operatorname{Tor}(h_n, h_m) \approx & \\ 0 \to & \pi \otimes \pi_{m+1}(X) & \to & \pi_{m+n+2}(X * L) & \xrightarrow{t} & \operatorname{Tor}(\pi, \pi_m(X)) & \to 0 \end{array}$$

where the top sequence splits and the vertical homomorphisms are the Hurewicz homomorphisms. Since X is $(m-1)$-connected, h_m is an isomorphism, and $\operatorname{Tor}(h_n, h_m)$ is an isomorphism. Since the top sequence splits, there is a homomorphism s: $\operatorname{Tor}(\pi, \tilde{H}_m(X)) \to H_{m+n+2}(X * L)$ such that $r \circ s$ is the identity

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on $\operatorname{Tor}(\pi, \tilde{H}_m(X))$. Let $s_1: \operatorname{Tor}(\pi, \pi_m(X)) \to \pi_{m+n+2}(X * L)$ be defined by $s_1 = h_{m+n+2} \circ s \circ (\operatorname{Tor}(h_n, h_m))^{-1}$. Clearly, $t \circ s_1$ is the identity on $\operatorname{Tor}(\pi, \pi_m(X))$.

Thus, if $Y = Y_p^n = e^{n+1} \cup_f S^n$, the pseudo-projective space obtained by adjoining an $(n + 1)$ cell to S^n by $f: S^n \to S^n$ of degree p, then (see **8**, p. 321)

$$\tilde{H}_i(Y_p^n) \approx \begin{cases} Z_p & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, Y_p^n is a Moore space of type (Z_p, n). Consider $Y_p^n * S^m$ for $n \geqq 2$, and $m > 20$ (giving the computation of $\pi_{m+n+1+j}(Y_p^n * S^m)$ for $0 \leqq j \leqq 19$ for m in the stable range). An easy computation using the stable homotopy of S^m, $m > 20$ (see **16**, pp. 186–188), together with Theorem 7.1, implies the following theorems.

THEOREM 7.2.

$$\pi_{m+n+1+j}(Y_3^n * S^m) \approx \begin{cases} Z_3 & \text{if } j = 0, 3, 4, 7, 8, 10, 12, 13, 14, 15, 16, \\ 0 & \text{if } j = 1, 2, 5, 6, 9, 17, 18, 19, \end{cases}$$

and $0 \to Z_3 \to \pi_{m+n+12}(Y_3^n * S^m) \to Z_3 \to 0$ *is exact.*

THEOREM 7.3.

$$\pi_{m+n+j+1}(Y_5^n * S^m) \approx \begin{cases} Z_5 & \text{if } j = 0, 7, 8, 15, 16, \\ 0 & \text{if } 1 \leqq j \leqq 19, j \neq 7, 8, 15, 16. \end{cases}$$

THEOREM 7.4. *Suppose that*

$$\pi_i(X) = \begin{cases} 0 & \text{if } i = 1, 2, \ldots, m-1, m+2, m+3, \ldots, m+r-1, \\ \pi_m(X) = \pi_1, \\ \pi_{m+1}(X) = \pi_2. \end{cases}$$

Let Y be an $(n - 1)$-connected CW *complex* $(n \geqq 2)$. *Then the following sequence is exact:*

$$0 \to H_{n+j-1}(Y, \pi_2) \to \pi_{m+n+1+j}(X * Y) \to H_{n+j}(Y, \pi_1) \to 0$$

for $0 \leqq j \leqq \min(m - 1, r - 1)$.

COROLLARY 7.5. *Let X be such that*

$$\pi_1(X) = \begin{cases} \pi_1 & \text{if } i = m \geqq 2, \\ \pi_2 & \text{if } i = m + 1, \\ 0 & \text{otherwise;} \end{cases}$$

then the above sequences hold for $0 \leqq j \leqq m - 1$.

COROLLARY 7.6. *Let X be as in Theorem* 7.4 *or* 7.5. *Let* $Y = \mathrm{CP}^k$, *the k-dimensional complex projective space. Then*

$$\pi_{m+3+j}(X * \mathrm{CP}^k) \approx \begin{cases} \pi_2 & (j \text{ odd}, j \leqq \min\{2 \cdot k - 1, m - 1, r - 1\}), \\ \pi_1 & (j \text{ even}, j \leqq \min\{2 \cdot k - 2, m - 1, r - 1\}), \\ 0 & (j \geqq 2k, j \leqq \min(m - 1, r - 1)). \end{cases}$$

Proof.

$$H_q(\mathrm{CP}^k, \pi) \approx \begin{cases} \pi & \text{if } q \text{ is even, } 0 \leqq q \leqq 2k, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$0 \to H_{j+1}(\mathrm{CP}^k, \pi_2) \to \pi_{m+3+j}(X * \mathrm{CP}^k) \to H_{j+2}(\mathrm{CP}^k, \pi_1) \to 0$$

for $0 \leqq j \leqq \min(m-1, r-1)$, yields the result.

In order to obtain two-term conditions using the "nice" gaps in the homotopy of U, we must pass to the $(m-1)$-connective fibering $E(U, m)$ of U; see (**8**, pp. 156–158). This is an $(m-1)$-connected space such that

$$\pi_i(E(U,m)) \approx \pi_i(U) \qquad (i \geqq m).$$

Let m be an odd integer, $m > 2$, and consider $Y * E(U, m)$.

$$E_{p,q}{}^2(Y * E(U,m)) \approx \tilde{H}_{n+p}(Y; \pi_{m+q}(U)) \approx \begin{cases} H_{n+p}(Y) & \text{if } q \text{ is even,} \\ 0 & \text{if } q \text{ is odd,} \end{cases}$$

for $q \leqq m-2$.

THEOREM 7.7. *Let* $Y = \mathrm{CP}^k$, $X = E(U,m)$ *for* m *odd,* $m > 2$, *and* $2k \leqq m-2$. *Then, for* $0 \leqq j \leqq m-2$,

$$\pi_{m+3+j}(\mathrm{CP}^k * E(U,m)) \approx \begin{cases} 0 & \text{if } j \text{ is odd,} \\ Z^i & \text{if } j = 2i-2, j \leqq 2k-2, \\ Z^k & \text{if } j \text{ is even, } 2k-2 \leqq j < m-2. \end{cases}$$

Proof. This follows since $E^\infty(\mathrm{CP}^k * E(U, m)) = E^2(\mathrm{CP}^k * E(U, m))$ and $A/Z \approx Z \Rightarrow A \approx Z \oplus Z$, provided A is an abelian group.

THEOREM 7.8. *Let* $Y = Y_p{}^n$, $n \geqq 2$. *Then* $\pi_{m+n+j+1}(Y_p{}^n * E(U, m)) \approx Z_p \otimes \pi_{m+j}(U)$ *for* $0 \leqq j \leqq m-2$.

Finally, consider $V_{5,2}$, the space of all unit tangent vectors on S^4. It is well known that (see **8**, p. 323)

$$\tilde{H}_i(V) \approx \begin{cases} Z & \text{if } i = 7, \\ Z_2 & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 7.8. *Let* $Y = V_{5,2}$, $X = E(U, m)$ *for* m *odd,* $m > 2$. *Then, if* $0 \leqq j \leqq m-2$,

$$\pi_{m+4+j}(V_{5,2} * E(U,m)) \approx \begin{cases} Z_2 & \text{if } j = 0, 2, \\ Z \oplus Z_2 & \text{if } j \text{ is even, } j > 2, \\ 0 & \text{if } j \text{ is odd.} \end{cases}$$

8. Examples of two-term conditions in $\mathfrak{C}(X, Y, v)$ and $\mathfrak{C}^2(X, Y)$. Throughout this section, let X be either a *connected*, k-dimensional, CW complex or a path-connected, k-dimensional, metric space. Correspondingly, let $\bar{H}^i(X; G)$ denote cellular cohomology if X is a CW complex or Čech

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cohomology if X is a metric space. Finally, let $\mathfrak{C}(X, Y)$ denote either $\mathfrak{C}(X, Y, v)$ (v constant) or $\check{\mathfrak{C}}^2(X, Y)$, depending on X. Denote $\pi_i(M(X, Y), v)$ by $\bar{\pi}_i$.

Let F_4 denote the exceptional Lie group of dimension 52. The first 23 homotopy groups of F_4 have been computed in (**13**). There it is shown that

$$\pi_i(F_4) = 0 \qquad (i = 1, 2, 4, 5, 6, 7, 10, 12, 13, 19).$$

This fact, plus two-term conditions in $\mathfrak{C}(X, Y)$, implies the following result.

THEOREM 8.1. *Let* $\dim X \leqq 4$ *and* $\pi_i = \pi_i(F_4)$. *Then various homotopy groups of* $M(X; F_4)$ *are given by:*

$$\bar{\pi}_1 \approx \bar{H}^2(X; \pi_3), \qquad \bar{\pi}_2 \approx \bar{H}^1(X; \pi_3), \qquad \bar{\pi}_3 \approx \bar{H}^0(X; \pi_3),$$
$$\bar{\pi}_4 \approx \bar{H}^4(X; \pi_8), \qquad \bar{\pi}_9 \approx \bar{H}^2(X; \pi_{11}) \oplus \bar{H}^0(X; \pi_9),$$

$$\bar{H}^1(X; \pi_8) \to \bar{H}^3(X; \pi_9) \to \bar{\pi}_6 \to \bar{H}^2(X; \pi_8) \to \bar{H}^4(X; \pi_9) \to \bar{\pi}_5 \to \bar{H}^3(X; \pi_8) \to 0,$$

$$0 \to \bar{H}^4(X; \pi_{14}) \to \bar{\pi}_{10} \to \bar{H}^1(X; \pi_{11}) \to 0$$

are exact.

Proof. The proof follows easily from the fact that

$$\bar{E}_{p,q}{}^2 \approx \bar{H}^q(X; \pi_{p+q}(F_4))$$

implies TTC $\{0, 6; 2\}$, $\{9, 10; 2\}$ (see Figure 8.1) on $\mathfrak{C}(X, Y)$, from Theorem 6.3; and from extended TTC's of (**5**).

In (**13**), it is also shown that $\pi_i(\mathrm{Spin}(7)) = 0$ if $i = 1, 2, 4, 5, 6, 12$, $\pi_3(\mathrm{Spin}(7)) \approx Z$.

THEOREM 8.2. *Let* $\dim X \leqq 4$, *and* $\pi_i = \pi_i(\mathrm{Spin}(7))$. *Then* (see Figure 8.2)

$$\bar{\pi}_1 \approx \bar{H}^2(X; \pi_3), \quad \bar{\pi}_2 \approx \bar{H}^1(X; \pi_3), \quad \pi_3 \approx Z \oplus \bar{H}^4(X; \pi_7),$$

and $\bar{H}^2(X; \pi_7) \to \bar{H}^4(X; \pi_8) \to \bar{\pi}_4 \to \bar{H}^3(X; \pi_7) \to 0$ *is exact.*

Next, let $Y = Y_5{}^n * S^m$ for $n \geqq 2$, $m > 20$. The computations of Theorem 7.3 imply the following result.

THEOREM 8.3. *Let* $\dim X \leqq 7$. *Then* $M(X, Y_5{}^n * S^m)$ *is* $(m + n - 7)$-*connected and* (see Figure 8.3), *if* $\bar{H}^i(X; Z_5) \equiv \bar{H}^i$, *then*

$$\bar{\pi}_{m+n-6+j} \approx \bar{H}^{(7-j)} \qquad (0 \leqq j < 7),$$
$$\bar{\pi}_{m+n+1} \approx \bar{\pi}_{m+n+9} \approx \bar{H}^0 + \bar{H}^7,$$
$$\bar{\pi}_{m+n+8} \approx \bar{H}^0 \oplus \bar{H}^1.$$

Furthermore, the following sequences are exact:

$$0 \to \bar{H}^2 \to \bar{\pi}_{m+n+7} \to \bar{H}^1 \to \bar{H}^3 \to \bar{\pi}_{m+n+6} \to \bar{H}^2 \to \bar{H}^4 \to \bar{\pi}_{m+n+5} \to \bar{H}^3 \to \bar{H}^5$$
$$\to \bar{\pi}_{m+n+4} \to \bar{H}^4 \to \bar{H}^6 \to \bar{\pi}_{m+n+3} \to \bar{H}^5 \to \bar{H}^7 \to \bar{\pi}_{m+n+2} \to \bar{H}^6 \to 0,$$

$$\bar{H}^2 \to \bar{H}^4 \to \bar{\pi}_{m+n+13} \to \bar{H}^3 \to \bar{H}^5 \to \bar{\pi}_{m+n+12} \to \bar{H}^4 \to \bar{H}^6 \to \bar{\pi}_{m+n+11}$$
$$\to \bar{H}^5 \to \bar{H}^7 \to \bar{\pi}_{m+n+10} \to \bar{H}^6 \to 0.$$

This follows essentially by (**5**, Gap Theorem I).

In (**15**), it is shown that

$$H^*(G_{k+2,k}) \approx \Lambda(x_{2k+1}, x_{2k+3}),$$

where $G_{k+2,k}$ is the complex Stiefel manifold of left cosets $U(k+2)/U(k)$ and $\Lambda(x_{2k+1}, x_{2k+3})$ is the exterior algebra on generators x_{2k+1}, x_{2k+3} of dimension $2k+1$ and $2k+3$, respectively. Let $k = 1$, then $G_{3,1} = G$ has homology

$$\tilde{H}_i(G) \approx \begin{cases} Z & \text{if } q = 3, 5, 8, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = G * K(\pi, m)$ for π abelian and $m \geqq 11$. Then, it is shown in (**17**) that

$$\begin{aligned} \pi_{m+4+j}(G * K) &\approx \tilde{H}_{3+j}(G; \pi) \qquad (0 \leqq j \leqq m-2), \\ &\approx \begin{cases} \pi & \text{if } j = 0, 2, 5, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

THEOREM 8.4. *Let* $\dim X \leqq 4$. *Then* $M(X, G * K)$ *is* $(m-1)$*-connected and, if* $\bar{H}^i(X; \pi) \equiv \bar{H}^i$, *then*

$$\begin{aligned} \bar{\pi}_{m+i} &\approx \bar{H}^{(4-i)} && (i = 0, 1), \\ \bar{\pi}_{m+6} &\approx \bar{H}^0 \oplus \bar{H}^3, && \\ \bar{\pi}_{m+7+i} &\approx \bar{H}^{(2-i)} && (i = 0, 1, 2), \\ \bar{\pi}_{m+n} &= 0 && (10 \leqq n \leqq m-2). \end{aligned}$$

Furthermore, the following sequences are exact:

$$0 \to \bar{H}^3 \to \bar{\pi}_{m+3} \to \bar{H}^1 \to \bar{H}^4 \to \bar{\pi}_{m+2} \to \bar{H}^2 \to 0,$$

$$0 \to \bar{H}^4 \to \bar{\pi}_{m+5} \to \bar{H}^1 \to 0 \quad \text{(see Figure 8.4).}$$

Finally, let $Y = \mathrm{CP}^s * E(U, m)$ for m odd, $m > 2$, $2s \leqq m-2$. Define a function $f: Z \to Z$ *by*

$$f(i) = \begin{cases} i & \text{if } i \leqq s, \\ s & \text{if } i \geqq s. \end{cases}$$

THEOREM 8.5. *Let* $k \leqq 3$. *The first* $2m+1$ *homotopy groups* $\bar{\pi}_i$ *of*

$$M(X, \mathrm{CP}^s * E(U, m))$$

can be computed from:

$$M(X, \mathrm{CP}^s * E(U, m)) \text{ is } (m+2-k)\text{-connected},$$

$$\bar{\pi}_m \approx \bar{H}^3(X), \qquad \bar{\pi}_{m+1} \approx \bar{H}^2(X),$$

$$\bar{\pi}_{m+2j+1} \approx \bar{H}^0(X;\ Z^{f(j)}) \oplus \bar{H}^2(X;\ Z^{f(j+1)}) \qquad (j = 1, 2, \ldots, \tfrac{1}{2}(m-1)),$$

and

$$0 \to \bar{H}^4(X; Z^{f(j+1)}) \to \bar{\pi}_{m+2j} \to \bar{H}^1(X; Z^{f(j)}) \to 0 \qquad (j = 1, 2, \ldots, \tfrac{1}{2}(m - \tfrac{3}{2}))$$

(see Figure 8.5), *where* $Z^{f(j)}$ *is the direct sum of* $f(j)$ *copies of* Z.

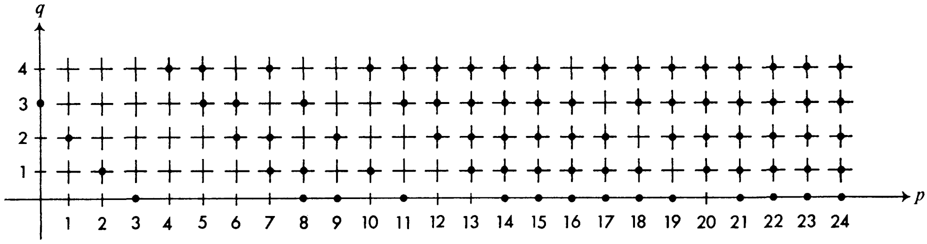

FIGURE 8.1. $\bar{E}^2(X, F_4)$

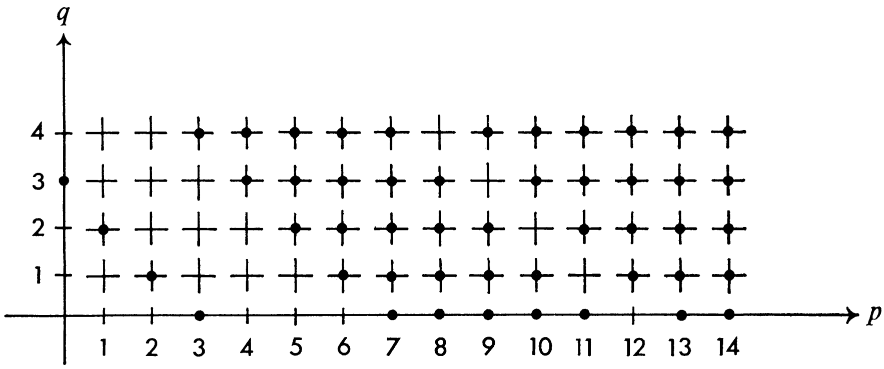

FIGURE 8.2. $\bar{E}^2(X, \mathrm{Spin}(7))$

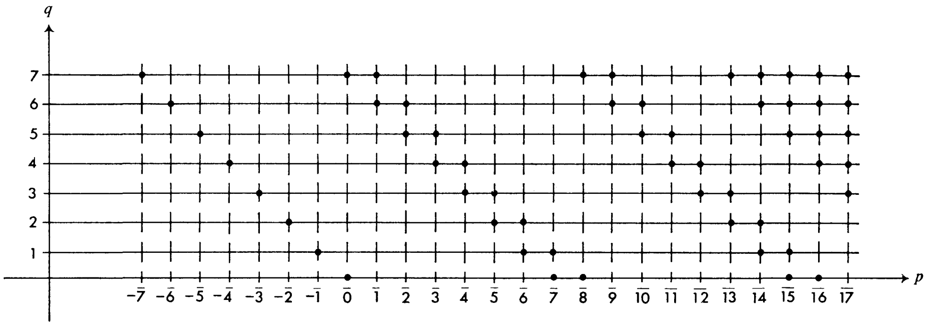

FIGURE 8.3. $\bar{E}^2(X, Y_5^n * S^m), \bar{j} = m + n + 1 + j$

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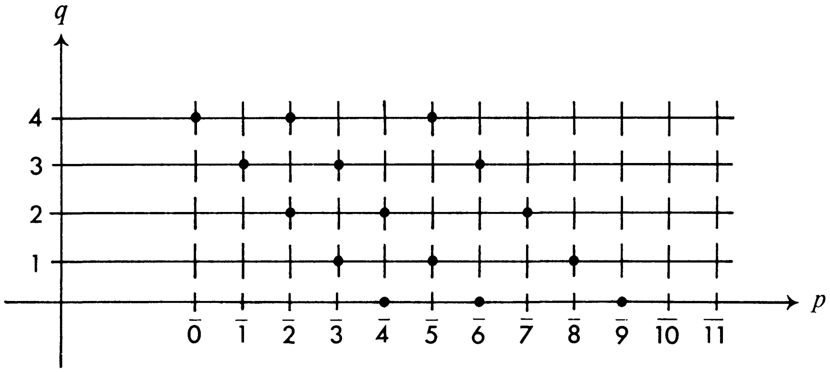

FIGURE 8.4. $\bar{E}^2(X, G_{3,1} * K(\pi, m)), \bar{j} = m + j$

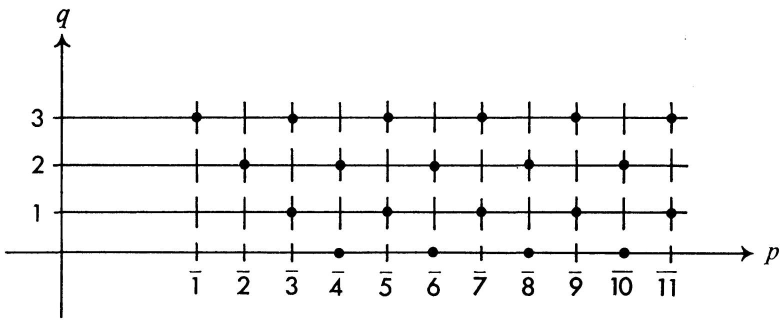

FIGURE 8.5. $\bar{E}^2(X, \mathrm{CP}^k * E(U, m)), \bar{j} = m + j - 1$

9. Computation of homotopy of $M(\mathrm{CP}^k, U)$, $M(V_{n+1,2}, U)$, and $M(L(p, q), S^n)$.

THEOREM 9.1.

$$\pi_i(M(\mathrm{CP}^k, U), v) \approx \begin{cases} Z^{k+1} & \textit{if } i \textit{ is odd}, \\ 0 & \textit{if } i \textit{ is even}, i > 0, \end{cases}$$

where v: $\mathrm{CP}^k \to U$ *is any map.*

This follows as an easy corollary to the following more general theorem (Theorem 9.2).

Definition. An arc-connected space X is said to be free in *even* (*odd*) *dimensions* if and only if $\bar{H}^i(X)$ is free abelian for all i and $\bar{H}^i(X) = 0$ for odd i ($\bar{H}^i(X) = 0$ for even $i > 0$).

Assume, as usual, that X is a CW complex or X is a metric space. Let

$$\bar{H}^*(X) = \sum \bar{H}^i(X) \quad \text{and} \quad \tilde{H}^*(X) = \tilde{H}^0(X) \oplus \sum_{i>0} \bar{H}^i(X),$$

where $\tilde{H}^0(X)$ is the reduced cellular (Čech) group if X is a CW complex (metric space).

THEOREM 9.2. (a) *Let X have finite dimension $2k$ and let X be free in even dimensions. Then, if $v: X \to U$ is constant,*

$$\pi_j(M(X, U), v) \approx \begin{cases} \bar{H}^*(X) & \text{if } j \text{ is odd,} \\ 0 & \text{if } j \text{ is even, } j > 0. \end{cases}$$

(b) *Let X have finite dimension $2k + 1$ and let X be free in odd dimensions. Then, if $v: X \to U$ is constant,*

$$\pi_i(M(X, U), v) \approx \begin{cases} Z & \text{if } j \text{ is odd,} \\ \bar{H}^*(X) & \text{if } j \text{ is even, } j > 0. \end{cases}$$

Proof. Case (a) gives rise to the following $\bar{E}^2$ (see Figure 9.1):

$$\bar{E}_{p,q}{}^2 = 0 \quad (\text{unless } p \text{ is odd } and\ q = 0, 2, 4, \ldots, 2k).$$

Thus, $d^i: \bar{E}_{p,q}{}^i \to \bar{E}_{p-1,q+i}{}^i$ is zero for all $i, p, q \Rightarrow \bar{E}_{p,q}{}^2 = \bar{E}_{p,q}{}^\infty$ for all p, q. Consider $\pi_p(M(X, U), v) \supset \pi_{p,0} \supset \pi_{p,1} \supset \ldots, \pi_{p,2k-1} \supset \pi_{p,2k} = 0$ such that

$$\frac{\pi_{p,q-1}}{\pi_{p,q}} \approx \bar{E}_{p,q}{}^2 \approx \bar{H}^q(X).$$

Note that if q is odd,

$$\pi_{p,q-1} = \pi_{p,q}, \qquad \pi_{p,2k-2} \approx \pi_{p,2k-1} \approx \bar{H}^{2k}(X),$$

and

$$\frac{\pi_{p,2k-3}}{\pi_{p,2k-1}} \approx \frac{\pi_{p,2k-3}}{\pi_{p,2k-2}} \approx \bar{H}^{2k-2}(X).$$

Since X is free, then $\pi_{p,2k-3} = \pi_{p,2k-2} = \bar{H}^{2k-2}(X) \oplus \bar{H}^{2k}(X)$. Continuation yields

$$\bar{\pi}_p \approx \bar{H}^*(X) \quad (\text{for } p \text{ odd}).$$

Note that this theorem is valid *for any basepoint* $v: X \to U$ provided X is a CW complex. This is not so with part (b), where the fact that $d^i: E_{p,0}{}^i \to E_{p-1,n}{}^i$ is zero is *needed*. The easy (and similar) proof of (b) is left to the reader.

THEOREM 9.3. *If n is even,*

$$\pi_i(M(V_{n+1,2}, U), v) \approx \begin{cases} Z & \text{if } i \text{ is even, } i > 0, \\ Z_2 \oplus Z & \text{if } i \text{ is odd, } i \geqq 1, \end{cases}$$

where $v: V_{n+1,2} \to U$ is constant. The reason this little result is included is that it is an example of a case where a two-term condition does not hold on $\mathfrak{C}^2(X, Y, v)$ but does hold on $\mathfrak{C}^4(X, Y, v)$ and results can still be obtained.

Proof. The proof for $n = 4$ will be given. The extension to any even n is obvious. $X = V_{5,2}$ has the following cohomology:

$$\tilde{H}^i(X) \approx \begin{cases} Z & \text{if } i = 7, \\ Z_2 & \text{if } i = 4, \\ 0 & \text{otherwise.} \end{cases}$$

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Then $\mathfrak{C}(V, U, v)$ satisfies a TTC$\{0, \infty; 4\}$ (see Figure 9.2) since

$$\bar{E}_{p,q}{}^2 = 0 \quad (\text{for } p \text{ odd, } q \neq 0, 4 \text{ or } p \text{ even, } q \neq 7).$$

If p is odd, then $\bar{E}_{p,0}{}^2 \approx \bar{H}^0 \approx \bar{H}^0(V) \approx Z$, $\bar{E}_{p,4}{}^2 \approx Z_2$; if p is even, then $\bar{E}_{p,7}{}^2 \approx Z$. Since v is constant and X is connected, Theorem 6.4 implies that $d^i\colon \bar{E}_{p,0}{}^i \to \bar{E}_{p-1,i}{}^i$ is zero. Thus,

$$\bar{E}_{p,0}{}^4 = \bar{E}_{p,0}{}^2 \approx Z, \quad \bar{E}_{p,4}{}^2 = \bar{E}_{p,4}{}^3, \quad \text{and} \quad \bar{E}_{p,7}{}^2 = \bar{E}_{p,7}{}^3.$$

Consider $d^3\colon \bar{E}_{p,4}{}^3 \to \bar{E}_{p,7}{}^3$, then $d^3\colon Z_2 \to Z$ must be zero. Thus,

$$\bar{E}_{p,4}{}^2 = \bar{E}_{p,7}{}^4 \quad \text{and} \quad \bar{E}_{p,7}{}^2 = \bar{E}_{p,7}{}^4,$$

and the theorem follows.

Let $X = L(p, q)$, the 3-dimensional Lens space associated with the integers (p, q), p *odd*. In the remainder of this section, computations of many of the first $n + 17$ homotopy groups of $M(L(p, q), S^n)$ for $n > 20$ will be given. This is an interesting example in which a two-term condition arises in

$$\mathfrak{C}^2(L(p, q), S^n, v)$$

[v constant] without the Gap Theorems (see **5**) holding in $L(p, q)$ or S^n. Let (A)–(I) represent the following statements:

(A): $[p \equiv 0\ (3)]$;
(B): $[p \equiv 0\ (5) \wedge p \not\equiv 0\ (3, 7, 11)]$;
(C): $[p \equiv 0\ (7) \wedge p \not\equiv 0\ (3, 5, 11)]$;
(D): $[p \equiv 0\ (11) \wedge p \not\equiv 0\ (3, 5, 7)]$;
(E): $[p \equiv 0\ (5, 7) \wedge p \not\equiv 0\ (3, 11)]$;
(F): $[p \equiv 0\ (5, 11) \wedge p \not\equiv 0\ (3, 7)]$;
(G): $[p \equiv 0\ (7, 11) \wedge p \not\equiv 0\ (3, 5)]$;
(H): $[p \equiv 0\ (5, 7, 11) \wedge p \not\equiv 0\ (3)]$;
(I): $[p \not\equiv 0\ (3, 5, 7, 11)]$.

The computation is given by Table 9.1 plus the fact that $M(L(p, q), S^n)$ is $(n - 4)$-connected. Generators for the groups in Table 9.1 are given by generators in the corresponding groups for S^n. The following is a sample calculation.

Let statement (C) hold, i.e., $p \equiv 0\ (7)$ and $p \not\equiv 0\ (3, 5, 11)$. Computations from (**9**) show that

$$E_{s,t}{}^2(L(p, q), S^n) \approx \begin{cases} \pi_s(S^n) & (t = 0), \\ T_p(\pi_{s+1}(S^n)) & (t = 1), \\ (\pi_{s+2}(S^n))_p & (t = 2), \\ \pi_{s+3}(S^n) & (t = 3), \end{cases}$$

where $G_p = G/pG$ and $T_p(G) = \{g \in G | p \cdot g = 0\}$. The tables in (**16**) show that $E_{s,1}{}^2 = T_p(\pi_{s+1}(S^n)) = 0$ for $0 \leqq s \leqq n + 18$, unless $s = n + 10$, and $E_{s,2}{}^2 = (\pi_{s+2}(S^n))_p = 0$ for $0 \leqq s \leqq n + 17$, unless $s = n, n + 9$. Thus, E_2 is described in Figure 9.3.

This yields two-term conditions $\{0, n + 9; 2\}$ and $\{n + 11, n + 17; 2\}$ on $\mathfrak{C}^2$, and the results for statement (C) (Table 9.1) follow easily.

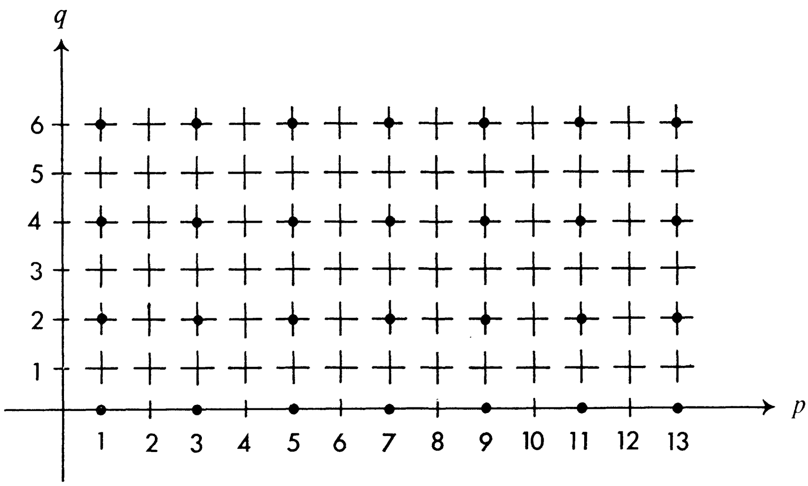

FIGURE 9.1. $\bar{E}^2(X, U, v)$

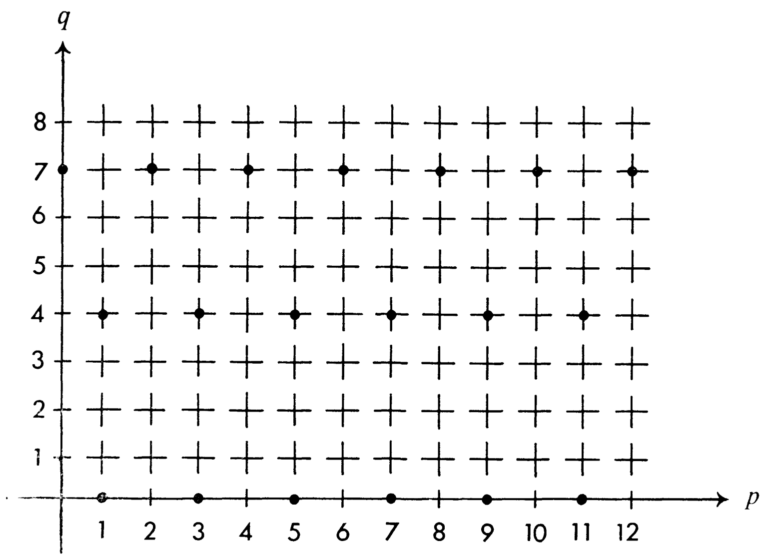

FIGURE 9.2. $\bar{E}^2(V_{5,2}, U, v)$

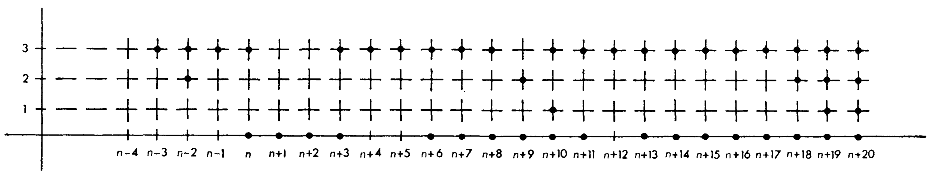

FIGURE 9.3. $E_2(L(p, q), S^n)$, $n > 20$, $p \equiv 0\ (7) \wedge p \not\equiv 0\ (3, 5, 11)$

TABLE 9.1. $\pi_{n-4+i}[S^{nL(p,q)}, v]$, $1 \leqq i \leqq 21$

i	$\pi_{n-4+i}[M(L(p, q), S^n), v]$, p odd, $n > 20$	
1	Z	
2	$Z_p \oplus Z_2$	
3	Z_2	
4	$Z \oplus Z_{24}$	
5	$Z_2 \oplus Z_3$	if A
	Z_2	if $\sim A$
6	$Z_2 \oplus Z_3$	if A
	Z_2	if $\sim A$
7	$Z_2 \oplus Z_{24}$	
8	Z_{240}	
9	Z_2^2	if $C \vee D \vee G \vee I$
10	Z_2^4	if $C \vee D \vee G \vee I$
11	$Z_{240} \oplus Z_6$	
12	$Z_{504} \oplus Z_2^2$	if $\sim A$
13	$Z_2^3 \oplus (Z_{504})_p$	if $C \vee E \vee G \vee H$
	Z_2^3	if $B \vee D \vee F \vee I$
14	$Z_6 \oplus Z_3$	if $B \vee D \vee E \vee I$
15	$Z_{504} \oplus Z_3$	
16	$Z_{480} \oplus Z_2$	if $\sim A$
	$0 \to Z_{480} \oplus Z_2 \to \bar{\pi}_{n+12} \to Z_3 \to 0$	if A
17	$Z_3 \oplus Z_2^2$	if $C \vee D \vee G \vee I$
18	Z_2^6	if $C \vee D \vee G \vee I$
	$0 \to Z_2^4 \to \bar{\pi}_{n+14} \to Z_{480} \oplus Z_2 \to 0$	if A
19	$Z_{480} \oplus Z_8 \oplus Z_2^2$	
20	$Z_2^3 \oplus Z_{264}$	if $\sim A$
21	$Z_2^4 \oplus \pi_{n+20}(S^n)$	if $B \vee C \vee E \vee I$

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University of Oregon,
Eugene, Oregon