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The Global Attractor of a Damped, Forced Hirota Equation in H^1

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Abstract. The existence of the global attractor of a damped forced Hirota equation in the phase space $H^1(\mathbb{R})$ is proved. The main idea is to establish the so-called asymptotic compactness property of the solution operator by energy equation approach.

1 Introduction

Our aim is to study the global attractor of the Cauchy problem for the following damped, forced Hirota equation in the phase space $H^1(\mathbb{R})$,

$$\begin{gathered} i\partial_t u + \alpha\partial_x^2 u + i\beta\partial_x^3 u + i\lambda u\partial_x(|u|^2) + i\mu|u|^2\partial_x u + \eta|u|^2 u + i\gamma u = f, \\ u(x,0) = u_0(x) \in H^1(\mathbb{R}), \quad (x,t) \in \mathbb{R}\times\mathbb{R}. \end{gathered} \tag{1.1}$$

where $\alpha, \beta, \lambda, \mu, \eta, \gamma$ are real constants, $\beta\cdot\alpha \neq 0$, $\gamma > 0$, and f is time-independent and belongs to $H^1(\mathbb{R})$. The parameter $\gamma > 0$ can be looked on as a damping coefficient.

The Hirota equation (1.1) is a typical model of mathematical physics. It encompasses the well-known nonlinear Schrödinger equation and the modified KdV equation, and in particular contains the nonlinear derivative Schrödinger equation. Hasegawa and Kodama [10,11,17] proposed (1.1) as a model for propagation of pulse in optical fibers.

If $\gamma = 0$ and $f = 0$, for global well-posedness of the Cauchy problem (1.1), Laurey [19] obtained global well-posedness in $H^1(\mathbb{R})$ and $H^s(\mathbb{R})(s \geq 2)$. Huo and Guo [12] obtained local well-posedness in $H^s(\mathbb{R})(s \geq \frac{1}{4})$ and a global result in $H^s(\mathbb{R})(s \geq 1)$. From the mathematical point of view, the extra term with the factor γ accounts for a weak dissipation with no regularization, or smoothing property. Hence the well-posedness of (1.1) comes essentially from the dispersive regularization property of the equation (1.1). The proof of well-posedness is similar to that in [12]. We use the so-called the Fourier restriction norm (Bourgain function spaces) to consider the problem. This is one facet of the paper. The Fourier restriction norm method was first introduced by J. Bourgain [2,3] to study the KdV and nonlinear Schrödinger equations in the periodic case. It was simplified by Kenig, Ponce, and Vega [15,16].

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Another facet of this paper is to consider the long time behavior of the solutions, which is described by the global attractor. The existence of a global attractor for hyperbolic equations is usually obtained by the asymptotic compactness or the asymptotic smoothing properties of the solution operator, together with the existence of a bounded absorbing set. Those properties are usually proved by the energy equation method. For equations on unbounded domains, the use of energy equation is particularly suitable, since it does not depend on compact imbedding of function spaces. It does require, however, the weak continuity of the solution operator in the sense that if the initial datum u_{0n} converges to u_0 weakly, then the corresponding solution $u_n(t)$ converges weakly to $u(t)$, for all time t. This weak continuity is usually obtained by passing to the limit in the weak formulation of the equation and by using the uniqueness of the solution.

There are many papers on the existence of global attractors of equations such as the KdV equation or the Schrödinger equation [4–8, 14, 20–22]. However, there seem to be none considering the existence of global attractors of Hirota equations. Therefore, our result seems to be new.

To study well-posedness of the solution to problems (1.1), we use its equivalent integral formulation

$$u(t) = W(t)u_0 + i\int_0^t W(t-t')(i\lambda u\partial_x(|u|^2) + i\mu|u|^2\partial_x u + \eta|u|^2u + i\gamma u - f)(t')dt',$$

where $W(t) = \mathcal{F}_x^{-1}e^{-it(\alpha\xi^2-\beta\xi^3)}\mathcal{F}_x$ is the unitary operator associated with the corresponding linear equation. For simplicity, denote the phase function as $\phi(\xi) = \alpha\xi^2 - \beta\xi^3$.

It is important to point out that the above phase function $\phi(\xi)$ has non-zero singular points, which is different from the phase function related to the linear KdV equation and also makes the problem much more difficult. Therefore, we need to use Fourier restriction operators

$$P^N f = \int_{|\xi|\geq N} e^{ix\xi}\hat{f}(\xi)d\xi, \quad P_N f = \int_{|\xi|\leq N} e^{ix\xi}\hat{f}(\xi)d\xi, \ \forall N > 0$$

to eliminate the singularity.

The paper is constructed as follows. In Section 2, we introduce some notations, define the Bourgain space related to equation (1.1) and obtain some preliminary estimates. Then in Section 3, we investigate the local solution of the problem in H^s for $s \geq \frac{1}{4}$. In Section 4, we discuss further the global well-posedness in H^1 and the absorbing sets. Finally we give the asymptotic compactness and the existence of the global attractor in Section 5.

2 Function Spaces and Preliminary Estimates.

For convenience, we introduce some notations:

- $\mathcal{F}F_\rho(\xi,\tau) = \dfrac{f(\xi,\tau)}{(1+|\tau+\phi(\xi)|)^\rho}$, $\phi(\xi) = \alpha\xi^2 - \beta\xi^3$ or $\alpha\xi^2 + \beta\xi^3$.

- $\sigma = \tau - \beta\xi^3 + \alpha\xi^2$, $\sigma_j = \tau_j - \beta\xi_j^3 + \alpha\xi_j^2 (j = 1, 2)$, $\bar{\sigma}_3 = \tau_3 - \beta\xi_3^3 - \alpha\xi_3^2$.
- $\int_\star \cdot \, d\delta := \int_{\xi=\xi_1+\xi_2+\xi_3; \tau=\tau_1+\tau_2+\tau_3} \cdot \, d\tau_1 d\tau_2 d\tau_3 d\xi_1 d\xi_2 d\xi_3$.
- $D_x^s = \mathcal{F}_x^{-1} |\xi|^s \mathcal{F}_x$ for fraction s,
- $D_x^m = \mathcal{F}_x^{-1} (i\xi)^m \mathcal{F}_x$ for integral m.
- $J_x^s = \mathcal{F}_x^{-1} \langle\xi\rangle^s \mathcal{F}_x$, $a = \max\big(1, \big|\frac{2\alpha}{3\beta}\big|\big)$.
- $\|f\|_{L_x^p L_t^q} = (\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} |f(x,t)|^q dt)^{\frac{p}{q}} dx)^{1/p}$, $\quad \|f\|_{L_t^\infty H_x^s} = \big\| \|f\|_{H_x^s} \big\|_{L_t^\infty}$.
- $\int (J^2 u)\bar{v} dx = (J^2 u, v)_{L^2} = (u, v)_{H^1}$.

For $s, b \in \mathbb{R}$, $X_{s,b}$ denotes the completion of the Schwartz function space on $\mathbb{R}^2$ with respect to the norm

$$\|u\|_{X_{s,b}} = \|\langle\xi\rangle^s \langle\tau - \beta\xi^3 + \alpha\xi^2\rangle^b \mathcal{F}u\|_{L_\xi^2 L_\tau^2},$$

$$\|\bar{u}\|_{\bar{X}_{s,b}} = \|\langle\xi\rangle^s \langle\tau - \beta\xi^3 - \alpha\xi^2\rangle^b \mathcal{F}\bar{u}\|_{L_\xi^2 L_\tau^2},$$

where $\langle \cdot \rangle = (1 + |\cdot|)$. One can easily prove that $\|u\|_{X_{s,b}} = \|\bar{u}\|_{\bar{X}_{s,b}}$, which will be used later.

Let $\psi \in C_0^\infty(\mathbb{R})$ with $\psi = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $\operatorname{supp} \psi \subset [-1, 1]$. Denote $\psi_\delta(\cdot) = \psi(\delta^{-1}(\cdot))$ for some $\delta \in \mathbb{R}$.

For $T > 0$, we consider the localized Bourgain space $X_{s,b}^T$, which is endowed with the norm

$$\|u\|_{X_{s,b}^T} = \|u\|_{X_{s,b}^{[-T,T]}} = \|\psi_T u\|_{X_{s,b}}.$$

In our arguments, we shall use the trivial embedding relation $\|u\|_{X_{s_1,b_1}} \leq \|u\|_{X_{s_2,b_2}}$ whenever $s_1 \leq s_2$, $b_1 \leq b_2$. Denote $\hat{u}(\tau, \xi) = \mathcal{F}u$ by the Fourier transform in t and x of u and $\mathcal{F}_{(\cdot)} u$ by the Fourier transform in the $(\cdot)$ variable. Define the space

$$H_0^1(-r, r) := \{f \in H^1(\mathbb{R}), \operatorname{supp} f \subset (-r, r), r > 0\}.$$

Note that for all $r > 0$, the continuous injections hold

$$H_0^1(-r, r) \subseteq H^1(\mathbb{R}) \subseteq L^2(\mathbb{R}) \subseteq H^{-1}(\mathbb{R}) \subseteq H_0^{-1}(-r, r), \tag{2.1}$$

and the compact injections also hold

$$H_0^1(-r, r) \subset_c L^2(\mathbb{R}) \subset_c H_0^{-1}(-r, r). \tag{2.2}$$

We will often use Agmon's inequality, which reads

$$\|u\|_{L^\infty(\mathbb{R})} \leq \|u\|_{L^2(\mathbb{R})}^{1/2} \|u_x\|_{L^2(\mathbb{R})}^{1/2}.$$

Lemma 2.1 [23] *The group $\{W(t)\}_{-\infty}^{+\infty}$ satisfies*

$$\|W(t)u_0\|_{L_x^8 L_t^8} \leq C\|u_0\|_{L^2}.$$

Lemma 2.2 [12] *The group* $\{W(t)\}_{-\infty}^{+\infty}$ *satisfies*

$$\|D_x W(t)P^{2a}u_0\|_{L_x^\infty L_t^2} \le C\|u_0\|_{L^2}, \quad \|W(t)P^a u_0\|_{L_x^4 L_t^\infty} \le C\|u_0\|_{H^{1/4}},$$

$$\|D_x^{1/6} W(t)P^{2a}u_0\|_{L_x^6 L_t^6} \le C\|u_0\|_{L^2},$$

where the constant C *depends on* β *and* α.

Lemma 2.3 [15] *If* $\rho > \frac{1}{2}$ *for any fixed* N *with* $0 < N < +\infty$, *it holds that*

$$\|P_N F_\rho\|_{L_x^2 L_t^\infty} \le C\|f\|_{L_\xi^2 L_\tau^2},$$

where the constant C *depends on* N.

Lemma 2.4 [12] *If* $\rho > \frac{1}{2}\frac{4(q-2)}{3q}$, *for* $2 \le q \le 8$, *then* $\|F_\rho\|_{L_x^q L_t^q} \le C\|f\|_{L_\xi^2 L_\tau^2}$, *where the constant* C *depends on* β *and* α.

Lemma 2.5 [12]

(i) *Let* $\rho > \frac{\theta}{2}$ *with* $\theta \in [0,1]$. *Then* $\|D_x^\theta P^{2a} F_\rho\|_{L_x^{\frac{2}{1-\theta}} L_t^2} \le C\|f\|_{L_\xi^2 L_\tau^2}$.

(ii) *Let* $\rho > \frac{1}{2}$. *Then* $\|D_x^{-\frac{1}{4}} P^{2a} F_\rho\|_{L_x^4 L_t^\infty} \le C\|f\|_{L_\xi^2 L_\tau^2}$, *where the constant* C *depends on* β *and* α.

Lemma 2.6 [12] *Assume* f, f_1, f_2, *and* f_3 *belong to Schwartz space on* $\mathbb{R}^2$. *Then*

$$\int_\star \overline{\hat{f}}(\xi,\tau)\hat{f}_1(\xi_1,\tau_1)\hat{f}_2(\xi_2,\tau_2)\hat{f}_3(\xi_3,\tau_3)\,d\delta = \int \bar{f} f_1 f_2 f_3(x,t)\,dxdt.$$

Lemma 2.7 [15,16] *Let* $s \in \mathbb{R}$, $\frac{1}{2} < b < b' < 1$, $0 < \delta \le 1$. *Then*

$$\|\psi_\delta(t)W(t)u_0\|_{X_{s,b}} \le C\delta^{\frac{1}{2}-b}\|u_0\|_{H^s},$$

$$\|\psi_\delta(t)\int_0^t S(t-\tau)F(\tau)\,d\tau\|_{X_{s,b}} \le C\delta^{\frac{1}{2}-b}\|F\|_{X_{s,b-1}},$$

$$\|\psi_\delta(t)\int_0^t S(t-\tau)F(\tau)\,d\tau\|_{L_t^\infty H_x^s} \le C\delta^{\frac{1}{2}-b}\|F\|_{X_{s,b-1}},$$

$$\|\psi_\delta(t)F\|_{X_{s,b-1}} \le C\delta^{b'-b}\|F\|_{X_{s,b'-1}}.$$

3 Local Solutions in H^s

In this section, we obtain, by the contraction mapping principle, the local well-posedness of the problem, which is given by Theorem 3.2. The contraction argument provides the local solution, once we prove that the following estimates hold for some $b > \frac{1}{2}$:

(3.1) $$\|\partial_x(|u|^2 u)\|_{X_{s,b-1}} \le C\|u\|_{X_{s,b}}^3,$$

(3.2) $$\||u|^2\partial_x u\|_{X_{s,b-1}} \le C\|u\|_{X_{s,b}}^3,$$

(3.3) $$\|(|u|^2 u)\|_{X_{s,b-1}} \le C\|u\|_{X_{s,b}}^3.$$

In fact, (3.1), (3.2), and (3.3) can be shown in the following theorem.

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Theorem 3.1 *If $s \geq \frac{1}{4}$, $\frac{1}{2} < b < \frac{7}{9}$, $b' > \frac{1}{2}$, then*

$$\|\partial_x(u_1 u_2 \bar{u}_3)\|_{X_{s,b-1}} \leq C\|u_1\|_{X_{s,b'}}\|u_2\|_{X_{s,b'}}\|u_3\|_{X_{s,b'}}, \tag{3.4}$$

$$\|\partial_x(u_1) u_2 \bar{u}_3\|_{X_{s,b-1}} \leq C\|u_1\|_{X_{s,b'}}\|u_2\|_{X_{s,b'}}\|u_3\|_{X_{s,b'}}, \tag{3.5}$$

$$\|u_1 u_2 \bar{u}_3\|_{X_{s,b-1}} \leq C\|u_1\|_{X_{s,b'}}\|u_2\|_{X_{s,b'}}\|u_3\|_{X_{s,b'}}. \tag{3.6}$$

Remark The proofs of (3.4) and (3.6) can be found in [12]. In fact, (3.6) holds if $s \geq -\frac{1}{4}$ [13]. The proof of (3.5) is similar to that of (3.4).

Proof Here we only prove (3.5). By duality and the Plancherel identity, it suffices to show that

$$\begin{aligned}
\Upsilon &= \int_\star \langle\xi\rangle^s|\xi_1|\frac{\bar{f}(\tau,\xi)}{\langle\sigma\rangle^{1-b}}\mathcal{F}u_1(\tau_1,\xi_1)\mathcal{F}u_2(\tau_2,\xi_2)\mathcal{F}\bar{u}_3(\tau_3,\xi_3)\,d\delta \\
&= \int_\star \frac{\langle\xi\rangle^s|\xi_1|}{\langle\sigma\rangle^{1-b}\prod_{j=1}^2\langle\xi_j\rangle^s\langle\sigma_j\rangle^{b'}\langle\xi_3\rangle^s\langle\bar{\sigma}_3\rangle^{b'}}\bar{f}(\tau,\xi)f_1(\tau_1,\xi_1)f_2(\tau_2,\xi_2)f_3(\tau_3,\xi_3)\,d\delta \\
&\leq C\|f\|_{L_2}\prod_{j=1}^3\|f_j\|_{L_2},
\end{aligned}$$

for all $\bar{f} \in L^2$, $\bar{f} \geq 0$, where

$$f_j = \langle\xi_j\rangle^s\langle\sigma_j\rangle^{b'}\hat{u}_j,\ j = 1,2; \quad f_3 = \langle\xi_3\rangle^s\langle\bar{\sigma}_3\rangle^{b'}\hat{\bar{u}}_3;$$
$$\xi = \xi_1 + \xi_2 + \xi_3;, \quad \tau = \tau_1 + \tau_2 + \tau_3.$$

We may assume $f_j \geq 0$, $j = 1,2,3$. Let

$$\mathcal{F}F_\rho^j(\xi,\tau) = \frac{f_j(\xi,\tau)}{(1+|\tau-\beta\xi^3+\alpha\xi^2|)^\rho}, \quad j = 1,2,$$
$$\mathcal{F}F_\rho^3(\xi,\tau) = \frac{f_3(\xi,\tau)}{(1+|\tau-\beta\xi^3-\alpha\xi^2|)^\rho},$$

Let

$$K(\xi,\xi_1,\xi_2,\xi_3) = \frac{\langle\xi\rangle^s|\xi_1|}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s}.$$

In order to obtain the boundedness of integral Υ, we split the integration domain into several pieces.

Case 1: Assume $|\xi| \leq 6a$.

1.1 If $|\xi_1| \leq 2a$, then $K(\xi,\xi_1,\xi_2,\xi_3) \leq C$. Using Lemma 2.4 and Lemma 2.6, the integral Υ restricted to this domain is bounded by

$$C\|F_{1-b}\|_{L_x^2L_t^2}\|F_{b'}^1\|_{L_x^6L_t^6}\|F_{b'}^2\|_{L_x^6L_t^6}\|F_{b'}^3\|_{L_x^6L_t^6} \leq C\|f\|_{L_\xi^2L_\tau^2}\|f_1\|_{L_\xi^2L_\tau^2}\|f_2\|_{L_\xi^2L_\tau^2}\|f_3\|_{L_\xi^2L_\tau^2}.$$

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1.2 Assume $|\xi_1| \geq 2a$.

1.2.1 If $|\xi_2| \leq 2a$ or $|\xi_3| \leq 2a$(without loss of generality, we can assume $|\xi_2| \leq 2a$), then $K(\xi, \xi_1, \xi_2, \xi_3) \leq C|\xi_1|$. The integral Υ restricted to this domain is bounded by

$$C\|F_{1-b}\|_{L_x^3 L_t^3} \|D_x P^{2a} F_{b'}^1\|_{L_x^\infty L_t^2} \|P_{2a} F_{b'}^2\|_{L_x^2 L_t^\infty} \|F_{b'}^3\|_{L_x^6 L_t^6} \leq C\|f\|_{L_\xi^2 L_\tau^2} \|f_1\|_{L_\xi^2 L_\tau^2} \|f_2\|_{L_\xi^2 L_\tau^2} \|f_3\|_{L_\xi^2 L_\tau^2},$$

which follows from Lemmas 2.3–2.6 for $b < \frac{7}{9}$.

1.2.2 If $|\xi_2| \geq 2a$ and $|\xi_3| \geq 2a$, then we use $s \geq \frac{1}{4}$ to bound $K(\xi, \xi_1, \xi_2, \xi_3)$ by

$$K(\xi, \xi_1, \xi_2, \xi_3) \leq C \frac{|\xi_1|}{\langle \xi_2 \rangle^{\frac{1}{4}} \langle \xi_3 \rangle^{\frac{1}{4}}}.$$

The integral Υ restricted to this domain is bounded by

$$C\|F_{1-b}\|_{L_x^2 L_t^2} \|D_x P^{2a} F_{b'}^1\|_{L_x^\infty L_t^2} \|P^{2a} D_x^{-\frac{1}{4}} F_{b'}^2\|_{L_x^4 L_t^\infty} \|P^{2a} D_x^{-\frac{1}{4}} F_{b'}^3\|_{L_x^4 L_t^\infty} \leq C\|f\|_{L_\xi^2 L_\tau^2} \|f_1\|_{L_\xi^2 L_\tau^2} \|f_2\|_{L_\xi^2 L_\tau^2} \|f_3\|_{L_\xi^2 L_\tau^2},$$

which follows from Lemmas 2.4–2.6.

Case 2: Assume $|\xi| \geq 6a$.

2.1 If $|\xi_1| \leq 2a$, then $|\xi| \leq 3\max\{|\xi_2|, |\xi_3|\}$ (without loss of generality, we can assume $2a \leq \frac{1}{3}|\xi| \leq |\xi_3|$). It follows that $K(\xi, \xi_1, \xi_2, \xi_3) \leq C$. Similarly to Case 1.1, we can obtain the boundedness of the integral Υ.

2.2 Assume $|\xi_1| \geq 2a$, we distinguish the different situations.

2.2.1 If $|\xi_2| \leq 2a$ or $|\xi_3| \leq 2a$ (without loss of generality, we can assume $|\xi_3| \leq 2a$), then $|\xi| \leq 3\max\{|\xi_1|, |\xi_2|\}$. It follows that $K(\xi, \xi_1, \xi_2, \xi_3) \leq C|\xi_1|$. The integral Υ restricted to this domain is bounded by

$$C\|F_{1-b}\|_{L_x^3 L_t^3} \|D_x P^{2a} F_{b'}^1\|_{L_x^\infty L_t^2} \|F_{b'}^2\|_{L_x^6 L_t^6} \|P_{2a} F_{b'}^3\|_{L_x^2 L_t^\infty} \leq C\|f\|_{L_\xi^2 L_\tau^2} \|f_1\|_{L_\xi^2 L_\tau^2} \|f_2\|_{L_\xi^2 L_\tau^2} \|f_3\|_{L_\xi^2 L_\tau^2},$$

which follows from Lemmas 2.3–2.6 for $b < \frac{7}{9}$.

2.2.2 Assume $|\xi_2| \geq 2a$ and $|\xi_3| \geq 2a$. We discuss three situations separately.

(i) If $|\xi| \leq 3\max\{|\xi_1|, |\xi_2|, |\xi_3|\} = 3|\xi_1|$, then we can obtain the boundedness of the integral Υ similarly to Case 1.2.2.

(ii) If $|\xi| \leq 3\max\{|\xi_1|, |\xi_2|, |\xi_3|\} = 3|\xi_2|$, by symmetry, we have the following three cases.
If $s \geq 1$, then $K(\xi, \xi_1, \xi_2, \xi_3) \leq C$. Similarly to Case 1.1 we can obtain the boundedness of the integral Υ.

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If $\frac{1}{2} \leq s \leq 1$, then

$$K(\xi, \xi_1, \xi_2, \xi_3) \leq C \frac{|\xi_1|^{1/2}}{\langle \xi_3 \rangle^{1/4}}.$$

By Lemma 2.4–Lemma 2.6 for $b < \frac{7}{9}$, the integral Υ restricted to this domain is bounded by

$$C\|F_{1-b}\|_{L_x^3 L_t^3} \|D_x^{\frac{1}{2}} P^{2a} F_{b'}^1\|_{L_x^4 L_t^2} \|F_{b'}^2\|_{L_x^6 L_t^6} \|D_x^{-\frac{1}{4}} P^{2a} F_{b'}^3\|_{L_x^4 L_t^\infty} \\ \leq C\|f\|_{L_\xi^2 L_\tau^2} \|f_1\|_{L_\xi^2 L_\tau^2} \|f_2\|_{L_\xi^2 L_\tau^2} \|f_3\|_{L_\xi^2 L_\tau^2}.$$

If $\frac{1}{4} \leq s \leq \frac{1}{2}$, then

$$K(\xi, \xi_1, \xi_2, \xi_3) \leq C \frac{\langle \xi \rangle^s |\xi_1|^{1-s}}{\langle \xi_2 \rangle^{1/4} \langle \xi_3 \rangle^{\frac{1}{4}}}.$$

We obtain the boundedness of the integral Υ as follows,

$$C\|D_x^s P^{6a} F_{1-b}\|_{L_x^{\frac{2}{1-s}} L_t^2} \|D_x^{1-s} P^{2a} F_{b'}^1\|_{L_x^{\frac{2}{s}} L_t^2} \|D_x^{-\frac{1}{4}} P^{2a} F_{b'}^2\|_{L_x^4 L_t^\infty} \|D_x^{-\frac{1}{4}} P^{2a} F_{b'}^3\|_{L_x^4 L_t^\infty} \\ \leq C\|f\|_{L_\xi^2 L_\tau^2} \|f_1\|_{L_\xi^2 L_\tau^2} \|f_2\|_{L_\xi^2 L_\tau^2} \|f_3\|_{L_\xi^2 L_\tau^2},$$

which follows from Lemma 2.4–Lemma 2.6 for $1 - b \geq \frac{s}{2}$.

(iii) If $|\xi| \leq 3 \max\{|\xi_1|, |\xi_2|, |\xi_3|\} = 3|\xi_3|$, then we can obtain the result similarly to case 2.2.2(ii).

Therefore, this completes the proof of Theorem 3.1. ■

Next, we prove the local well-posedness of Cauchy problem (1.1) in $H^s (s \geq \frac{1}{4})$. Assume $\gamma \in \mathbb{R}$ and $f = f(x, t) \in X_{s,b-1}^T (s \geq \frac{1}{4})$ for some $T > 0$.

For $u_0 \in H^s(\mathbb{R}) (s \geq \frac{1}{4})$, we define the operator

$$\Phi(u) = \psi_T(t) W(t) u_0 + i\psi_T(t) \int_0^t W(t - t')(i\lambda u \partial_x(|u|^2) + i\mu |u|^2 \partial_x u \\ + \eta |u|^2 u + i\gamma u - f)(t')\, dt',$$

and the set

$$\mathcal{B} = \{u \in X_{s,b}^T \cap H^1 : \|u\|_{X_{s,b}^T \cap H^s} \leq 4CT^{\frac{1}{2}-b} \|u_0\|_{H^s}\}.$$

In order to show that Φ is a contraction mapping on $\mathcal{B}$, we first prove $\Phi(\mathcal{B}) \subset \mathcal{B}$. From Lemma 2.7 and Theorem 3.1 for $\frac{1}{2} < b < b' < 1$, it follows that

$$\|\Phi(u)\|_{X_{s,b}^T} \leq CT^{\frac{1}{2}-b} \|u_0\|_{H^s} + CT^{b'-b}(\|u\|_{X_{s,b}^T}^3 + \|u\|_{X_{s,b}^T} + \|f\|_{X_{s,b-1}^T}).$$

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Therefore, if we fix T such that

$$8CT^{b'-b}T^{1-2b}\|u_0\|_{H^s}^2 \le \tfrac{1}{4}, \quad 4CT^{b'-b}T^{\frac{1}{2}-b} \le \tfrac{1}{4},$$
$$CT^{b'-b}\|f\|_{X_{s,b-1}^T} \le CT^{\frac{1}{2}-b}\|u_0\|_{H^s},$$

then $\Phi(\mathcal{B}) \subset \mathcal{B}$.

For $u, v \in \mathcal{B}$, in an analogous way to above, it follows that

$$\begin{aligned}\|\Phi(u) - \Phi(v)\|_{X_{s,b}^T} &\le CT^{b'-b}(\max(\|u\|_{X_{s,b}^T}, \|v\|_{X_{s,b}^T})^2 + 1)\|u - v\|_{X_{s,b}^T}\\ &\le CT^{b'-b}(8T^{1-2b}\|u_0\|_{H^s}^2 + 4T^{\frac{1}{2}-b})\|u - v\|_{X_{s,b}^T}\\ &\le \tfrac{1}{2}\|u - v\|_{X_{s,b}^T}.\end{aligned}$$

Therefore, Φ is a contraction mapping on $\mathcal{B}$. There exists a unique solution to Cauchy problem (1.1) in $X_{s,b}^T(s \ge \frac{1}{4})$ for $T > 0$. We state the result as the following theorem.

Theorem 3.2 *Let $s \ge \frac{1}{4}$, $\frac{1}{2} < b < \frac{7}{9}$. Let $u_0 \in H^s$, $f \in X_{s,b-1}$ and $\gamma \in \mathbb{R}$. Then there exists a constant $T > 0$, and Cauchy problem (1.1) admits a unique local solution $u(x,t) \in C([0,T]; H^s) \cap X_{s,b}^T$. Moreover, given $t \in (0,T)$, the map $(\gamma, f, u_0) \to u(t)$ is continuous from $\mathbb{R} \times X_{s,b-1}^T \times H^s$ to $C([0,T]; H^s)$.*

4 Global Well-Posedness in H^1 and Absorbing Sets

In this section, for $\gamma \in \mathbb{R}$, $f \in H^1$, and $u_0 \in H^1$, we will obtain the global well-posedness of the problem (1.1). This is achieved with the help of H^1 energy inequality as in [9, 19]. Similarly, we establish some energy-type equation for the solutions of (1.1).

Lemma 4.1 *Let $u_0 \in H^\infty(\mathbb{R})$, $f(x) \in H^\infty(\mathbb{R})$, and $u \in C([0,T]; H^\infty(\mathbb{R}))$ be the solution of (1.1). Then*

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \gamma\|u\|_{L^2}^2 = \operatorname{Im}\int f\bar{u}\,dx, \tag{4.1}$$

$$-\frac{1}{2}\frac{d}{dt}\int u\bar{u}_x dx - \lambda \operatorname{Im}\int (u\bar{u}_x)^2 dx + \gamma\int u\bar{u}_x dx = \Re\int f\bar{u}_x\,dx, \tag{4.2}$$

$$\begin{aligned}-\frac{1}{2}\frac{d}{dt}\|u_x\|_{L^2}^2 - \eta \operatorname{Im}\int (u\bar{u}_x)^2 dx - \Big(\lambda + \frac{\mu}{2}\Big)\int (\partial_x|u|)^2|u_x|^2 dx - \gamma\\ \int |u_x|^2\,dx = -\operatorname{Im}\int f_x\bar{u}_x dx,\end{aligned} \tag{4.3}$$

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\|u\|_{L^4}^4 + \frac{3\beta}{2}\int (\partial_x|u|)^2|u_x|^2 dx + \alpha \operatorname{Im}\int (uu_x)^2 dx + \gamma\|u\|_{L^4}^4\\ = \operatorname{Im}\int f|u|^2\bar{u}\,dx,\end{aligned} \tag{4.4}$$

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$$\frac{d}{dt}I_1(u) + 2\gamma I_1(u) = K_1(f,u), \tag{4.5}$$

where

$$I_1(u) = -\eta\lambda\|u_x\|_{L^2}^2 - \lambda(\lambda + \frac{\mu}{2})\|u\|_{L^4}^4 + (3\beta\eta - \alpha(2\lambda+\mu))\int uu_x\,dx,$$

$$K_1(f,u) = 2(3\beta\eta - \alpha(2\lambda+\mu))\,\Re\int f\bar{u}_x dx - 6\eta\lambda\,\mathrm{Im}\int f_x\bar{u}_x dx$$
$$- 2\lambda(2\lambda+\mu)\,\mathrm{Im}\int f|u|^2\bar{u}\,dx.$$

Proof We obtain (4.1) by multiplying (1.1) by $\bar{u}$ and (4.2) by $\bar{u}_x$, integrating and taking the imaginary part and the real part, respectively. Similarly, we obtain (4.3) by multiplying (1.1) by $\bar{u}_{xx}$ and (4.4) by $|u|^2\bar{u}$, integrating and taking the imaginary part. We obtain (4.5) by taking the following linear arrangement of the previous equations:

$$2(3\beta\eta - \alpha(2\lambda+\mu))\,(4.2) + 6\eta\lambda\,(4.3) - 2\lambda(2\lambda+\mu)\,(4.3). \qquad \blacksquare$$

Now choose a smooth function $\psi \in \mathcal{S}(\mathbb{R})$ such that $\psi \geq 0$ and $\int_{-\infty}^{+\infty}\psi(x)\,dx = 1$. Define $\psi_\varepsilon = \frac{1}{\varepsilon}\psi(\frac{x}{\varepsilon})$, for $\varepsilon > 0$. We regularize $u_0 \in H^1(\mathbb{R})$, $f(x) \in H^1(\mathbb{R})$: ${u_0}^\varepsilon = u_0 * \psi_\varepsilon$, $f^\varepsilon = f * \psi_\varepsilon$. The smooth functions ${u_0}^\varepsilon$ and f^ε converge respectively to u_0 and f in $H^1(\mathbb{R})$ as $\varepsilon \to 0$. We consider the solution of (1.1) $u^\varepsilon \in C([0,T_\varepsilon];H^\infty)$ with data $u^\varepsilon(0) = {u_0}^\varepsilon$ and forcing term f^ε. By Theorem 3.2 (the continuity with respect to the data of the local solution), it follows that with initial condition $u^\varepsilon(0) = {u_0}^\varepsilon$ and forcing term f^ε the solution $u^\varepsilon(t)$ converges to the solution $u(t)$ with data u_0 and forcing term f in $X^T_{1,b-1} \cap H^1$ for some appropriate $T > 0$.

By Lemma 4.1, for a smooth initial condition ${u_0}^\varepsilon = u_0 * \psi_\varepsilon$ and a smooth forcing term $f^\varepsilon = f * \psi_\varepsilon$, the local solution u^ε given by Theorem 3.2 satisfies the following energy-type equation:

$$\frac{d}{dt}I_j(u^\varepsilon(t)) + 2\gamma I_j(u^\varepsilon(t)) = K_j(f^\varepsilon, u^\varepsilon(t)), \quad j = 0,1, t \in [-T,T], \tag{4.6}$$

where $I_0(u^\varepsilon) = \|u^\varepsilon\|_{L^2}^2$, $K_0(f^\varepsilon,u^\varepsilon) = \int f^\varepsilon\bar{u}^\varepsilon\,dx$. So I_1 is defined as (4.5). We integrate (4.6) to obtain

$$I_j(u^\varepsilon(t)) + 2\gamma\int_0^t I_j(u^\varepsilon(t'))\,dt' = I_j u^\varepsilon(0) + \int_0^t K_j(f^\varepsilon, u^\varepsilon(t'))\,dt', \quad j = 0,1, t \in [-T,T]. \tag{4.7}$$

Taking the limit $\varepsilon \to 0$ in (4.7) and using the continuity of the solution with respect to the data and forcing term, in particular using

$$I_j(u(t)) = \lim_{\varepsilon\to 0} I_j(u^\varepsilon(t)) \text{ and } K_j(f,u) = \lim_{\varepsilon\to 0} K_j(f^\varepsilon, u^\varepsilon(t)) \quad j = 0,1, t \in [-T,T],$$

we obtain that

$$
\begin{aligned}
(4.8)\quad I_j(u(t)) + 2\gamma \int_0^t I_j(u(t'))\,dt' \\
= I_j u(0) + \int_0^t K_j(f, u(t'))\,dt', \quad j = 0, 1, t \in [-T, T].
\end{aligned}
$$

From the energy-type equation (4.8), one can extend the solution $u(t)$ to obtain a global one for any $T > 0$ with $u \in X_{1,b}^T \cap C([-T, T]; H^1(\mathbb{R}))$. One can also check that for each $T > 0$ and each initial condition $u_0 \in H^1(\mathbb{R})$, there exists a constant $C = C(\|u_0\|_{H^1}, T)$ such that

$$\|u\|_{X_{1,b}^T} \leq C(\|u_0\|_{H^1}, T).$$

In fact, energy-type equation (4.8) provides an estimate of the solution in $H^1(\mathbb{R})$ norm at each instant in time. What is more, we can divide each interval $[-T, T]$ into small enough subintervals as required in the proof of Theorem 3.2. Therefore, this can be obtained by a straightforward and classical procedure, so we omit the details here.

We state one of our main results as follows.

Theorem 4.2 *Let $\gamma \in \mathbb{R}$, $f \in H^1(\mathbb{R})$ and $u_0 \in H^1(\mathbb{R})$, b real and close to $\frac{1}{2}$. Then for all $T > 0$, problem* (1.1) *admits a unique global solution $u(x, t) \in C(\mathbb{R}; H^1)$, which belongs to $X_{1,b}^T$. Moreover, the map, which associates the datum (γ, f, u_0) to the corresponding unique solution $u(t)$, is continuous from $\mathbb{R} \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$ into $C([-T, T]; H^1(\mathbb{R})) \times X_{1,b}^T$ with, in particular,*

$$\|u\|_{X_{1,b}^T} \leq C(\lambda, \|u_0\|_{H^1(\mathbb{R})}, \|f\|_{H^1(\mathbb{R})}, T).$$

Furthermore, the solution $u(t)$ satisfies the energy equation

$$(4.9)\qquad \frac{d}{dt} I_j(u(t)) + 2\gamma I_j(u(t)) = K_j(f, u(t)), \quad j = 0, 1, \text{ for all } t \in \mathbb{R},$$

where I_j, K_j are defined as (4.5) *and* (4.6).

Thanks to Theorem 4.2, we can define a group associated with equation (1.1) as follows.

Definition 4.3 For $\gamma \in \mathbb{R}$, $f \in H^1(\mathbb{R})$ fixed, we denote $\{S(t)\}_{t\in\mathbb{R}}$ by the group in $H^1(\mathbb{R})$ defined $S(t)u_0 = u(t)$, where $u(t)$ is the unique solution of equation (1.1) and belongs to $X_{1,b}^T$ for all $T > 0$.

From now on, we are interested in the long time behavior of equation (1.1) taking the dissipation into account. We assume that γ is positive and the forcing term f belongs to $H^1(\mathbb{R})$; we shall obtain the existence of bounded absorbing sets for the solution operator $\{S(t)\}_{t\in\mathbb{R}}$ with the help of the energy-type equation. We first obtain an absorbing ball in $L^2(\mathbb{R})$; then we prove the absorbing ball in $H^1(\mathbb{R})$. We just outline it here, since this is a standard procedure.

By applying the Cauchy–Schwartz inequality and the Young inequality to the term on the right-hand side of (4.1), we obtain that

$$\frac{d}{dt}\|u(t)\|_{L^2}^2+\gamma\|u(t)\|_{L^2}^2\le\frac{1}{\gamma}\|f\|_{L^2}^2. \tag{4.10}$$

Integrating with respect to time, we have

$$\|u(t)\|_{L^2}^2\le\|u_0\|_{L^2}^2e^{-\gamma t}+\frac{1}{\gamma^2}\|f\|_{L^2}^2(1-e^{-\gamma t}).$$

Therefore, we deduce that for u,

$$\lim_{t\to\infty}\sup\|u(t)\|_{L^2}\le\rho_0=\frac{1}{\gamma}\|f\|_{L^2}, \tag{4.11}$$

uniformly bounded in $L^2(\mathbb{R})$. For the absorbing ball in $H^1(\mathbb{R})$, we first estimate the following terms by Agmon inequality and Cauchy inequality,

$$\int|u|^4\,dx\le\|u\|_{L^2}^2\|u\|_{L^\infty}^2\le\|u\|_{L^2}^3\|u_x\|_{L^2}\le\frac{1}{2\varepsilon_1}\|u\|_{L^2}^6+\frac{\varepsilon_1}{2}\|u_x\|_{L^2}^2,$$

$$\int u\bar{u}_x\,dx\le\|u\|_{L^2}\|u_x\|_{L^2}\le\frac{1}{2\varepsilon_2}\|u\|_{L^2}^2+\frac{\varepsilon_2}{2}\|u_x\|_{L^2}^2.$$

$$\Re\int f\bar{u}_x\,dx\le\|f\|_{L^2}\|u_x\|_{L^2}\le\frac{1}{2\varepsilon_3}\|f\|_{L^2}^2+\frac{\varepsilon_3}{2}\|u_x\|_{L^2}^2,$$

$$\operatorname{Im}\int f_x\bar{u}_x\,dx\le\|f_x\|_{L^2}\|u_x\|_{L^2}\le\frac{1}{2\varepsilon_4}\|f_x\|_{L^2}^2+\frac{\varepsilon_4}{2}\|u_x\|_{L^2}^2,$$

$$\begin{aligned}\operatorname{Im}\int f|u|^2\bar{u}\,dx&\le\|f\|_{L^2}\|u\|_{L^2}\|u\|_{L^\infty}^2\le\|f\|_{L^2}\|u\|_{L^2}^2\|u_x\|_{L^2}\\&\le\frac{1}{2\varepsilon_5}\|f\|_{L^2}^2\|u\|_{L^2}^4+\frac{\varepsilon_5}{2}\|u_x\|_{L^2}^2,\end{aligned}$$

where

$$\varepsilon_1=\frac{|\eta\lambda|}{2|\lambda(\lambda+\frac{\mu}{2})|},\quad\varepsilon_2=\frac{|\eta\lambda|}{2|(3\beta\eta-\alpha(2\lambda+\mu))|},\quad\varepsilon_3=\frac{|\gamma|\cdot|\eta\lambda|}{6|(3\beta\eta-\alpha(2\lambda+\mu))|},$$

$$\varepsilon_4=\frac{|\gamma|}{18},\quad\varepsilon_5=\frac{|\gamma|\cdot|\eta\lambda|}{12|\lambda(\lambda+\frac{\mu}{2})|}.$$

Therefore,

$$|\eta\lambda|\cdot\|u_x\|_{L^2}^2-|\lambda\Big(\lambda+\frac{\mu}{2}\Big)|\cdot\|u\|_{L^4}^4-\Big|(3\beta\eta-\alpha(2\lambda+\mu))\int uu_x\,dx\Big|\le I_1(u).$$

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Then,

$$(4.12)\quad \frac{|\eta\lambda|}{2}\|u_x\|_{L^2}^2-\frac{|\lambda(\lambda+\frac{\mu}{2})|^2}{|\eta\lambda|}\|u\|_{L^2}^6-\frac{|(3\beta\eta-\alpha(2\lambda+\mu))|^2}{|\eta\lambda|}\|u\|_{L^2}^2$$
$$\le I_1(u)\le \frac{3|\eta\lambda|}{2}\|u_x\|_{L^2}^2+\frac{|\lambda(\lambda+\frac{\mu}{2})|^2}{|\eta\lambda|}\|u\|_{L^2}^6+\frac{|(3\beta\eta-\alpha(2\lambda+\mu))|^2}{|\eta\lambda|}\|u\|_{L^2}^2.$$

Now similarly to the above, using Agmon's and Cauchy's inequalities to estimate $K_1(f,u)$, we obtain

$$(4.13)\quad K_1(f,u)\le \frac{6|(3\beta\eta-\alpha(2\lambda+\mu))|^2}{|\gamma|\cdot|\eta\lambda|}\|f\|_{L^2}^2+54\frac{|\eta\lambda|}{|\gamma|}\|f_x\|_{L^2}^2$$
$$+\frac{24|\lambda(\lambda+\frac{\mu}{2})|^2}{|\gamma|\cdot|\eta\lambda|}\|f\|_{L^2}^2\|u\|_{L^2}^4+\frac{|\gamma|\cdot|\eta\lambda|}{2}\|u_x\|_{L^2}^2.$$

By (4.12) and (4.13), we have

$$K_1(f,u)\le \gamma I_1(u)+\frac{6|(3\beta\eta-\alpha(2\lambda+\mu))|^2}{|\gamma|\cdot|\eta\lambda|}\|f\|_{L^2}^2$$
$$+54\frac{|\eta\lambda|}{|\gamma|}\|f_x\|_{L^2}^2+\frac{24|\lambda(\lambda+\frac{\mu}{2})|^2}{|\gamma|\cdot|\eta\lambda|}\|f\|_{L^2}^2\|u\|_{L^2}^4$$
$$+\frac{|\lambda(\lambda+\frac{\mu}{2})|^2}{|\eta\lambda|}\|u\|_{L^2}^6+\frac{|(3\beta\eta-\alpha(2\lambda+\mu))|^2}{|\eta\lambda|}\|u\|_{L^2}^2.$$

Therefore, from the energy-type equation (4.9) for $j=1$, it follows that

$$\frac{d}{dt}I_1(u)+\gamma I_1(u)\le \frac{6|(3\beta\eta-\alpha(2\lambda+\mu))|^2}{|\gamma|\cdot|\eta\lambda|}\|f\|_{L^2}^2+54\frac{|\eta\lambda|}{|\gamma|}\|f_x\|_{L^2}^2$$
$$+\frac{24|\lambda(\lambda+\frac{\mu}{2})|^2}{|\gamma|\cdot|\eta\lambda|}\|f\|_{L^2}^2\|u\|_{L^2}^4$$
$$+\frac{|\lambda(\lambda+\frac{\mu}{2})|^2}{|\eta\lambda|}\|u\|_{L^2}^6+\frac{|(3\beta\eta-\alpha(2\lambda+\mu))|^2}{|\eta\lambda|}\|u\|_{L^2}^2.$$

By the Gronwall lemma and (4.10), we obtain

$$\lim_{t\to\infty}\sup I_1(u)\le \frac{6|(3\beta\eta-\alpha(2\lambda+\mu))|^2}{\gamma^2\cdot|\eta\lambda|}\|f\|_{L^2}^2+54\frac{|\eta\lambda|}{\gamma^2}\|f_x\|_{L^2}^2$$
$$+\frac{24|\lambda(\lambda+\frac{\mu}{2})|^2}{\gamma^5\cdot|\eta\lambda|}\|f\|_{L^2}^2\|f\|_{L^2}^4$$
$$+\frac{|\lambda(\lambda+\frac{\mu}{2})|^2}{\gamma^7\cdot|\eta\lambda|}\|f\|_{L^2}^6+\frac{|(3\beta\eta-\alpha(2\lambda+\mu))|^2}{\gamma^3\cdot|\eta\lambda|}\|f\|_{L^2}^2.$$

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Finally, by (4.12) we obtain

$$
\lim_{t\to\infty}\sup \|u_x\|^2_{L^2(\mathbb{R})} \le \rho_1 = \frac{12|(3\beta\eta - \alpha(2\lambda+\mu))|^2}{\gamma^2\cdot|\eta\lambda|^2}\|f\|^2_{L^2} + \frac{48|\lambda(\lambda+\frac{\mu}{2})|^2}{\gamma^5\cdot|\eta\lambda|^2}\|f\|^2_{L^2}\|f\|^4_{L^2} + \frac{108}{\gamma^2}\|f_x\|^2_{L^2} + \frac{4|\lambda(\lambda+\frac{\mu}{2})|^2}{\gamma^7\cdot|\eta\lambda|^2}\|f\|^6_{L^2} + \frac{4|(3\beta\eta-\alpha(2\lambda+\mu))|^2}{\gamma^3\cdot|\eta\lambda|^2}\|f\|^2_{L^2}. \tag{4.14}
$$

Therefore, we have the following result.

Theorem 4.4 *Let $\gamma > 0$, $f \in H^1(\mathbb{R})$. Then the solution operator $\{S(t)\}_{t\in\mathbb{R}}$ associated with equation* (1.1) *possesses a bounded absorbing set in $H^1(\mathbb{R})$, with the radius of the absorbing ball given by* (4.11) *and* (4.14).

5 Asymptotic Compactness and the Global Attractor

From Theorem 4.4, it follows that there exists a bounded set in $H^1(\mathbb{R})$ which is absorbing for the solution operator $\{S(t)\}_{t\in\mathbb{R}}$. Therefore, to obtain the existence of the global attractor, it suffices to prove the *asymptotic compactness property*. If $\{u_{0n}\}$ is a sequence and bounded in $H^1(\mathbb{R})$, and $\{t_n\}$ is a sequence satisfying $t_n \to \infty$, then $\{S(t_n)u_{0n}\}_n$ is precompact in $H^1(\mathbb{R})$. We shall use it later, so we state it below as a lemma [1, 18, 24].

Lemma 5.1 *Let E be a complete metric space and let $\{S(t)\}_{t\in\mathbb{R}}$ be a group of continuous(nonlinear) operators in E. If $\{S(t)\}_{t\in\mathbb{R}}$ possesses a bounded absorbing set B in E and is asymptotically compact in E, then $\{S(t)\}_{t\in\mathbb{R}}$ possesses the global attractor $A = \bigcap_s \overline{\bigcup_{t\ge s} S(t)B}$.*

In this section, we will show that the asymptotic compactness property follows from the energy-type equation (4.9). The present case fits the abstract framework given by I. Moise, R. Rosa and X. Wang [21]. The only delicate point is the weak continuity of the solution operator. We need enough regularity to pass the weak limit in the equation and we need the uniqueness to obtain the weak convergence to the right solution.

Lemma 5.2 *The solution operator $\{S(t)\}_{t\in\mathbb{R}}$ is weakly continuous in $H^1(\mathbb{R})$ in the sense that if u_{0n} converges weakly to some u_0 in $H^1(\mathbb{R})$ as $n \to \infty$, then $S(t)u_{0n}$ converges to $S(t)u_0$ weakly in $H^1(\mathbb{R})$ for all $t \in \mathbb{R}$.*

Proof Let $u_{0n} \rightharpoonup u_0$ weakly in $H^1(\mathbb{R})$. We fix T and consider $u_n = S(t)u_{0n}$ for $t \in [-T, T]$. Note that $\{u_{0n}\}_n$ is bounded in $H^1(\mathbb{R})$, since it has a weak limit in the phase space. From Theorem 3.2, it follows that u_n', the time-derivative of u_n, satisfies

$$
\{u_n\}_n \text{ is bounded in } X^T_{1,b} \cap C([-T,T];H^1(\mathbb{R})). \tag{5.1}
$$

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Then from equation (1.1), we deduce that

$$\{u_n'\}_n \text{ is bounded in } C([-T,T];H^{-2}(\mathbb{R})). \tag{5.2}$$

From (5.1), it follows that

$$\{u_{n'}\}_n \rightharpoonup^* u \text{ weakly star in } X_{1,b}^T, \tag{5.3}$$

for some $u \in X_{1,b}^T \cap C([-T,T];H^1(\mathbb{R}))$ and some subsequence $\{n'\}$.

Moreover, from (5.2) it follows that for any $v \in H^2(\mathbb{R})$ and $t, t+t' \in [-T,T]$,

$$\begin{aligned}(u_n(t+t') - u_n(t), v)_{L^2} &= \int_t^{t+t'} (u_n'(s), v)_{L^2}\, ds \\ &\le \int_t^{t+t'} \|u_n'(s)\|_{H^{-2}} \|v\|_{H^2}\, ds \\ &\le Ct'\|v\|_{H^2},\end{aligned} \tag{5.4}$$

where C is a constant and independent of n. Let $v = J^{-2}(u_n(t+t') - u_n(t))$, by (5.4) we obtain

$$\begin{aligned}\|u_n(t+t') - u_n(t)\|_{H^{-1}}^2 &\le Ct'\|u_n(t+t') - u_n(t)\|_{L^2} \\ &\le Ct'\|u_n\|_{L^\infty([-T,T];L^2)} \le Ct',\end{aligned} \tag{5.5}$$

for a larger constant C.

Let $\psi_r(s)$ be defined as above. Then $\psi_r u_n$ belongs to $H_0^1(-r,r)$. From (5.1) and (5.5), it follows that $\{\psi_r u_n\}_n$ is equibounded and equicontinuous in

$$C([-T,T];H_0^1(-r,r))$$

for any $r > 0$. Moreover, From the continuous injections (2.1) and the compact injections(2.2), we obtain that $\{\psi_r u_n\}_n$ is equibounded and equicontinuous in

$$C([-T,T];H_0^{-1}(-r,r)),$$

and is precompact in $H_0^{-1}(-r,r)$. Therefore, by the Arzela–Ascoli theorem, we deduce that the sequence $\{\psi_r u_n\}_n$ for each $r > 0$ is precompact in

$$C([-T,T];H_0^{-1}(-r,r)).$$

By a diagonalization process, we can chose a subsequence $\{\psi_r u_{n'}\}_{n'}$ and an element $u \in C([-T,T];H_0^{-1}(-r,r))$ such that

$$\{u_{n'}\}_n \to u \text{ strongly in } C([-T,T];H_0^{-1}(-r,r)),\ \forall r > 0. \tag{5.6}$$

The weak-star convergence (5.3) and the strong convergence (5.6) allow us to pass the limit in either the weak or the mild formulation of equation (1.1) to deduce that

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the limit function u is the solution of Cauchy problem (1.1). For this passage to the limit, we do not need the weak-star convergence (5.3) in $X^T_{1,b}$. Thus the weak-star convergence (5.3) in $X^T_{1,b}$ is only needed to assume that the limit u belongs to $X^T_{1,b}$, in which case u must be the uniqueness of the solution in Theorem 3.2. Hence we have $u(t) = S(t)u_0$. By contradiction, one can deduce that in fact the whole sequence u_n converges to u in the sense of (5.3) and (5.6).

It still remains to show that $u_n(t)$ converges weakly to $u(t)$ in $H^1(\mathbb{R})$ for any $t \in [-T, T]$. We know that the convergence is strong in $H_0^{-1}(-r, r)$, for each $r > 0$. Therefore, taking $v \in C_c^\infty(\mathbb{R})$, we obtain that for large enough r, $J^2 v$ belongs to $H_0^{-1}(-r, r)$, so that

$$(u_n, v)_{H^1(\mathbb{R})} = (u_n, J^2 v) \to (u, J^2 v) = (u, v)_{H^1(\mathbb{R})}.$$

Then, from (5.1) and the density of $C_c^\infty(\mathbb{R})$ in $H^1(\mathbb{R})$, it follows that for every $v \in H^1(\mathbb{R})$, $(u_n, v)_{H^1(\mathbb{R})} \to (u, v)_{H^1(\mathbb{R})}$. This proves the desired weak continuity in $H^1(\mathbb{R})$. ■

With the previous lemma in mind we can prove that the solution operators $\{S(t)\}_{t\in\mathbb{R}}$ are asymptotic compact by the framework summarized in [21].

Lemma 5.3 *If the sequence $\{u_{0n}\}$ is bounded in $H^1(\mathbb{R})$ and the sequence $\{t_n\}$ satisfies $t_n \to \infty$, then there exist $u \in H^1(\mathbb{R})$ and a subsequence $\{n'\}$ such that $S(t_{n'})u_{0n'} \to u$ strongly in $H^1(\mathbb{R})$.*

Proof The proof is standard; we refer to [21, 22]. ■

This gives the asymptotic compactness property of the solution operator and, hence, the existence of the global attractor. Therefore, we have the following result.

Theorem 5.4 *Let $\gamma > 0$, $f \in H^1(\mathbb{R})$. Then the solution operator $\{S(t)\}_{t\in\mathbb{R}}$ associated with equation (1.1) in $H^1(\mathbb{R})$ possesses a connected global attractor in $H^1(\mathbb{R})$.*

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