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Abstract

Suppose that G is a connected reductive group over a p-adic field F, that K is a hyperspecial maximal compact subgroup of $G(F)$, and that V is an irreducible representation of K over the algebraic closure of the residue field of F. We establish an analogue of the Satake isomorphism for the Hecke algebra of compactly supported, K-biequivariant functions $f : G(F) \to \operatorname{End} V$. These Hecke algebras were first considered by Barthel and Livné for GL_2. They play a role in the recent mod p and p-adic Langlands correspondences for $\mathrm{GL}_2(\mathbb{Q}_p)$, in generalisations of Serre's conjecture on the modularity of mod p Galois representations, and in the classification of irreducible mod p representations of unramified p-adic reductive groups.

1. Introduction

1.1 Statement of the theorem

Let F be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformiser ϖ, and residue field k of order q. Suppose that G is a connected reductive group over F that is unramified (i.e. quasi-split and split over an unramified extension) and that K is a hyperspecial maximal compact subgroup. Fix any maximal split torus S in G such that the apartment corresponding to S contains the hyperspecial point in the reduced building corresponding to K. Since G is quasi-split, $T = Z_G(S)$ is a maximal torus of G.

With these assumptions, it is known that G extends to a smooth $\mathcal{O}$-group scheme [Tit79, § 3.8.1], which we will also denote by G, whose special fibre is a connected reductive group over k and is such that $K = G(\mathcal{O})$. The tori S and T extend to smooth $\mathcal{O}$-subgroup schemes $S \subset T$ of G, which reduce to a maximal split torus and its centraliser in the special fibre of G. The relative root systems of S in G in the two fibres are naturally identified with each other. We denote by $\Phi \subset X^*(S)$ the set of roots, by Φ^+ a choice of positive roots, and by W the Weyl group. There is a closed $\mathcal{O}$-subgroup scheme $B = T \ltimes U$ of G whose fibres are the Borel subgroups associated to Φ^+.

Suppose that V is an irreducible representation of $G(k)$ over $\bar{k}$, which we shall also consider as a representation of K via the reduction homomorphism $K = G(\mathcal{O}) \to G(k)$. The *Hecke algebra* $\mathcal{H}_G(V)$ *of* V is the $\bar{k}$-algebra of compactly supported functions $f : G(F) \to \operatorname{End}_{\bar{k}} V$ satisfying $f(k_1 g k_2) = k_1 f(g) k_2$ for all $k_1, k_2 \in K$ and $g \in G(F)$, where the multiplication is given by convolution. We remark that, by Frobenius reciprocity, it follows that $\mathcal{H}_G(V) \cong \operatorname{End}_{\bar{k}G(F)}(\text{c-Ind}_K^{G(F)} V)$, where $\text{c-Ind}_K^{G(F)} V = \{\psi : G(F) \to V \mid \psi(kg) = k\psi(g)\ \forall k \in K,\ g \in G;\ \operatorname{supp} \psi \text{ is compact}\}$ is the compactly induced representation (see [BL94, Proposition 5]).

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It is known that the $T(k)$-representation $V^{U(k)}$ is one-dimensional (see Lemma 2.5). The corresponding Hecke algebra $\mathcal{H}_T(V^{U(k)})$ consists of $T(\mathcal{O})$-biequivariant, compactly supported functions $\varphi : T(F) \to \operatorname{End}_{\bar{k}}(V^{U(k)}) = \bar{k}$.

Let $\operatorname{ord}_F : F^\times \twoheadrightarrow \mathbb{Z}$ denote the valuation of F. For $\chi \in X^*(S)$ and $t \in T(F)$, we define $(\operatorname{ord}_F \circ \chi)(t)$ to be $(1/n) \operatorname{ord}_F(n\chi(t))$, where $n > 0$ is chosen so that $n\chi$ extends to an F-rational character of T. Since $X_F^*(T) \to X^*(S)$ is injective with finite cokernel, this does not depend on any choices.

Definition 1.1. Let T^- denote the following submonoid of $T(F)$:

$$T^- = \{t \in T(F) : (\operatorname{ord}_F \circ \alpha)(t) \leqslant 0 \ \forall \alpha \in \Phi^+\}.$$

Let $\mathcal{H}_T^-(V^{U(k)})$ denote the subalgebra of $\mathcal{H}_T(V^{U(k)})$ consisting of those $\varphi : T(F) \to \bar{k}$ that are supported on T^-.

Theorem 1.2. *Suppose that V is an irreducible representation of $G(k)$ over $\bar{k}$. Then*

$$\begin{aligned} \mathcal{S} : \mathcal{H}_G(V) &\to \mathcal{H}_T(V^{U(k)}) \\ f &\mapsto \left(t \mapsto \sum_{u \in U(F)/U(\mathcal{O})} f(tu)\Big|_{V^{U(k)}} \right) \end{aligned}$$

is an injective $\bar{k}$-algebra homomorphism with image $\mathcal{H}_T^-(V^{U(k)})$.

Note that as f is compactly supported, the sum over $U(F)/U(\mathcal{O})$ has only finitely many non-zero terms and $\mathcal{S}f$ is compactly supported. Since $T(F)$ normalises $U(F)$, the image of $\mathcal{S}f$ is contained in $V^{U(k)}$.

It is easy to see that $\lambda \mapsto \lambda(\varpi)$ yields an isomorphism $X_*(S) \to T(F)/T(\mathcal{O})$ which sends the antidominant coweights $X_*(S)_- = \{\lambda \in X_*(S) : \langle \lambda, \alpha \rangle \leqslant 0 \ \forall \alpha \in \Phi^+\}$ to $T^-/T(\mathcal{O})$ (Lemma 2.1).

Corollary 1.3. *$\mathcal{H}_G(V)$ is commutative and isomorphic to $\bar{k}[X_*(S)_-]$. In particular, it is noetherian.*

At least when G is split and the derived subgroup of G is simply connected, there is another argument to show that $\mathcal{H}_G(V)$ is commutative, which uses an analogue of a Gelfand involution; see the end of § 2.1.

1.2 Comparison with the classical Satake isomorphism

Recall that the classical Satake isomorphism is given by the formula

$$\begin{aligned} \mathbb{C}[K \backslash G(F)/K] &\xrightarrow{\sim} \mathbb{C}[X_*(S)]^W \\ f &\mapsto \left(t \mapsto \delta(t)^{1/2} \int_{U(F)} f(tu)\, du \right), \end{aligned}$$

where δ is the modulus character of the Borel subgroup and the Haar measure du on $U(F)$ satisfies $\int_{U(\mathcal{O})} du = 1$ (see [Car79, Gro98]). The relevance of the factor $\delta^{1/2}$ is to make the image of the Satake transform W-invariant. Leaving it out still yields an algebra homomorphism $\mathcal{S}'$ into $\mathbb{C}[X_*(S)]$, which now also makes sense over $\mathbb{Z}$ and is obviously compatible with $\mathcal{S}$ when V

is the trivial representation, as shown in the following diagram.

$$\begin{tikzcd} \mathbb{Z}[K\backslash G(F)/K] \arrow[r, hook, "\mathcal{S}'"] \arrow[d] & \mathbb{Z}[X_*(S)] \arrow[d] \\ \mathcal{H}_G(1) = \bar{k}[K\backslash G(F)/K] \arrow[r, hook, "\mathcal{S}"] & \bar{k}[X_*(S)] \end{tikzcd}$$

In this case (when V is trivial), there is a simple explanation of why the image of $\mathcal{S}$ is supported on antidominant coweights. The image of the Satake transform is W-invariant, and the modulus character is a power of p which, among the W-conjugates of a given coweight, is biggest on the antidominant one.

The proof of Theorem 1.2 follows the same steps as the classical proof, but there are two complications. Firstly, it is harder to determine the space of Hecke operators supported on a given double coset. This requires an argument using the Bruhat–Tits building (Proposition 3.8). Secondly, for general V it is subtle to prove that the image of $\mathcal{S}$ is contained in $\mathcal{H}_T^-(V^{U(k)})$. We first show that the image is supported on 'almost antidominant' coweights and then use that $\mathcal{S}$ is a homomorphism to conclude. This extra step is really necessary, as one can see by considering the Hecke bimodule $\mathrm{Hom}_{G(F)}(\text{c-Ind}_K^{G(F)} V_1, \text{c-Ind}_K^{G(F)} V_2)$ whose support under the Satake map may extend slightly beyond the antidominant coweights [Her10, § 6].

1.3 Comparison with the p-adic Satake isomorphism

Schneider and Teitelbaum [ST06] constructed p-adic Satake maps, and their p-adic completions, for the Hecke algebras associated to an irreducible representation of $G_{/F}$. In Proposition 2.10 we establish a compatibility between Schneider and Teitelbaum's p-adic Satake map and the mod p Satake map $\mathcal{S}$, in the case where V extends to a representation of $G_{/k}$. (This is satisfied, for example, if the derived subgroup of $G_{/\bar{k}}$ is simply connected.) In this case, V is a submodule of the reduction of a K-stable lattice in some irreducible representation of $G_{/F}$. Note that V does not necessarily equal the reduction; in fact, this cannot usually be achieved.

1.4 The W-regular case

The refined Cartan decomposition says that the $\lambda(\varpi)$ for $\lambda \in X_*(S)_-$ form a system of coset representatives for $K\backslash G(F)/K$. We will see in the proof of Theorem 1.2 that $\mathcal{H}_G(V)$ has a natural $\bar{k}$-basis $\{T_\lambda : \lambda \in X_*(S)_-\}$. The Hecke operator T_λ is characterised by having support $K\lambda(\varpi)K$ and by $T_\lambda(\lambda(\varpi)) \in \mathrm{End}_{\bar{k}} V$ being a projection. More obviously (see Lemma 2.1), $\mathcal{H}_T^-(V^{U(k)})$ has a $\bar{k}$-basis $\{\tau_\lambda : \lambda \in X_*(S)_-\}$ where τ_λ is supported on $\lambda(\varpi)T(\mathcal{O})$ and $\tau_\lambda(\lambda(\varpi)) = 1$.

We will say that an irreducible representation V of $G(k)$ over $\bar{k}$ is *W-regular* if the 'extremal weight subspaces' $wV^{U(k)} \subset V$ for $w \in W$ are distinct.

Proposition 1.4. *Suppose that V is W-regular. Then for each $\lambda \in X_*(S)_-$ we have $\mathcal{S}T_\lambda = \tau_\lambda$. In particular, $T_\lambda * T_{\lambda'} = T_{\lambda+\lambda'}$ for all $\lambda, \lambda' \in X_*(S)_-$.*

For general V and for $\lambda \in X_*(S)_-$, the proof of Theorem 1.2 shows that

$$\tau_\lambda = \sum_{\substack{\mu \in X_*(S)_- \\ \mu \geqslant_{\mathbb{R}} \lambda}} d_\lambda(\mu) \mathcal{S}T_\mu$$

where $d_\lambda(\mu) \in \bar{k}$ and $d_\lambda(\lambda) = 1$. In the classical setting, the work of Lusztig and Kato shows that the $d_\lambda(\mu)$ are Kazhdan–Lusztig polynomials in $q = |k|$ (see [Gro98, HKP10, Kat82]).

In [Her10, § 5] we use their results to compute $d_\lambda(\mu)$ in all cases, at least when G is split and its derived subgroup is simply connected. It turns out that $d_\lambda(\mu)$ does not depend on V but only on the stabiliser of the subspace $V^{U(k)}$ in W.

1.5 Satake parameters

Let $\Delta \subset \Phi^+$ denote the set of simple roots. Let $\widehat{S}$ be the torus dual to S (over $\bar{k}$). For each subset $J \subset \Delta$, define the torus $\widehat{S}_J$ by the exact sequence

$$\mathbb{G}_m^J \to \widehat{S} \to \widehat{S}_J \to 1,$$

where the first map is given by $\prod_{\delta \in J} \delta$. The closed points of the 'toric' variety Spec $\mathcal{H}_G(V)$ have the following concrete description. Classically, only one torus ($\widehat{S} = \widehat{S}_\varnothing$) is needed.

Corollary 1.5. *The $\bar{k}$-algebra homomorphisms $\mathcal{H}_G(V) \to \bar{k}$ are parameterised by pairs (J, s_J) where $J \subset \Delta$ and $s_J \in \widehat{S}_J(\bar{k})$.*

In [Her10, § 4] we give an alternative parameterisation, analogous to the classical parameterisation by unramified characters of T.

1.6 Example: $G = \mathrm{GL}_n$

We suppose that $S = T$ is the diagonal torus and that B is the Borel subgroup of upper-triangular matrices. Then the $\lambda_i(x) = \mathrm{diag}(1, \ldots, 1, x, \ldots, x)$ (with i non-trivial entries) generate $X_*(S)_-$, and we denote by T_i the corresponding Hecke operator T_{λ_i}. Theorem 1.2 shows that $\mathcal{H}_G(V)$ is the localised polynomial algebra $\bar{k}[T_1, \ldots, T_{n-1}, T_n^{\pm 1}]$.

1.7 Previous work

The Hecke algebras $\mathcal{H}_G(V)$ were first calculated by Barthel and Livné in the case where $G = \mathrm{GL}_2$; see [BL94, BL95]. (We follow their strategy for computing $\mathcal{H}_G(V)$ as a vector space. However, they used explicit methods to determine the algebra structure.) This was important for their (partial) classification of irreducible smooth representations π of $\mathrm{GL}_2(F)$ over $\bar{k}$ that have a central character, which was completed by Breuil in the case where $F = \mathbb{Q}_p$ (see [Bre03]) and which plays a crucial role for mod p and p-adic local Langlands correspondences for $\mathrm{GL}_2(\mathbb{Q}_p)$. In [Her10] we extend the work of Barthel and Livné, giving a classification of irreducible, admissible representations of $\mathrm{GL}_n(F)$ over $\bar{k}$ in terms of supersingular representations. Our proofs depend heavily on the methods developed in this paper.

We also remark that Schein independently determined the Hecke algebras for GL_n by explicit methods [Sch09], after we had done this in a similar manner.

In another direction, Gross showed that the classical Satake isomorphism can be defined over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$; see [Gro98, § 3] and also [Laz99, § 1.2].

1.8 Algebraic modular forms

Suppose that $F = \mathbb{Q}_p$ and that G arises by base change from a connected reductive $\mathbb{Q}$-group G such that $G(\mathbb{R})$ is compact. Given a compact open subgroup $K_{\mathbb{A}} = K \times K^p$ in $G(\mathbb{A}^\infty)$, we can consider Gross's space $M(K_{\mathbb{A}}, V^*)$ of algebraic modular forms of level $K_{\mathbb{A}}$ and weight V^*, the linear dual of V (see [Gro99]). The Hecke algebra $\mathcal{H}_G(V)$ acts naturally on this space, and there is a simple result concerning compatibility of the action of the T_λ on $M(K_{\mathbb{A}}, V^*)$ with classical Hecke operators. In forthcoming joint work with Emerton and Gee, we use it to prove strong new results on the weights in a Serre-type conjecture for rank-three unitary groups.

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1.9 Organisation of the paper

In § 2 we discuss the proofs of the main results. Technical parts of the arguments requiring buildings are discussed in § 3. We include the proofs of some well-known results since we did not find an appropriate reference for them.

For a reader who is inexperienced with algebraic groups, we recommend assuming first that $G = \mathrm{GL}_n$ or, more generally, that G is split with simply connected derived subgroup. Many arguments simplify in these settings.

2. Proofs

2.1 The Satake isomorphism for $\mathcal{H}_G(V)$

Lemma 2.1. *The map $\zeta : T(F) \to X_*(T)$ given by*

$$\langle \zeta(t), \chi \rangle = \mathrm{ord}_F(\chi(t)) \quad \text{for } \chi \in X^*(T)$$

induces isomorphisms of abelian groups

$$S(F)/S(\mathcal{O}) \xrightarrow{\sim} T(F)/T(\mathcal{O}) \xrightarrow{\sim} X_*(S). \tag{2.2}$$

Moreover, $T^-/T(\mathcal{O})$ (see Definition 1.1) corresponds to $X_*(S)_-$ under the isomorphism. A 'splitting' of (2.2) is provided by $X_*(S) \to S(F)$, $\lambda \mapsto \lambda(\varpi)$.

Note that $\chi(t) \in (F^{\mathrm{nr}})^\times$ since T splits over an unramified extension, so $\mathrm{ord}_F(\chi(t)) \in \mathbb{Z}$.

Proof. We consider the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & S(\mathcal{O}) \arrow[r] \arrow[d, hook] & S(F) \arrow[r] \arrow[d, hook] & X_*(S) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & T(\mathcal{O}) \arrow[r] & T(F) \arrow[r, "\zeta"] & X_*(T) &
\end{tikzcd}$$

Note that ζ lands in the $\mathrm{Gal}(\overline{F}/F)$-invariant part of $X_*(T)$, that is, in $X_*(S)$. As $T_{/\mathcal{O}^{\mathrm{nr}}}$ is split (see Lemma 3.2), $\ker \zeta = T(F) \cap T(\mathcal{O}^{\mathrm{nr}}) = T(\mathcal{O})$. All the claims follow immediately. $\square$

We need to introduce a partial order $\leqslant_{\mathbb{R}}$ on $X_*(S)_{\mathbb{R}}$. First, note that $X^*(S)_{\mathbb{R}} = \mathbb{R}\langle\Phi\rangle \oplus X^*(G_{/F})_{\mathbb{R}}$, where $X^*(G_{/F}) = \mathrm{Hom}_F(G_{/F}, \mathbb{G}_m)$. Since Φ is a root system in $\mathbb{R}\langle\Phi\rangle$, for every $\alpha \in \Phi$ there is a 'coroot' $\alpha^\vee \in (\mathbb{R}\langle\Phi\rangle)^*$, characterised by $s_\alpha(x) = x - \langle x, \alpha^\vee\rangle\alpha$. For $y, y' \in (\mathbb{R}\langle\Phi\rangle)^*$, we say that $y \geqslant_{\mathbb{R}} y'$ if $y - y'$ is a non-negative real linear combination of the positive coroots.

Definition 2.3. Suppose that $\lambda, \lambda' \in X_*(S)_{\mathbb{R}}$. We say that $\lambda \geqslant_{\mathbb{R}} \lambda'$ if $\lambda - \lambda'$ lies in the direct summand $(\mathbb{R}\langle\Phi\rangle)^*$ and $\lambda - \lambda' \geqslant_{\mathbb{R}} 0$.

Alternatively, one could use the relative coroots in $X_*(S)$ as defined in [Spr98, § 15.3].

Lemma 2.4. *Suppose that $\lambda \in X_*(S)$. Then $\{\lambda' \in X_*(S)_- : \lambda' \geqslant_{\mathbb{R}} \lambda\}$ is finite.*

Proof. By the definition of $\geqslant_{\mathbb{R}}$ we may project onto $(\mathbb{R}\langle\Phi\rangle)^*$. The projections $\bar\lambda$ and $\bar\lambda'$ lie in the coweight lattice for the root system Φ in $(\mathbb{R}\langle\Phi\rangle)^*$, and $\bar\lambda$ is antidominant. In this setting the result is well known. $\square$

Next, we will study the invariants of an irreducible $G(k)$-representation V over $\bar k$ under the unipotent radical of a parabolic subgroup.

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LEMMA 2.5. *Suppose that V is an irreducible representation of $G(k)$ over $\bar{k}$. Then $V^{U(k)}$ is one-dimensional. Suppose that $P = L \ltimes N$ is a parabolic subgroup of $G_{/k}$ and denote by $\overline{P} = L \ltimes \overline{N}$ the opposite parabolic.*

(i) *$V^{N(k)}$ is an irreducible representation of $L(k)$.*

(ii) *The natural map $V^{N(k)} \to V \to V_{\overline{N}(k)}$ is an isomorphism of $L(k)$-representations.*

Assertion (i) was first proved by Smith [Smi82] in the case where $G_{/k}$ is semisimple and simply connected. Cabanes [Cab84] provided a general proof, using (B, N)-pairs. Below we give a proof that generalises the proof for the simply connected case found in [Hum06, § 5.10].

Proof. Let us first assume that the derived subgroup of $G_{/\bar{k}}$ is simply connected. By conjugating, we may assume that $P = L \ltimes N$ is a standard Levi decomposition, i.e. $P \supset B$ and $T \subset L$. Let $\mathbb{G}$ be the split k-form of $G_{/\bar{k}}$, and fix a split maximal torus $\mathbb{T}$ and a Borel subgroup $\mathbb{B}$ containing it. Let $\phi \in \mathrm{Gal}(\bar{k}/k)$ denote the Frobenius element. There is a finite-order automorphism $\pi \in \mathrm{Aut}_k(\mathbb{G}, \mathbb{B}, \mathbb{T})$ and an isomorphism $f : G(\bar{k}) \to \mathbb{G}(\bar{k})$ respecting maximal tori and Borel subgroups such that $f \circ \phi = (\pi \circ \phi) \circ f$. In particular, $G(k) = \mathbb{G}(\bar{k})^{\pi\circ\phi}$. Let $\mathbb{L} \ltimes \mathbb{N}$ be the parabolic subgroup of $\mathbb{G}$ corresponding to $L \ltimes N$ in G.

Since $\mathbb{G}'$ is simply connected, a (slight extension of a) result of Steinberg shows that V is isomorphic to the restriction to $G(k)$ of an irreducible representation $F(\nu)$ of the algebraic group $\mathbb{G}$ whose highest weight $\nu \in X^*(\mathbb{T})$ is q-restricted, i.e. which satisfies $0 \leqslant \langle \nu, \beta^\vee \rangle < q$ for all simple roots β of $\mathbb{G}$; see [Her09, Proposition A.1.3]. Moreover, $V^{U(k)} \cong F(\nu)_\nu$ (the weight space of weight ν) is one-dimensional.

(i) This is [Hum06, Corollary 5.10]. Even though $\mathbb{G}$ is assumed to be semisimple in that reference, the proof goes through word for word. From the proof we see that $F(\nu)^{\mathbb{N}} = F(\nu)^{N(k)}$ is the sum of weight spaces $F(\nu)_{\nu'}$ with $\nu - \nu' \in \mathbb{Z}_{\geqslant 0}\Theta^+$, where Θ^+ denotes the positive roots of $(\mathbb{T}, \mathbb{L})$. This is an irreducible $L(k)$-representation since ν is also q-restricted for $\mathbb{L}$ and $\mathbb{L}'$ is simply connected (as $\mathbb{G}'$ is simply connected).

(ii) Since $(V^*)^{\overline{N}(k)} \cong \mathrm{Hom}_{\bar{k}}(V_{\overline{N}(k)}, \bar{k})$, it follows that $V_{\overline{N}(k)} \cong ((V^*)^{\overline{N}(k)})^*$ is irreducible as a $L(k)$-representation. It thus suffices to show that $V^{N(k)} \to V_{\overline{N}(k)}$ is non-zero or, equivalently, that $V^{N(k)}$ pairs non-trivially with $(V^*)^{\overline{N}(k)}$ under the duality $V \times V^* \to \bar{k}$. By (i), $V^{N(k)}$ contains the highest-weight space $L(\nu)_\nu$ and $(V^*)^{\overline{N}(k)}$ contains the lowest-weight space $(L(\nu)^*)_{-\nu}$. Since these spaces pair non-trivially, this completes the proof. (One can even see directly in this way that the pairing on $V^{N(k)} \times (V^*)^{\overline{N}(k)}$ is non-degenerate, i.e. that the map $V^{N(k)} \to V_{\overline{N}(k)}$ is an isomorphism.)

We remark that this argument shows that $V^{N(k)}$ is a direct summand of V as a $L(k)$-representation, which is also clear from the proof in [Hum06].

Let us now reduce the general case to the previous one. For ease of notation we will be writing G for its special fibre $G \times_{\mathcal{O}} k$, and similarly for S, T, etc. We pick a z-extension of G. This is an exact sequence

$$1 \to R \to \widetilde{G} \xrightarrow{\pi} G \to 1 \tag{2.6}$$

of affine algebraic k-groups, where $\widetilde{G}$ is reductive with $\widetilde{G}'$ simply connected and R a central torus (even an induced torus). Exactness means that the first map is a closed embedding, the second map is faithfully flat, and the first map is the kernel of the second. The notion of a z-extension goes back to Langlands in the characteristic-zero case; for the general case, see [Col08, § 3.1].

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By [Bor91, Theorem 22.6] we have that: (a) $\widetilde{T} = \pi^{-1}(T)$ is a maximal torus of $\widetilde{G}$; (b) the maximal split subtorus $\widetilde{S} \subset \widetilde{T}$ satisfies $\pi(\widetilde{S}) = S$; (c) $X^*(S) \hookrightarrow X^*(\widetilde{S})$ induces a bijection $\alpha \mapsto \widetilde{\alpha} = \alpha \circ \pi$ on relative roots; and (d) $U_{\widetilde{\alpha}}$ maps isomorphically to U_α for any $\alpha \in \Phi$. Let $\Theta \subset \Phi$ be the set of roots of (S, L). Since $\widetilde{L} = \langle \widetilde{T}, U_{\widetilde{\alpha}} : \alpha \in \Theta \rangle$ and $\widetilde{N} \cong \prod_{\Phi^+ - \Theta^+} U_{\widetilde{\alpha}}$ (in any fixed order), and similarly for L and N, the map π induces

$$1 \to R \to \widetilde{L} \to L \to 1, \quad \widetilde{N} \xrightarrow{\sim} N.$$

As R is connected, $H^1(\mathrm{Gal}(\bar{k}/k), R(\bar{k})) = 0$ by Lang's theorem, so that

$$\widetilde{G}(k) \twoheadrightarrow G(k), \quad \widetilde{L}(k) \twoheadrightarrow L(k), \quad \widetilde{N}(k) \xrightarrow{\sim} N(k).$$

Thus V is an irreducible representation of $\widetilde{G}(k)$ on which $R(k)$ acts trivially. The result now follows from the previous case. $\square$

The following technical lemma is crucial in controlling the support of the image of the Satake map. Let Φ_{nd} denote the set of non-divisible roots in Φ. Recall that for any root $\beta \in \Phi_{\mathrm{nd}}$, there is a root subgroup U_β over F whose Lie algebra is the sum of weight spaces for the positive multiples of β. It extends to a smooth $\mathcal{O}$-subgroup scheme of G (see § 3).

Lemma 2.7. *Let α be a simple root (so that $\alpha \in \Phi^+_{\mathrm{nd}}$).*

(i) *The product map $\prod_{\beta \in \Phi^+_{\mathrm{nd}}, \beta \neq \alpha} U_\beta \to U$ is an isomorphism of $\mathcal{O}$-schemes onto a closed subgroup scheme U'. It is normal in U and independent of the order of the factors in the product. The product map induces an isomorphism of $\mathcal{O}$-group schemes $U_\alpha \ltimes U' \to U$.*

(ii) *Suppose that A is an abelian group and that $\phi : U(F)/U(\mathcal{O}) \to A$ is a function with finite support. Then*

$$\sum_{U(F)/U(\mathcal{O})} \phi(u) = \sum_{u_\alpha \in U_\alpha(F)/U_\alpha(\mathcal{O})} \; \sum_{u' \in U'(F)/U'(\mathcal{O})} \phi(u_\alpha u').$$

(iii) *Suppose that $\lambda \in X_*(S)$ and $\alpha \in \Phi_{\mathrm{nd}}$ are such that $\langle \lambda, \alpha \rangle > 1$. Let $t = \lambda(\varpi)$. Suppose that A is an abelian group of exponent p. Suppose that $\psi : U_\alpha(F)/tU_\alpha(\mathcal{O})t^{-1} \to A$ is a function with finite support such that ψ is left invariant under* $\ker(U_\alpha(\mathcal{O}) \to U_\alpha(k))$. *Then*

$$\sum_{u_\alpha \in U_\alpha(F)/tU_\alpha(\mathcal{O})t^{-1}} \psi(u_\alpha) = 0.$$

Proof. We will prove (i) and (iii) at the end of § 3. Part (ii) follows immediately from (i).

Note, however, that when G is split, the proof is easier. In that case, there are $\mathcal{O}$-group isomorphisms $x_\alpha : \mathbb{G}_a \xrightarrow{\sim} U_\alpha$ such that for $t \in T$ we have $t x_\alpha(a) t^{-1} = x_\alpha(\alpha(t)a)$ and for all α and β with $\alpha \neq -\beta$, $[x_\alpha(a), x_\beta(b)] = \prod_{i,j>0} x_{i\alpha + j\beta}(c_{i,j} a^i b^j)$ (in some order) with $c_{i,j} \in \mathcal{O}$; see [Jan03, II.1.2]. Then (iii) is obvious since U_α is abelian and $tU_\alpha(\mathcal{O})t^{-1}$ is a proper subgroup of $\ker(U_\alpha(\mathcal{O}) \to U_\alpha(k))$ of p-power index. Part (i) follows as in the general case, except that instead of Bruhat–Tits theory one can appeal to [Jan03, II.1.7]. $\square$

Proof of Theorem 1.2. We will use the refined Cartan decomposition (Lemma 3.5)

$$G(F) = \coprod_{\lambda \in X_*(S)_-} K \lambda(\varpi) K.$$

Step 0. Let $f \in \mathcal{H}_G(V)$. Since K is compact open in $G(F)$, f is supported on a finite number of cosets in $G(F)/K$. By the Iwasawa decomposition (Lemma 3.4), f is supported on a finite number

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of cosets in $B(F)/B(\mathcal{O})$. Thus $\mathcal{S}f$ is supported on a finite number of cosets in $T(F)/T(\mathcal{O})$, and for each $t \in T(F)$ the sum $\sum_{u \in U(F)/U(\mathcal{O})} f(tu)|_{V^{U(k)}}$ is zero outside a finite number of terms. As T normalises U, it follows that the image of $\sum_{u \in U(F)/U(\mathcal{O})} f(tu)|_{V^{U(k)}}$ is contained in $V^{U(k)}$. It is clear that $\mathcal{S}$ is $\bar{k}$-linear.

Step 1. We show that the space of functions in $\mathcal{H}_G(V)$ supported on any single double coset is one-dimensional. The argument is analogous to that for [BL94, Lemma 7] but requires technical input from Bruhat–Tits theory. Suppose that $f \in \mathcal{H}_G(V)$ is supported on the double coset KtK with $t = \lambda(\varpi)$ for some $\lambda \in X_*(S)_-$. Let $P_\lambda = L_\lambda \ltimes U_\lambda$ denote the parabolic subgroup of $G_{/k}$ defined by $\lambda \in X_*(S)$ (see [Spr98, 13.4.2, 15.4.4]). Note that $L_\lambda = L_{-\lambda}$ and that $P_{-\lambda} = L_\lambda \ltimes U_{-\lambda}$ is the opposite parabolic subgroup. It follows immediately from the definitions that the possible values for $f(t)$ consist of all the $\phi \in \operatorname{End}_{\bar{k}} V$ such that

$$k_1\phi = \phi k_2 \quad \text{whenever } k_1, k_2 \in K \text{ and } k_1 t = t k_2.$$

Note that $k_1 \in K \cap tKt^{-1}$, $k_2 \in K \cap t^{-1}Kt$ and $k_1 = tk_2t^{-1}$. Proposition 3.8 implies that, equivalently, ϕ has to factor through an $L_\lambda(k)$-equivariant map $V_{U_\lambda(k)} \to V^{U_{-\lambda}(k)}$, and Lemma 2.5 shows that the space of such ϕ is one-dimensional (Schur's lemma).

Again by Lemma 2.5, there is a function in $\mathcal{H}_G(V)$ that is supported on KtK and maps t to the endomorphism

$$V \twoheadrightarrow V_{U_\lambda(k)} \xleftarrow{\sim} V^{U_{-\lambda}(k)} \hookrightarrow V. \tag{2.8}$$

We denote it by T_λ. Obviously, it is a projection.

Step 2. Let us verify that $\mathcal{S}$ is a homomorphism. This imitates the classical argument. Suppose that $f_i : G(F) \to \operatorname{End}_{\bar{k}} V$ $(i = 1, 2)$ are elements of $\mathcal{H}_G(V)$. Let $v \in V^{U(k)}$. Then

$$\begin{aligned}
\mathcal{S}(f_1 * f_2)(t)v &= \sum_{u \in U(F)/U(\mathcal{O})} \sum_{g \in G/K} f_1(tug) f_2(g^{-1})v \\
&= \sum_{u \in U(F)/U(\mathcal{O})} \sum_{b \in B(F)/B(\mathcal{O})} f_1(tub) f_2(b^{-1})v \\
&= \sum_{u \in U(F)/U(\mathcal{O})} \sum_{\tau \in T(F)/T(\mathcal{O})} \sum_{\nu \in U(F)/U(\mathcal{O})} f_1(tu\tau\nu) f_2(\nu^{-1}\tau^{-1})v \\
&= \sum_{\tau \in T(F)/T(\mathcal{O})} \sum_{\nu \in U(F)/U(\mathcal{O})} \sum_{u \in U(F)/U(\mathcal{O})} f_1(t\tau\nu) f_2(\nu^{-1}\tau^{-1}u)v \\
&= \sum_{\tau \in T(F)/T(\mathcal{O})} \sum_{\nu \in U(F)/U(\mathcal{O})} \sum_{u \in U(F)/U(\mathcal{O})} f_1(t\tau\nu) f_2(\tau^{-1}u)v \\
&= \sum_{\tau \in T(F)/T(\mathcal{O})} \sum_{\nu \in U(F)/U(\mathcal{O})} f_1(t\tau\nu)(\mathcal{S}f_2)(\tau^{-1})v \\
&= \sum_{\tau \in T(F)/T(\mathcal{O})} (\mathcal{S}f_1)(t\tau)(\mathcal{S}f_2)(\tau^{-1})v \\
&= (\mathcal{S}f_1 * \mathcal{S}f_2)(t)v.
\end{aligned}$$

Note that when we sum over quotients, the summand does not depend on the representative chosen provided that we respect the stated order of summation. The first and the last three equalities come from the definitions, the second comes from the Iwasawa decomposition $G(F) = B(F)K$ (Lemma 3.4), and the third follows from the fact that $B = T \ltimes U$. For the fourth equality, we replaced $(\tau^{-1}u\tau)\nu$ by ν, and for the fifth we replaced $(\tau\nu^{-1}\tau^{-1})u$ by u.

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Step 3. We show that $(\mathcal{S}T_\lambda)(\mu(\varpi)) = 0$ for $\mu \in X_*(S)$ unless $\mu \geqslant_{\mathbb{R}} \lambda$ and that $(\mathcal{S}T_\lambda)(\lambda(\varpi)) = 1$. The argument is the classical one. By Lemma 3.6, $K\lambda(\varpi)K \cap \mu(\varpi)U \neq \varnothing$ implies $\mu \geqslant_{\mathbb{R}} \lambda$ and $K\lambda(\varpi)K \cap \lambda(\varpi)U = \lambda(\varpi)U(\mathcal{O})$. Since $U_{-\lambda}(k) \subset U(k)$ and $T_\lambda(\lambda(\varpi))$ is a projection onto $V^{U_{-\lambda}(k)}$, we see that $(\mathcal{S}T_\lambda)(\lambda(\varpi)) = 1$.

Step 4. We show that $(\mathcal{S}f)(\mu(\varpi)) = 0$ if $\langle \mu, \alpha \rangle > 1$ for some simple root α. Let $t' = \mu(\varpi)$. By Lemma 2.7(i) and (ii), $U = U_\alpha \ltimes U'$ for some normal $\mathcal{O}$-subgroup scheme U' and, for $v \in V^{U(k)}$,

$$\begin{aligned}(\mathcal{S}f)(t')v &= \sum_{u_\alpha \in U_\alpha(F)/U_\alpha(\mathcal{O})} \sum_{u' \in U'(F)/U'(\mathcal{O})} f(t'u_\alpha u')v \\ &= \sum_{u_\alpha \in U_\alpha(F)/t'U_\alpha(\mathcal{O})t'^{-1}} \Bigg(\sum_{u' \in U'(F)/U'(\mathcal{O})} f(u_\alpha t' u')v \Bigg).\end{aligned}$$

By Lemma 2.7(iii), this sum is zero since $\langle \mu, \alpha \rangle > 1$, $\bar{k}$ is of characteristic p, and the function of u_α defined by the expression in parentheses is left invariant under $\ker(U_\alpha(\mathcal{O}) \to U_\alpha(k))$.

Step 5. We show that $(\mathcal{S}T_\lambda)(\mu(\varpi)) = 0$ if $\mu \notin X_*(S)_-$. Suppose that this is not the case. Let $M_\lambda = \{\mu \in X_*(S) : (\mathcal{S}T_\lambda)(\mu(\varpi)) \neq 0\}$. Note that this is a finite set by Step 0. Label the simple roots as $(\alpha_i)_{i=1}^r$ so that $\langle \mu, \alpha_1 \rangle > 0$ for some $\mu \in M_\lambda$. Define a homomorphism of abelian groups

$$\begin{aligned} o : X_*(S) &\to \mathbb{Z}^r \\ \mu &\mapsto (\langle \mu, \alpha_i \rangle)_{i=1}^r. \end{aligned}$$

Note that this is injective on M_λ: if $o(\mu_1) = o(\mu_2)$ for $\mu_i \in M_\lambda$, then $\mu_1 - \mu_2 \in X_F^*(G)^\perp$ (as $\mu_i \geqslant_{\mathbb{R}} \lambda$ by Step 3) and $\mu_1 - \mu_2 \in (\mathbb{R}\langle \Phi \rangle)^\perp$, so $\mu_1 = \mu_2$. Let μ be the element of M_λ such that $o(\mu)$ is greatest in the lexicographic order of $\mathbb{Z}^r$. In particular, $\langle \mu, \alpha_1 \rangle > 0$. We show that $\mathcal{S}(T_\lambda^2) = (\mathcal{S}T_\lambda)^2$ is non-zero on $2\mu(\varpi)$. Consider

$$(\mathcal{S}T_\lambda)^2(2\mu(\varpi)) = \sum_{\mu' \in X_*(S)} \mathcal{S}T_\lambda(\mu'(\varpi))\mathcal{S}T_\lambda((2\mu - \mu')(\varpi)).$$

If the term indexed by μ' is non-zero, then $\mu', 2\mu - \mu' \in M_\lambda$ and hence $o(\mu') \leqslant o(\mu)$ and $o(2\mu - \mu') \leqslant o(\mu)$. But since the sum of these inequalities yields an equality, it follows easily that $\mu' = \mu$. So $(\mathcal{S}T_\lambda)^2(2\mu(\varpi)) = ((\mathcal{S}T_\lambda)(\mu(\varpi)))^2 \neq 0$. Since $\langle 2\mu, \alpha_1 \rangle > 1$, we get a contradiction by Step 4 with $f = T_\lambda^2$.

Step 6. It remains to show that $\mathcal{S}$ is injective and maps onto $\mathcal{H}_T^-(V^{U(k)})$. This is again classical. By Step 1 the T_λ, $\lambda \in X_*(S)_-$, form a $\bar{k}$-basis of $\mathcal{H}_G(V)$, and by Lemma 2.1 the τ_μ, $\mu \in X_*(S)_-$, form a $\bar{k}$-basis of $\mathcal{H}_T^-(V^{U(k)})$. By Step 3, we may write $\mathcal{S}T_\lambda = \sum_{\mu \geqslant_{\mathbb{R}} \lambda} a_\lambda(\mu)\tau_\mu$ with $a_\lambda(\mu) \in \bar{k}$ and $a_\lambda(\lambda) = 1$. Since $\{\mu \in X_*(S)_- : \mu \geqslant_{\mathbb{R}} \lambda\}$ is finite by Lemma 2.4, the claims follow. $\square$

Suppose now that G is split and that G' is simply connected. We give a sketch of a simpler proof that $\mathcal{H}_G(V)$ is commutative. By [Jan03, II.1.16], there is a 'transpose' involution ${}^\tau : G \to G$ that induces the identity on T. (When $G = \mathrm{GL}_n$, one can take the usual transpose map.) Let ${}^\tau V$ be the dual $\mathrm{Hom}_{\bar{k}}(V, \bar{k})$ with $G(k)$-action $(g\psi)(v) := \psi({}^\tau g \cdot v)$. Since G' is simply connected, V extends to a representation of the algebraic group $G_{/\bar{k}}$. By using a weight-space decomposition of V, it follows that V and ${}^\tau V$ are isomorphic as $G(k)$-representations [Jan03, II.2.12(2)]. Fix a $G(k)$-linear isomorphism $\kappa : V \xrightarrow{\sim} {}^\tau V$.

An element $\varphi \in \mathrm{End}_{\bar{k}} V$ induces an endomorphism of ${}^\tau V$ and hence an endomorphism ${}^\tau\varphi \in \mathrm{End}_{\bar{k}} V$ by means of κ. Given $f \in \mathcal{H}_G(V)$, we define $f^* : G \to \mathrm{End}_{\bar{k}} V$ by $f^*(g) := {}^\tau f({}^\tau g)$. It is easy to check that $f^* \in \mathcal{H}_G(V)$ and that $f_1^* * f_2^* = (f_2 * f_1)^*$. It remains to show that * acts trivially or, equivalently, that $T_\lambda^* = T_\lambda$ for all $\lambda \in X_*(S)_-$. As ${}^\tau$ preserves $K = G(\mathcal{O})$ and $\lambda(\varpi)$, it follows that T_λ^* has the same support as T_λ. Moreover, it is clear that $T_\lambda^*(\lambda(\varpi))$ is a linear projection. Hence $T_\lambda^* = T_\lambda$.

2.2 Comparison with the p-adic Satake map

We will explain a result concerning compatibility with the p-adic Satake isomorphism of Schneider and Teitelbaum [ST06, § 3]. It will be convenient to state the result of Schneider and Teitelbaum in a slightly different form. To keep the notation simple, let us assume in this subsection that $G_{/F}$ is split (just as in [ST06]).

Let E be the (absolutely) irreducible representation of $G_{/F}$ of highest weight $\nu \in X^*(T)$. Then $E^{U(\mathcal{O})}$ is the highest-weight space of E; in particular, it is one-dimensional and $T(\mathcal{O})$ acts on it via ν. (This is because $E^{U(\mathcal{O})} \subset E^{\mathfrak{u}}$, where $\mathfrak{u} = \operatorname{Lie} U(\mathcal{O}) = \operatorname{Lie} U(F) = \operatorname{Lie}(U_{/F})$; but $E^{\mathfrak{u}} = E^U$ since $U_{/F}$ is connected.) Consider the p-adic Hecke algebra

$$\widetilde{\mathcal{H}}_G(E) = \operatorname{End}_{G(F)}(\text{c-Ind}_K^{G(F)} E),$$

which we again think of as an algebra (under convolution) of functions $f : G(F) \to \operatorname{End}_F(E)$ with compact support such that $f(k_1 g k_2) = k_1 f(g) k_2$ for all $k_1, k_2 \in K$ and $g \in G(F)$.

Lemma 2.9 [ST06, Lemma 1.4]. *The map*

$$\iota : \widetilde{\mathcal{H}}_G(1) \to \widetilde{\mathcal{H}}_G(E)$$

with $(\iota\phi)(g) = \phi(g) g \in \operatorname{End}_F(E)$ *is an algebra isomorphism.*

The point is that for $f \in \widetilde{\mathcal{H}}_G(E)$ and $g \in G(F)$, we have $g^{-1} f(g) \in \operatorname{End}_F(E)^{K \cap g^{-1} K g} = \operatorname{End}_F(E)^G = F$ by considering the action of the Lie algebra as above. Note that the lemma depends crucially on E being a representation not just of K but of $G(F)$, thus the analogue does not work for the characteristic p Hecke algebras.

Fix a K-stable norm $\|\cdot\|_E$ on E such that $\|E\|_E = |F|$. Equivalently, this corresponds to a choice of K-stable $\mathcal{O}$-lattice $E_0 \subset E$ given by $E_0 = \{x \in E : \|x\|_E \leqslant 1\}$. Then $\widetilde{\mathcal{H}}_G(E)$ carries a submultiplicative sup-norm, where $\operatorname{End}_F(E)$ is given the operator norm with respect to $\|\cdot\|_E$. Similarly, we have the Hecke algebra $\widetilde{\mathcal{H}}_T(E^{U(\mathcal{O})})$, likewise equipped with a sup-norm. The p-adic Satake map is then the following isometric isomorphism of normed F-algebras:

$$\begin{aligned} \widetilde{\mathcal{S}} : \widetilde{\mathcal{H}}_G(E) &\xrightarrow{\sim} \widetilde{\mathcal{H}}_T(E^{U(\mathcal{O})})^{W,*} \\ f &\mapsto \bigg(t \mapsto \sum_{U(F)/U(\mathcal{O})} f(tu) \Big|_{E^{U(\mathcal{O})}} \bigg). \end{aligned}$$

To define the right-hand side, let $\delta : B(F) \to q^{\mathbb{Z}} \subset \mathbb{R}^\times$ be the modulus character of the Borel subgroup. (Note that our δ is inverse to the one in [ST06].) Then $\widetilde{\mathcal{H}}_T(E^{U(\mathcal{O})})^{W,*}$ is the subalgebra of those $\varphi \in \widetilde{\mathcal{H}}_T(E^{U(\mathcal{O})})$ for which $\varphi\nu^{-1}\delta^{1/2} : T(F)/T(\mathcal{O}) \to \overline{F}$ is W-invariant. This condition does not depend on the choice of square root of δ (see [ST06, Example 2 in § 2]). To prove that $\widetilde{\mathcal{S}}$ is an algebra isomorphism, one reduces to the $E = 1$ case by applying Lemma 2.9 to both sides, in which case it is equivalent to the classical Satake isomorphism. That $\widetilde{\mathcal{S}}$ is an isometry follows from Lemma 3.6. For details, see [ST06, § 3]. Note that the map $S_\nu : \widetilde{\mathcal{H}}_G(1) \to F[X_*(S)]$ in [ST06, p. 653] is related to the one above via $S_\nu(\psi) = \varpi^{\operatorname{ord}\nu} \nu^{-1} \widetilde{\mathcal{S}}(\iota\psi)$.

From now on, suppose that E_0 is a $G_{/\mathcal{O}}$-stable $\mathcal{O}$-lattice and that $E_0 \otimes_{\mathcal{O}} \bar{k}$ contains $F(\nu)$, the irreducible representation of $G_{/\bar{k}}$ of highest weight ν, as a *subobject*. For example, we could take the dual Weyl module $E_0 = H^0_{\mathcal{O}}(\lambda)$, in the notation of [Jan03, II.8.6(1)] (see also [Jan03, II.8.8(1), II.2.4]). Suppose, moreover, that ν is q-restricted, i.e. that $0 \leqslant \langle \nu, \alpha^\vee \rangle < q$ for all simple roots α. Then $F(\nu)$ is irreducible as a representation of $G(k)$, and we denote it by V. (See the proof of Lemma 2.5; if $(G_{/\bar{k}})'$ is not simply connected, this follows by a z-extension argument.) Let $\widetilde{\mathcal{H}}_G(E)_0 \subset \widetilde{\mathcal{H}}_G(E)$ denote the elements with sup-norm at most 1. In particular, $\operatorname{im}(f) \subset \operatorname{End}_{\mathcal{O}}(E_0)$ for $f \in \widetilde{\mathcal{H}}_G(E)_0$, and we can consider the reduction $\overline{f} : G(F) \to \operatorname{End}_{\bar{k}}(E_0 \otimes \bar{k})$. Similarly, we have $\widetilde{\mathcal{H}}_T(E^{U(\mathcal{O})})_0^{W,*} \subset \widetilde{\mathcal{H}}_T(E^{U(\mathcal{O})})^{W,*}$. By considering the weight-space decomposition of E_0, it is clear that $E_0 \cap E^{U(\mathcal{O})}$ reduces to $V^{U(k)} \subset E_0 \otimes_{\mathcal{O}} \bar{k}$.

Proposition 2.10. *With the above notation, we have the following commutative diagram.*

$$\begin{tikzcd}
\widetilde{\mathcal{H}}_G(E)_0 \arrow[r, "\widetilde{\mathcal{S}}", "\sim"'] \arrow[d, "\alpha"'] & \widetilde{\mathcal{H}}_T(E^{U(\mathcal{O})})_0^{W,*} \arrow[d, "\beta"] \\
\mathcal{H}_G(V) \arrow[r, "\mathcal{S}", "\sim"'] & \mathcal{H}_T^-(V^{U(k)})
\end{tikzcd}$$

Here $(\alpha f)(g) = \overline{f(g)}|_V$ and $(\beta\varphi)(t) = \overline{\varphi(t)}$. The vertical maps are well-defined and induce isomorphisms after base-extending from $\mathcal{O}$ to $\bar{k}$.

Proof. For $\lambda \in X_*(S)_-$, consider $\widetilde{T}_\lambda \in \widetilde{\mathcal{H}}_G(E)$ defined by: (i) $\operatorname{supp} \widetilde{T}_\lambda = K\lambda(\varpi)K$; and (ii) $\widetilde{T}_\lambda(\lambda(\varpi)) = \varpi^{-\langle \lambda, \nu \rangle}\lambda(\varpi)$. We claim that the $\widetilde{T}_\lambda$ form an $\mathcal{O}$-basis of $\widetilde{\mathcal{H}}_G(E)_0$ and that $\alpha(\widetilde{T}_\lambda) = T_\lambda$. On the ν'-weight space of E, for $\nu' \leqslant \nu$, $\varpi^{-\langle \lambda, \nu \rangle}\lambda(\varpi)$ acts as the scalar $\varpi^{\langle \lambda, \nu' - \nu \rangle}$. Thus $\overline{\widetilde{T}_\lambda(\lambda(\varpi))}$ is the linear projection onto the ν'-weight spaces of $E_0 \otimes_k \bar{k}$ for the weights ν' satisfying $\langle \lambda, \nu' - \nu \rangle = 0$. Thus it preserves any $G_{/\bar{k}}$-subrepresentation and, in particular, V. By (2.8) and the description of $V^{U_{-\lambda}(k)}$ given in the proof of Lemma 2.5, the claim follows and we see that α is well-defined.

Similarly, for $\lambda \in X_*(S)_-$, consider $\widetilde{\tau}_\lambda \in \widetilde{\mathcal{H}}_T(E^{U(\mathcal{O})})^{W,*}$ defined by: (i) $T^- \cap \operatorname{supp} \widetilde{\tau}_\lambda = \lambda(\varpi)T(\mathcal{O})$; and (ii) $\widetilde{\tau}_\lambda(\lambda(\varpi)) = 1$. We claim that the $\widetilde{\tau}_\lambda$ form an $\mathcal{O}$-basis of $\widetilde{\mathcal{H}}_T(E^{U(\mathcal{O})})_0^{W,*}$ and that $\beta(\widetilde{\tau}_\lambda) = \tau_\lambda$. Recall that $\delta^{1/2}(\mu(\varpi)) = q^{-\langle \mu, \rho \rangle}$ for $\mu \in X_*(S)$, where $\rho = \frac{1}{2}\sum_{\Phi^+} \alpha$ (see [Gro98, (3.3)]). Thus, for $\varphi \in \widetilde{\mathcal{H}}_T(E^{U(\mathcal{O})})^{W,*}$,

$$\varphi(w(\lambda(\varpi))) = \varphi(\lambda(\varpi))\varpi^{\langle w\lambda - \lambda, \nu \rangle} q^{\langle w\lambda - \lambda, \rho \rangle} \quad \text{for all } w \in W.$$

Since $w\lambda \geqslant_{\mathbb{R}} \lambda$ and the second exponent is positive if $w\lambda \neq \lambda$, it follows that $\operatorname{supp}(\overline{\varphi}) \subset T^-$ whenever $\|\varphi\| \leqslant 1$. By the same reasoning, $\|\widetilde{\tau}_\lambda\| \leqslant 1$. The claim follows and we see that β is well-defined.

This completes the proof, since the diagram obviously commutes. $\square$

Remark 2.11. Note that this argument yields another proof that $\operatorname{im}(\mathcal{S}) \subset \mathcal{H}_T^-(V^{U(k)})$ in the case where V arises from a representation of $G_{/\bar{k}}$ (which does not always happen if $(G_{/\bar{k}})'$ is not simply connected), after the surjectivity of the map α has been established.

2.3 The W-regular case

For the proof of Proposition 1.4 we will need a lemma. Let $\overline{\Phi}$ denote the set of absolute roots of $G_{/\bar{k}}$ with respect to $T_{/\bar{k}}$. Since $G_{/k}$ is quasi-split, W is a subgroup of the absolute Weyl group $\overline{W}$ and the restriction homomorphism $X^*(T_{/\bar{k}}) \twoheadrightarrow X^*(S_{/\bar{k}})$ is W-equivariant. Moreover, $\overline{\Phi}$ maps onto Φ under this map; in particular, Φ^+ determines a system of positive roots $\overline{\Phi}^+$ in $\overline{\Phi}$.

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Lemma 2.12.

(i) *Suppose that $\eta \in X^*(T_{/\bar{k}})_+$ and $w \in W$. There are simple reflections $s_i \in W$ such that*

$$\eta \gneqq s_1\eta \gneqq \cdots \gneqq s_l \cdots s_2 s_1 \eta = w\eta. \tag{2.13}$$

(ii) *Suppose that $\eta \in X^*(T_{/\bar{k}})_+$ and $\alpha \in \Phi$ is simple. If $\eta - s_\alpha\eta \geqslant 0$, then $\eta - s_\alpha$ is the sum of simple roots $\overline{\beta}_i \in \overline{\Phi}$ such that $\overline{\beta}_i|_S = \alpha$.*

Proof. (i) Let us write $w = s_l \cdots s_1$ as a reduced product of simple reflections in W. We will show that

$$\eta \geqslant s_1\eta \geqslant \cdots \geqslant s_l \cdots s_2 s_1 \eta = w\eta, \tag{2.14}$$

which implies (2.13) since every time there is an equality, the corresponding simple reflection s_i can be omitted. We claim that $\ell_{\overline{W}}(w) = \sum \ell_{\overline{W}}(s_i)$, where $\ell_{\overline{W}}$ denotes the length in $\overline{W}$. Once we establish this, we are done: by writing each s_i as a reduced product of simple reflections in $\overline{W}$, we are reduced to proving the analogue of (2.14) in $\overline{W}$, where it is easy and well known.

Recall that the length of w in W (respectively, $\overline{W}$) equals the number of non-divisible positive roots α in Φ (respectively, $\overline{\Phi}$) such that $w(\alpha) < 0$ (see, for example, [Bou02, § VI.1.6, Corollary 2]). In particular, a simple reflection $s_\alpha \in W$ stabilises $\Phi^+ - \{\alpha\}$. Say $\alpha_i \in \Phi$ is the simple root corresponding to $s_i \in W$. Since $w = s_l \cdots s_1$ is of length l in W, it sends precisely the following l non-divisible positive roots of Φ to a negative root: $\alpha_1, s_1\alpha_2, \ldots, s_1 \cdots s_{l-1}\alpha_l$. Letting $A_i = \{\overline{\beta} \in \overline{\Phi}^+ : \overline{\beta}|_S \in \mathbb{Z}_{>0}\alpha_i\}$, we see that w sends precisely the following positive roots of $\overline{\Phi}$ to a negative root: $A_1 \cup s_1A_2 \cup \cdots \cup s_1 \cdots s_{l-1}A_l$. Clearly, $\ell_{\overline{W}}(s_i) = |A_i|$, which implies the claim.

(ii) Write $\eta - s_\alpha\eta = \overline{\beta}_1 + \cdots + \overline{\beta}_r$ with $\overline{\beta}_i \in \overline{\Phi}$ simple. Now restrict to S. On the left-hand side we get an integer multiple of α and on the right-hand side a sum of simple roots $\overline{\beta}_i|_S$ in Φ. Thus $\overline{\beta}_i|_S = \alpha$ for all i. □

Proof of Proposition 1.4. By Step 3 of the proof of Theorem 1.2, we know that $(\mathcal{S}T_\lambda)(\lambda(\varpi)) = 1$. It thus suffices to show that for any given $\mu \in X_*(S) - \{\lambda\}$, each term in the sum defining $(\mathcal{S}T_\lambda)(\mu(\varpi))$ vanishes.

Let $t' = \mu(\varpi)$ and $t = \lambda(\varpi)$. Choose $0 \neq v \in V^{U(k)}$.

Step 1. We will show that if $T_\lambda(g)v \neq 0$, then $g \in KtI$, where $I = \mathrm{red}^{-1}(B(k))$ is an Iwahori subgroup. Let $W_\lambda \leqslant W$ be the Weyl group of $(S_{/k}, L_\lambda)$ (generated by simple reflections associated to simple roots $\alpha \in \Phi$ with $\langle \lambda, \alpha \rangle = 0$). For each $w \in W$, choose a representative $\dot{w} \in N(S)(k)$ and a lift of it, $\dot{w} \in G(\mathcal{O}) = K$. Then

$$G(k) = \coprod_{W_\lambda \backslash W} P_\lambda(k)\dot{w}B(k)$$

by [Bor91, 21.16(3)]. By Proposition 3.8,

$$K = \coprod_{W_\lambda \backslash W} (K \cap t^{-1}Kt)\dot{w}I$$

and thus

$$KtK = \coprod_{W_\lambda \backslash W} Kt\dot{w}I.$$

So if $T_\lambda(g)v \neq 0$, then $g = kt\dot{w}i$ for some $k \in K$, $w \in W$ and $i \in I$. Thus $T_\lambda(t)\dot{w}v \neq 0$. We will show that $w \in W_\lambda$. Recalling the definition of T_λ in (2.8), we may, by the proof of Lemma 2.5, reduce to the case where $(G_{/\bar{k}})'$ is simply connected. (The lifted Levi subgroup equals $L_{\widetilde{\lambda}}$ for any lift $\widetilde{\lambda} \in X_*(\widetilde{S})$ of λ. We can lift $\dot{w}$ since the Weyl group of $(S_{/k}, G_{/k})$ and that of its lift $(\widetilde{S}, \widetilde{G})$ can be naturally identified with each other by [Bor91, 22.6].) Since $(G_{/\bar{k}})'$ is simply connected, there is a q-restricted weight $\nu \in X^*(T)_+$ such that $V \cong F(\nu)$ as $G(k)$-representations. But we saw in the proof of Lemma 2.5 that $T_\lambda(t)$ is the projection onto the weight spaces for $\nu' \in X^*(T)$ such that $\nu - \nu'$ is a sum of simple roots of $(T_{/\bar{k}}, L_{\lambda/\bar{k}})$, i.e. a sum of simple roots $\overline{\beta} \in \overline{\Phi}$ such that $\langle \overline{\beta}, \lambda \rangle = 0$. Since $T_\lambda(t)\dot{w}v \neq 0$, it follows that $\nu - w\nu$ is a sum of simple roots $\overline{\beta} \in \overline{\Phi}$ such that $\langle \overline{\beta}, \lambda \rangle = 0$.

By Lemma 2.12(i), there are simple reflections $s_i \in W$ corresponding to simple roots $\alpha_i \in \Phi$ such that

$$\nu \gneq s_1\nu \gneq s_2 s_1 \nu \gneq \cdots \gneq s_l \cdots s_1 \nu = w\nu.$$

By Lemma 2.12(ii), the ith and $(i+1)$st term in this sequence differ by a sum of simple roots $\overline{\beta}_{ij} \in \overline{\Phi}$ such that $\overline{\beta}_{ij}|_S = \alpha_i$. Thus $\langle \alpha_i, \lambda \rangle = \langle \overline{\beta}_{ij}, \lambda \rangle = 0$. It follows that $s_i \in W_\lambda$ for all i. Since V is W-regular, we see that $w = s_l \cdots s_1 \in W_\lambda$ and $g \in KtI$.

(We remark that we actually used only the fact that $\mathrm{Stab}_W(\nu) \subset W_\lambda$.)

Step 2. We show that $KtI \cap t'U(F) = \varnothing$. Suppose not. We use the Iwahori decomposition

$$I = (I \cap \overline{U}(F))(I \cap T(F))(I \cap U(F)),$$

where $\overline{U}$ is the unipotent radical of the opposite Borel subgroup (Lemma 3.10). Since t contracts $I \cap \overline{U}(F)$, we find that $tIt^{-1} \subset IU(F)$. Thus

$$\varnothing \neq (KtI \cap t'U(F))t^{-1} \subset KU(F) \cap t't^{-1}U(F).$$

Therefore $K \cap t't^{-1}U(F) \neq \varnothing$ and so $t't^{-1} \in T(\mathcal{O})$, which contradicts the assumption that $\mu \neq \lambda$. □

2.4 Satake parameters

Proof of Corollary 1.5. By Corollary 1.3, we need to classify algebra homomorphisms $\theta : \bar{k}[X_*(S)_-] \to \bar{k}$, i.e. monoid homomorphisms $X_*(S)_- \to \bar{k}$ where $\bar{k}$ is considered with its multiplicative structure. Then $M := \theta^{-1}(\bar{k}^\times)$ satisfies

$$\lambda_1 + \lambda_2 \in M \iff \lambda_1 \in M \quad \text{and} \quad \lambda_2 \in M. \tag{2.15}$$

Let $X_*(S)_0 := \{\lambda \in X_*(S) : \langle \lambda, \alpha \rangle = 0 \ \forall \alpha \in \Phi\}$. Since this is a subgroup of $X_*(S)_-$, we have that $X_*(S)_0 \subset M$. For $\delta \in \Delta$, choose $\lambda_\delta \in X_*(S)_-$ such that $\langle \lambda_\delta, \delta' \rangle$ is zero if $\delta' \in \Delta - \{\delta\}$ and negative if $\delta' = \delta$.

We claim that $M = J^\perp \cap X_*(S)_-$ (a 'facet' of $X_*(S)_-$), where $J = \{\delta : \lambda_\delta \notin M\}$. (Note that J is independent of the choice of the λ_δ, since $X_*(S)_0 \subset M$.) Suppose that $\lambda \in X_*(S)_-$. Then there is an $n \in \mathbb{Z}_{>0}$ such that $n\lambda = \sum n_\delta \lambda_\delta + \lambda_0$ for some $n_\delta \in \mathbb{Z}_{\geqslant 0}$ and some $\lambda_0 \in X_*(S)_0$. Then, from (2.15), we see that $\lambda \in M$ if and only if $n_\delta \neq 0$ implies $\delta \notin J$ if and only if $\lambda \in J^\perp$.

Next, we show that the subgroup of $X_*(S)$ generated by M equals $J^\perp$. One inclusion being obvious, suppose that $\lambda \in J^\perp$. Then $\lambda + n \sum_{\delta \notin J} \lambda_\delta$ is in $X_*(S)_-$ (and hence in $M = J^\perp \cap X_*(S)_-$) for some $n \in \mathbb{Z}_{>0}$, which implies that λ is in the subgroup generated by M.

As $\bar{k}^\times$ is a group, $\theta|_M$ extends uniquely to a group homomorphism $\tilde{\theta} : J^\perp \to \bar{k}^\times$. Taking character groups in the exact sequence defining $\widehat{S}_J$, we find that $X^*(\widehat{S}_J) = J^\perp$. Thus $\tilde{\theta}$ corresponds to an element of $X_*(\widehat{S}_J) \otimes \bar{k}^\times \cong \widehat{S}_J(\bar{k})$.

All pairs (J, s_J) with $s_J \in \widehat{S}_J(\bar{k})$ are obtained in this way, because $J^\perp \cap X_*(S)_-$ satisfies (2.15), which allows us to extend a homomorphism $J^\perp \cap X_*(S)_- \to \bar{k}$ by zero to a monoid homomorphism $X_*(S)_- \to \bar{k}$. □

3. Buildings arguments

The main goal of this section is to prove Proposition 3.8 and Lemma 2.7. We also justify some basic results about unramified groups using the work of Bruhat and Tits [BT72, BT84]. Although most of these results are well known, we could not find a good reference for their proofs.

In what follows, references to [BT72] (Bruhat–Tits part I) and [BT84] (Bruhat–Tits part II) will be given in the form I.4.4.4 or II.5.1.40, for example.

We will keep as much as possible to the notation of [BT72, BT84]. In particular, K now denotes the p-adic field and $\overline{K}$ its residue field, N denotes $N(S)$, Z denotes the centraliser $Z(S)$ of S in G, and ${}^vW$ denotes the Weyl group. Group schemes over $\mathcal{O}$ are denoted by fraktur letters ($\mathfrak{G}, \mathfrak{T}, \dots$), their generic fibres by the corresponding roman letters ($G, T, \dots$), and their special fibres by overlined characters ($\overline{\mathfrak{G}}, \overline{\mathfrak{T}}, \dots$). Note that 'fixer' is a synonym for 'pointwise stabiliser'. An $\mathcal{O}$-group scheme is said to be *connected* if its two fibres are connected. The connected component of a smooth $\mathcal{O}$-group scheme is defined fibrewise (II.1.2.12). As in § 2, we are assuming that the valuation ord_K surjects onto the integers.

Let $\mathcal{I}$ denote the *reduced* building of G. The general construction in I.6 and I.7 produces $\mathcal{I}$ starting with a valuation of the 'root datum' $(T(K), (U_a(K))_a)$. Such a valuation is constructed for quasi-split groups by descent from the split case (II.4.2) and in general by étale descent from the quasi-split case (II.5.1). The apartment A of S is an affine space under the vector space V which is the quotient of $X_*(S)_\mathbb{R}$ dual to $\mathbb{R}\langle\Phi\rangle \subset X^*(S)_\mathbb{R}$.

Lemma 3.1. *Suppose that $\mathfrak{G}$ is a smooth $\mathcal{O}$-group scheme with generic fibre G. Then $\mathfrak{G} \times \overline{K}$ is reductive if and only if $\mathfrak{G} \cong \mathfrak{G}_x^0$ for some hyperspecial point x. In this case, G is unramified and $\mathfrak{G} \times \overline{K}$ is connected.*

Recall that a point $x \in \mathcal{I}$ is *hyperspecial* if G splits over K^{nr} and x is a special point inside the building of $G \times K^{\mathrm{nr}}$ (see [Tit79, 1.10]).

Proof. The first statement is II.5.1.40. (Note that in II.5, the superscript ♮ refers to the objects over the base field; the other objects live over the strict henselisation of the base field.)

Let us show that G is quasi-split. Without loss of generality, assume that x lies in the apartment of S. The canonical extension $\mathfrak{S}$ of S (the split torus over $\mathcal{O}$ with generic fibre S) is a closed subscheme of $\mathfrak{G}_x^0$, and its reduction $\overline{\mathfrak{S}}$ is a maximal $\overline{K}$-split torus in $\overline{\mathfrak{G}}_x^0$ (see II.5.1.11). The Lie algebra $\mathrm{Lie}\,\mathfrak{G}_x^0$ is a free $\mathcal{O}$-module of finite rank (since $\mathfrak{G}_x^0$ is a smooth group scheme), and we can consider its decomposition under $\mathfrak{S}$. Note that the character groups $X^*(\mathfrak{S})$, $X^*(\mathfrak{S}_{\overline{K}})$ and $X^*(S)$ are naturally isomorphic. Since $\overline{K}$ is a finite field, $\overline{\mathfrak{G}}_x^0$ is quasi-split and

$$\mathrm{rank}\,\overline{\mathfrak{G}}_x^0 = \dim_{\overline{K}}(\mathrm{Lie}\,\overline{\mathfrak{G}}_x^0)^{\overline{\mathfrak{S}}=1} = \dim_K(\mathrm{Lie}\,G)^{S=1} = \dim Z \geqslant \mathrm{rank}\,G.$$

(Here 'rank' denotes the absolute rank of an algebraic group.) On the other hand, any split torus in the special fibre of $\mathfrak{G}_x^0 \times \mathcal{O}^{\mathrm{nr}}$ can be lifted to a split torus in the generic fibre, as explained in the proof of II.4.6.4, so that $\mathrm{rank}\,\overline{\mathfrak{G}}_x^0 \leqslant \mathrm{rank}\,G$. Thus equality holds, and so Z is a maximal torus of G, i.e. G is quasi-split.

The connectedness of $\mathfrak{G} \times \overline{K}$ follows from base change to the strict henselisation and II.4.6.22. □

Assume, from now on, that $\mathfrak{G} \cong \mathfrak{G}_x^0$ for some hyperspecial point x. Then $\underline{K} := \mathfrak{G}_x^0(\mathcal{O})$ is a hyperspecial maximal compact subgroup of $G(K)$.

Let us summarise some results in II.4.6 on the structure of $\mathfrak{G} \cong \mathfrak{G}_x^0$. Fix an apartment A of $\mathcal{I}$ containing x. Let S be the corresponding maximal split torus of G and let $T = Z$ (a maximal torus, since G is quasi-split). Let Φ be the set of roots of (G, S), and let Φ^{red} denote the subset of non-divisible roots. For $a \in \Phi$, let U_a denote the corresponding root subgroup. In particular, $U_{2a} \subset U_a$ whenever $\{a, 2a\} \subset \Phi$. Fix a Steinberg–Chevalley valuation $\varphi = (\varphi_a)_{a\in\Phi}$ of the 'root datum' $(T(K), (U_a(K))_{a\in\Phi})$, as constructed in II.4.1–4.2. Here $\varphi_a : U_a(K) \to \mathbb{R} \cup \{\infty\}$; it yields a filtration of each root subgroup, $U_{a,k} = \{u \in U_a(K) : \varphi_a(u) \geqslant k\}$ (see II.4.3.1(1)). Let $\Gamma_a = \varphi_a(U_a - \{1\})$ and $\Gamma'_a = \{\varphi_a(u) : u \in U_a - \{1\}, \varphi_a(u) = \max \varphi_a(uU_{2a})\} \subset \Gamma_a$; these are discrete subsets of $\mathbb{R}$.

By II.4.4.18, there are smooth prolongations $\mathfrak{S}$ of S (the split torus over $\mathcal{O}$ with generic fibre S) and $\mathfrak{T}$ of T (denoted there by $\mathfrak{T}^R$). Then $\mathfrak{S}$ is a closed subgroup scheme of $\mathfrak{T}$.

Lemma 3.2. *$\mathfrak{T}$ is connected (i.e. its special fibre is connected).*

Proof. Let K^{nr} be the maximal unramified extension of K with ring of integers $\mathcal{O}^{\mathrm{nr}}$. Since $T \times K^{\mathrm{nr}}$ is split, it has a canonical prolongation $\mathfrak{T}^{\mathrm{nr}}$ to $\mathcal{O}^{\mathrm{nr}}$ (the split torus over $\mathcal{O}^{\mathrm{nr}}$ with generic fibre $T \times K^{\mathrm{nr}}$). As remarked in II.5.1.9 (top of [BT84, p. 149]), $\mathfrak{T}^{\mathrm{nr}}$ descends to the torus $\mathfrak{T}$ defined in II.4.4. Since $\mathfrak{T}^{\mathrm{nr}}$ is connected, this completes the proof. To justify that remark in II.5.1.9, one uses the last item in II.4.4.12(i) and the fact that $\mathfrak{T}^{\mathrm{nr}}$ is étoffé (II.1.7) to see that $\mathcal{O}[\mathfrak{T}] = \{f \in K[T] : f(\mathfrak{T}^{\mathrm{nr}}(\mathcal{O}^{\mathrm{nr}})) \subset \mathcal{O}^{\mathrm{nr}}\} = \mathcal{O}[\mathfrak{T}']$, where $\mathfrak{T}'$ is the torus descended from $\mathfrak{T}^{\mathrm{nr}}$. $\square$

From II.4.6.4 it follows that $\overline{\mathfrak{S}}$ is a maximal split torus of $\overline{\mathfrak{G}}_x^0$ and that $\overline{\mathfrak{T}}$ is the centraliser of $\overline{\mathfrak{S}}$ (a maximal torus, as $\overline{\mathfrak{G}}_x^0 \times \overline{K}$ is quasi-split). By considering the Lie algebra of $\mathfrak{G}_x^0$, we see that the root systems of (S, G) and $(\overline{\mathfrak{S}}, \overline{\mathfrak{G}}_x^0)$ are naturally identified with each other.

Recall that $\mathfrak{G}_x^0$ is the smooth $\mathcal{O}$-group scheme $\mathfrak{G}_f^0$ with generic fibre G associated to the optimal, quasi-concave function $f : \Phi \to \mathbb{R}$ defined by

$$f(a) = \min\{k \in \Gamma'_a : a(x - \varphi) + k \geqslant 0\}$$

(see II.4.6.26). For all non-divisible roots $a \in \Phi$, there is a smooth $\mathcal{O}$-group scheme $\mathfrak{U}_{f,a}$ with generic fibre U_a (see II.4.5), which we denote by $\mathfrak{U}_{x,a}$. It is a closed subgroup scheme of $\mathfrak{G}_x^0$, and $\overline{\mathfrak{U}}_{x,a}$ is the root subgroup of a in $\overline{\mathfrak{G}}_x^0$ (see II.4.6.4). The product map $\prod_a \mathfrak{U}_{x,a} \to \mathfrak{G}_x^0$, where a runs over all positive, non-divisible roots in any order, is an isomorphism onto a closed subgroup scheme $\mathfrak{U}^+$ (see II.4.6.2). Let U^+ denote its generic fibre. By II.4.4.19, $\mathfrak{T}$ normalises each $\mathfrak{U}_{x,a}$, and the product map yields an isomorphism of the semidirect product $\mathfrak{T} \ltimes \mathfrak{U}^+$ onto a closed subgroup scheme of $\mathfrak{G}_x^0$ whose fibres are the Borel subgroups associated to Φ^+. (Note that this is stated in II.3.8.2 only for a group scheme whose connected component is $\mathfrak{G}_x^0$, but it implies the assertion here: the scheme $\mathfrak{T} \times \mathfrak{U}^+$ is connected because it is the product of connected group schemes [Gro70, Exp. VI_A, Lemme 2.1.2].)

Lemma 3.3. *Suppose that F is a facet of A whose closure contains the hyperspecial point x. Then $\widehat{\mathfrak{G}}_F = \mathfrak{G}_F^0$. In particular, $\widehat{\mathfrak{G}}_x = \mathfrak{G}_x^0$.*

Note that $\widehat{\mathfrak{G}}_\Omega$ (of II.4.6.26) equals $\mathcal{G}_{\mathrm{pr}_{ss}^{-1}(\Omega)}$ in the notation of [Tit79, 3.4].

Proof. First, we show $\widehat{\mathfrak{G}}_F = \mathfrak{G}_F$ by showing that $\widehat{N}_F^1 = N_F^1$ (see II.4.6.26). Let $G(K)^1 = \{g \in G(K) : \mathrm{ord}_K(\chi(g)) = 0\ \forall \chi \in X_K^*(G)\}$. Note that $\ker \nu \cap G(K)^1 = H^1 \subset N_F^1 \subset \widehat{N}_F^1$ (by II.4.6.3), so it suffices to show that $\nu(N_F^1)=\nu(\widehat{N}_F^1)$. Identify A and V using the special point x as origin. Then $\widehat{N}_F^1$ is identified with a subgroup of ${}^vW$, namely the subgroup of elements fixing F. It is generated by those basic reflections r_a of ${}^vW$ such that F is contained in the hyperplane through x which is defined by $a\in\Phi$. But I.7.1.3 shows that $\nu(N_F^1)$ has the same description. (The point is that $\Gamma_a=\Gamma'_a\cup\frac{1}{2}\Gamma_{2a}$ (see I.6.2.1) and that $r_a=r_{2a}$.)

Finally, $\mathfrak{G}_F=\mathfrak{G}_F^0$ since $\mathfrak{T}$ is connected (II.4.6.2). (This is the only part that uses the assumption that x is hyperspecial, not just special.) □

Lemma 3.4 (Iwasawa decomposition [Tit79, 3.3.2]).

$$G(K)=T(K)U^+(K)\underline{K}.$$

Proof. We use the description of the building in terms of an affine Tits system. Associated to the valuation φ of the 'root datum' $(T(K),(U_a(K)))$ we have the apartment A, the set of affine roots $\alpha_{a,k}$ ($a\in\Phi$, $k\in\Gamma'_a$), the affine Weyl group W generated by the set of reflections in the boundary hyperplanes of the affine roots, and $\nu:N(K)\to\mathrm{Aff}(A)$ giving the action on the apartment with kernel H (see I.6.2). Let $N'=\nu^{-1}(W)$, $T'=N'\cap T(K)$, and $G'=\langle N',U_a(K)\rangle_{a\in\Phi}$. Fix a chamber $C\subset A$. Let $B=HU_C$ and let $\mathbf{S}$ be the set of reflections in the walls of C. By I.6.5, $(G',B,N',\mathbf{S})$ is a saturated Tits system of affine type such that the inclusion $G'\to G(K)$ is (B,N')-adapted of connected type and such that the condition $G'=\mathfrak{B}N'B$ in I.4.4(1) holds with $\mathfrak{B}=T'U^+(K)$.

Then $\mathcal{I}$ is naturally isomorphic to the building constructed out of this Tits system, whose facets are the 'parahoric' subgroups of $(G',B,N',\mathbf{S})$ (see I.2 and I.7.4.2). Let $\underline{K}'$ be the fixer of x in $G(K)$, so that $\underline{K}=\underline{K}'\cap G(K)^1$ by Lemma 3.3. By I.4.4.5, $\underline{K}'=(\nu^{-1}(\widehat{V})\cap\underline{K}')\underline{K}$, where $\widehat{V}$ consists of the translations in $\widehat{W}=\nu(N(K))$. As x is special, $\underline{K}'$ is a good maximally bounded subgroup of $G(K)$ (see I.4.4.6(i)) so that $G(K)=\widehat{\mathfrak{B}}\underline{K}'=\widehat{\mathfrak{B}}(\nu^{-1}(\widehat{V})\cap\underline{K}')\underline{K}$. The result follows from using the facts that $\widehat{\mathfrak{B}}=\nu^{-1}(\widehat{V})\mathfrak{B}$ (by I.4.1.5) and that $\nu^{-1}(\widehat{V})=T(K)$ (by I.6.2.10(i) and I.6.1.11(ii)). □

Lemma 3.5 (Cartan decomposition [Tit79, 3.3.3]).

$$G(K)=\coprod_{\lambda\in X_*(S)_-}\underline{K}\lambda(\varpi)\underline{K}.$$

Proof. We keep the notation of the previous proof. Let D be the 'Weyl' chamber in V corresponding to Φ^+ and let $\widehat{V}_D=\widehat{V}\cap\overline{D}$. By I.4.4.3(2), $G(K)=\underline{K}'\nu^{-1}(\widehat{V}_D)\underline{K}'$ and the set of double cosets biject with $\widehat{V}_D$. Since $\nu^{-1}(\widehat{V})\cap\underline{K}'=\ker\nu=H$, we have $\underline{K}'=H\underline{K}$ and $G(K)=\underline{K}\nu^{-1}(\widehat{V}_D)\underline{K}$. Besides, $G(K)^1\lhd G(K)$ and $H\subset T(K)$. Using these facts, it is easy to see that for $t_1,t_2\in\nu^{-1}(\widehat{V}_D)\subset T(K)$, $\underline{K}t_1\underline{K}=\underline{K}t_2\underline{K}$ if and only if $t_1t_2^{-1}\in H\cap G(K)^1=\ker\nu^1$ where ν^1 is the action map of $N(K)$ on the extended apartment (II.4.2.16). It follows that the set of double cosets $\underline{K}\backslash G(K)/\underline{K}$ bijects with $\widehat{V}_D^1=\nu^1(\nu^{-1}(\widehat{V}_D))$ (the analogue of $\widehat{V}_D$ for the extended building).

By I.4.2.16(3), $\langle\nu^1(t),c\rangle=-(\mathrm{ord}_K\circ c)(t)$ for $t\in T(K)$ and $c\in X_K^*(T)_{\mathbb{R}}=X^*(S)_{\mathbb{R}}$. By Lemma 2.1, $\nu^1(t)=-\zeta(t)$, where $\zeta:T(K)\to X_*(S)$ was defined there. The result follows from that lemma. □

Lemma 3.6. *Suppose that $\lambda\in X^*(S)_-$ and $\lambda'\in X^*(S)$.*

(i) *If $\underline{K}\lambda(\varpi)\underline{K}\cap\lambda'(\varpi)U^+(K)\neq\varnothing$, then $\lambda'\geqslant_{\mathbb{R}}\lambda$.*

(ii) $\underline{K}\lambda(\varpi)\underline{K}\cap\lambda(\varpi)U^+(K)=\lambda(\varpi)\mathfrak{U}^+(\mathcal{O})$.

Note that (i) is claimed without proof in [Car79, p. 148].

Proof. We keep the notation of Lemmas 3.4 and 3.5.

(i) For $t \in T(K)$ and $a \in \Phi$ we have $\langle \nu(t), a\rangle = -(\mathrm{ord}_K \circ a)(t)$ (see I.4.2.7(3)). Applying this with $t = (\lambda - \lambda')(\varpi)$, we see that the image of $\lambda' - \lambda \in X_*(S)_{\mathbb{R}}$ in the quotient space $V \cong (\mathbb{R}\langle\Phi\rangle)^*$ is $\nu((\lambda - \lambda')(\varpi))$.

Suppose that $\underline{K}\lambda(\varpi)\underline{K} \cap \lambda'(\varpi)U^+(K) \neq \varnothing$. On the one hand, $\underline{K}'\lambda(\varpi)\underline{K}' \cap HU^+(K)\lambda'(\varpi)\underline{K}' \neq \varnothing$. This implies that $\nu((\lambda - \lambda')(\varpi)) \geqslant_D 0$ by I.4.4.4(i), i.e. $\bar{\lambda}' \geqslant_{\mathbb{R}} \bar{\lambda}$. (Note that $\widehat{\mathfrak{B}}^0 = H\mathfrak{B}^0 = HU^+(K)$ by I.4.1.5 and I.6.5.) On the other hand, as $\underline{K}$ and $U^+(K)$ are contained in $G(K)^1$, $(\lambda' - \lambda)(\varpi) \in G(K)^1$, i.e. $\lambda' - \lambda \in X^*_K(G)^{\perp}_{\mathbb{R}}$. The assertion follows from the definition of $\leqslant_{\mathbb{R}}$ (Definition 2.3).

(ii) Note that the left-hand side is contained in

$$(\underline{K}'\lambda(\varpi)\underline{K}' \cap HU^+(K)\lambda(\varpi)\underline{K}') \cap \lambda(\varpi)U^+(K) = \lambda(\varpi)\underline{K}' \cap \lambda(\varpi)U^+(K)$$

by I.4.4.4(ii). As $U^+(K) \subset G(K)^1$, this is contained in $\lambda(\varpi)\underline{K} \cap \lambda(\varpi)U^+(K) = \lambda(\varpi)\mathfrak{U}^+(\mathcal{O})$. The opposite containment is obvious. □

LEMMA 3.7. *If $y \in A$ is hyperspecial, then $a(\varphi - y) \in \Gamma'_a$ for all $a \in \Phi$.*

Proof. We consider the $G(K)$-equivariant injection of buildings $j : \mathcal{I} \to \widetilde{\mathcal{I}}$, where $\widetilde{\mathcal{I}}$ is the building of G over K^{nr} (see II.5.1.24), or even just the restriction of j to apartments $A \to \widetilde{A}$ corresponding to S (respectively, T). Let $\widetilde{\Phi}$ denote the set of roots of (T, G). For $a \in \Phi$, let us say that an 'a-wall' is the boundary of an affine root defined by a in A. Similarly, we have the notion of an '$\widetilde{a}$-wall' in $\widetilde{A}$ for $\widetilde{a} \in \widetilde{\Phi}$. By II.5.1.20, the affine roots in A are precisely the intersections with A of the affine roots in $\widetilde{A}$.

As y is hyperspecial, for each $\widetilde{a} \in \widetilde{\Phi}$ there is an $\widetilde{a}$-wall passing through $j(y)$. By intersecting with A, we see that there is an a-wall passing through y for each $a \in \Phi$. Since the affine roots in A are defined to be the $\alpha_{a,k} = \{z \in A : a(z - \varphi) + k \geqslant 0\}$ for $a \in \Phi$ and $k \in \Gamma'_a$, the lemma follows. □

As in the proof of Theorem 1.2, we denote by $P_\lambda = L_\lambda \ltimes U_\lambda$ the parabolic subgroup of $\mathfrak{G} \times \overline{K}$ determined by $\lambda \in X_*(\overline{\mathfrak{S}}) = X_*(S)$.

PROPOSITION 3.8. *Suppose that $\lambda \in X_*(S)$. Let $t = \lambda(\varpi) \in S(K)$ and let* $\mathrm{red} : \mathfrak{G}(\mathcal{O}) \to \mathfrak{G}(\overline{K})$ *denote the reduction map. Then*

$$\mathrm{red}(\mathfrak{G}(\mathcal{O}) \cap t^{-1}\mathfrak{G}(\mathcal{O})t) = P_\lambda(\overline{K}).$$

Moreover,

$$\begin{aligned}\{(\mathrm{red}(g), \mathrm{red}(tgt^{-1})) &: g \in \mathfrak{G}(\mathcal{O}) \cap t^{-1}\mathfrak{G}(\mathcal{O})t\} \\ &= \{(g_+, g_-) \in P_\lambda(\overline{K}) \times P_{-\lambda}(\overline{K}) : [g_+] = [g_-] \in L_\lambda(\overline{K}) = L_{-\lambda}(\overline{K})\},\end{aligned} \tag{3.9}$$

where $[\,\cdot\,]$ denotes the projection to the Levi subgroup.

Note that this is actually obvious when $\mathfrak{G} = \mathrm{GL}_n$.

Proof. Let $\Omega = \{x, t^{-1}x\} \subset A$. By Lemma 3.3, $\mathfrak{G}(\mathcal{O}) \cap t^{-1}\mathfrak{G}(\mathcal{O})t$ is the fixer of Ω in $G(K)^1$; thus it equals $\widehat{N}^1_\Omega U_\Omega$ by I.7.4.4 and II.4.6.26.

Let us first show that $\nu(\widehat{N}^1_\Omega)$ is naturally isomorphic to ${}^vW_\lambda = \{w \in {}^vW : w\lambda = \lambda\}$. As x is special, $\nu(\widehat{N}^1_x)$ is isomorphic to ${}^vW$ via the forgetful map $\mathrm{Aff}(A) \to \mathrm{GL}(V)$ (see I.6.2.10). Suppose $n \in \widehat{N}^1_x$ and let $w = {}^v\nu(n)$. Then $n \in \widehat{N}^1_\Omega$ if and only if w fixes $x - t^{-1}x \in V$. By II.4.2.7(3), $\langle x - t^{-1}x, a\rangle = -\mathrm{ord}_K(a(t))$ for $a \in \Phi$. So w fixes $x - t^{-1}x$ if and only if $\lambda - w\lambda \in \langle\Phi\rangle^{\perp}$. But for all $w \in {}^vW$, $\lambda - w\lambda \in X^*_K(G)^{\perp}$. Thus $\lambda - w\lambda \in \langle\Phi\rangle^{\perp}$ is equivalent to $\lambda = w\lambda$.

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Next, we show that $\operatorname{red}(\widehat{N}^1_\Omega)$ equals the $\overline{K}$-points of the normaliser of $\overline{\mathfrak{S}}$ in L_λ. Note that $\widehat{N}^1_\Omega \supset G(K)^1 \cap \ker \nu = \mathfrak{T}(\mathcal{O})$ (by II.4.6.3(3)). Also, $\widehat{N}^1_\Omega \subset N(K) \cap \mathfrak{G}(\mathcal{O})$. If $n \in N(K) \cap \mathfrak{G}(\mathcal{O})$, then by considering the $\mathcal{O}^{\mathrm{nr}}$-points of $\mathfrak{S}$ we see that $\operatorname{red}(n) \in N(\overline{\mathfrak{S}})$ and that $\operatorname{red}(n)$ induces the same Weyl element on $X_*(\overline{\mathfrak{S}})$ as n on $X_*(\underline{S})$. From the previous paragraph, $\operatorname{red}(\widehat{N}^1_\Omega)/\overline{\mathfrak{T}}(\overline{K}) \cong {}^vW_\lambda$, which is precisely the Weyl group of $\overline{\mathfrak{S}}$ in L_λ.

To determine $\operatorname{red}(U_\Omega)$, let us compute $f_\Omega : \Phi \to \mathbb{R}$. By II.4.6.26, for $a \in \Phi$ we have

$$\begin{aligned} f_\Omega(a) &= f_x(a) + \max(a(x - t^{-1}x), 0) \\ &= f_x(a) + \max(-\langle a, \lambda\rangle, 0). \end{aligned}$$

As x and its translate $t^{-1}x$ are hyperspecial, $f'_\Omega = f_\Omega$ and $f'_x = f_x$ by Lemma 3.7.

Recall that $U_\Omega = \langle U_{f_\Omega,a}\rangle_{a\in\Phi^{\mathrm{red}}}$ (from II.4.6.3). Let us show that $\operatorname{red}(U_{f_\Omega,a})$ is trivial if $\langle a, \lambda\rangle < 0$ and equals $\overline{\mathfrak{U}}_{x,a}(\overline{K})$ otherwise. Note that $f^*_x(a) = f_x(a)+ \in \widetilde{\mathbb{R}}$ for any $a \in \Phi$, in the notation of II.4.6.9. If $\langle a, \lambda\rangle < 0$, then $f_\Omega(b) > f^*_x(b)$ for $b \in \{a, 2a\} \cap \Phi$, so that $U_{f_\Omega,a} \subset U_{f^*_x,a}$ and $\operatorname{red}(U_{f_\Omega,a}) = \{1\}$ as $\overline{\mathfrak{G}}^0_x$ is reductive (see II.4.6.10(ii)). Otherwise, $U_{f_\Omega,a} = U_{f_x,a} = \mathfrak{U}_{x,a}(\mathcal{O})$ so that $\operatorname{red}(U_{f_\Omega,a}) = \overline{\mathfrak{U}}_{x,a}(\overline{K})$.

Putting this all together, we see that $\operatorname{red}(\mathfrak{G}(\mathcal{O}) \cap t^{-1}\mathfrak{G}(\mathcal{O})t) = P_\lambda(\overline{K})$ by the rational Bruhat decomposition [Bor91, 21.15] applied to $L_\lambda(\overline{K})$.

To prove the final assertion, note first that $tU_{f_\Omega,a}t^{-1} = U_{f_{\Omega'},a}$ where $\Omega' = \{x, tx\}$. We show that the left-hand side of (3.9) is contained in the right-hand side. It suffices to show that t centralises $\widehat{N}^1_\Omega$ and $U_{f_\Omega,a}$ whenever $a \in \Phi^{\mathrm{red}}$ and $\langle a, \lambda\rangle = 0$. If $n \in \widehat{N}^1_\Omega$ with ${}^v\nu(n) = w$, then $ntn^{-1} = n\lambda(\varpi)n^{-1} = (w\lambda)(\varpi) = \lambda(\varpi) = t$ by the above. It is a standard fact that $\operatorname{im}(\lambda)$ centralises $U_a \supset U_{f_\Omega,a}$ if $\langle a, \lambda\rangle = 0$ (see, e.g., [Spr98, 15.4.4]).

To prove the opposite containment in (3.9), it is enough to show that the left-hand side contains $(g_+, 1)$ for all $g_+ \in U_\lambda(\overline{K})$. But this is clear since we have shown above that $\operatorname{red}(U_{f_\Omega,a}) = \overline{\mathfrak{U}}_{x,a}(\overline{K})$ and $\operatorname{red}(U_{f'_\Omega,a}) = \{1\}$ if $a \in \Phi^{\mathrm{red}}$ and $\langle a, \lambda\rangle > 0$. □

Lemma 3.10 (Iwahori decomposition). *Let $I \subset \mathfrak{G}(\mathcal{O})$ be the inverse image of $\overline{\mathfrak{T}}(\overline{K})\overline{\mathfrak{U}}^+(\overline{K})$ under the reduction map. Then I is an Iwahori subgroup and the product map $(I \cap U^-(K)) \times (I \cap T(K)) \times (I \cap U^+(K)) \to I$ is a bijection, for any chosen order of the factors. Moreover, T^- (see Definition 1.1) contracts $I \cap U^-(K)$ and expands $I \cap U^+(K)$.*

Proof. By II.4.6.33, there is a chamber $C \subset A$ with $x \in \overline{C}$ such that $I = \mathfrak{G}^0_C(\mathcal{O})$. Thus I is an Iwahori subgroup. We will use the notation of II.4.6.3. By II.4.6.7(i), $I = P^0_f = H^0U_f$ where $f = f'_C$. Also, $H^0 = \mathfrak{T}(\mathcal{O})$ as $\mathfrak{T}$ is connected. Note that $N_f \subset U_f \subset G(K)^1$. Since C is not contained in any walls, $N_f \leqslant H$ (see the proof of Lemma 3.3). Thus $N_f \subset G(K)^1 \cap H = \mathfrak{T}(\mathcal{O})$.

From I.6.4.9(iii), $I = H^0U_f = \mathfrak{T}(\mathcal{O})U^+_fU^-_f$. Note that $\mathfrak{T}(\mathcal{O}) \subset H = \ker\nu$ normalises each $U_{a,k}$ and therefore $U^\pm_f$: this follows from the definitions in I.6.2. The product map is injective since $U^- \times T \times U^+ \to G$ is an open immersion (the big cell). For the final claim, note that U^-_f is generated by the $U_{a,f(a)}$, $a \in \Phi^-$, and that $tU_{a,k}t^{-1} \subset U_{a,k}$ for $t \in T^-$ and $a \in \Phi^-$ (see II.4.2.7(2)). □

Proof of Lemma 2.7. (i) Let $\Psi = \{b \in \Phi^+ : b \notin \mathbb{Z}a\} \subset \Phi$. Since a is simple, Ψ is closed. Thus $\prod_{b\in\Phi^+_{\mathrm{nd}}, b\neq a} \mathfrak{U}_{x,b} \to \mathfrak{U}^+$ is an isomorphism (as $\mathcal{O}$-schemes) onto a closed subgroup scheme $\mathfrak{U}'$ of $\mathfrak{U}^+$ (by II.4.6.2). Now, $\mathfrak{U}'$ being normal in $\mathfrak{U}^+$ means that the conjugation map $\mathfrak{U}^+ \times \mathfrak{U}' \to \mathfrak{U}^+$ factors through $\mathfrak{U}'$, which can be checked on the generic fibre owing to the $\mathcal{O}$-flatness of $\mathfrak{U}^+ \times \mathfrak{U}'$ (see II.1.2.5). But there it is clear from $[U_b, U_c] \subset \langle U_{rb+sc} : r, s > 0\rangle$ (condition (DR2) in I.6.1.1).

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The product map $\mathfrak{U}_{x,a} \times \mathfrak{U}' \to \mathfrak{U}^+$ is an isomorphism of $\mathcal{O}$-schemes (II.4.6.2). As $\mathfrak{U}'$ is normal, it is an isomorphism of $\mathcal{O}$-group schemes $\mathfrak{U}_{x,a} \ltimes \mathfrak{U}' \to \mathfrak{U}^+$.

(iii) First, note that $\Gamma'_b = \mathbb{Z}$ for all $b \in \Phi$. This is clear when G is split (II.4.2.1), and the general case follows either by étale descent (II.5.1.19) or by quasi-split descent (II.4.2.21).

By II.4.5.1, $\mathfrak{U}_{x,a}(\mathcal{O}) = U_{a,f_x(a)} U_{2a,f_x(2a)}$, and this equals $U_{a,f_x(a)}$ as $f_x(2a) = 2f_x(a)$ (see condition (V4) in I.6.2.1). Next, from $\varphi_a(tut^{-1}) = \varphi_a(u) + (\mathrm{ord}_K \circ a)(t)$ (see II.4.2.7(2)) it follows that $tU_{a,f_x(a)}t^{-1} = U_{a,k}$ where $k = f_x(a) + \langle \lambda, a \rangle$. Let $l = f_x(a) + 1$ so that $k, l \in \Gamma'_a$ and $k > l$. Then $\mathrm{red}(U_{a,l}) = \{1\}$ since $f_x^*(a) = f_x(a) + \in \widetilde{\mathbb{R}}$, $\overline{\mathfrak{G}}_x^0$ is reductive, and $l > f_x(a)$ (by II.4.6.10(ii)).

Suppose first that $2a \notin \Phi$. Then $U_a(K)$ is abelian and

$$\sum_{U_a(K)/U_{a,k}} \psi(u_a) = \sum_{U_{a,l} \backslash U_a(K)} \ \sum_{U_{a,k} \backslash U_{a,l}} \psi(u_2 u_1).$$

We claim that $U_{a,k} \subset U_{a,l}$ is a proper subgroup of p-power index. This will finish the proof, since ψ is left $U_{a,l}$-invariant and the codomain A of ψ has exponent p. Since $k, l \in \Gamma'_a$ and $k > l$, it follows that $U_{a,k} \subsetneq U_{a,l}$. From II.4.3.2 we see that $U_a(K)$ is isomorphic to the additive group of a finite (unramified) extension L of K. Under this isomorphism, for any $r \in \Gamma_a$, $U_{a,r}$ corresponds to the $\mathcal{O}_L$-lattice $\{x \in L : \mathrm{ord}_K(x) \geqslant r\}$. Thus the index $[U_{a,l} : U_{a,k}]$ is a power of p.

Now suppose that $2a \in \Phi$. We know that $U_{2a}(K)$ is central in $U_a(K)$ with abelian quotient (condition (DR2) in I.6.1.1). Moreover, from the definitions, $U_{2a,2r} = U_{2a}(K) \cap U_{a,r}$ for all $r \in \mathbb{R}$. Note that

$$\sum_{U_a(K)/U_{a,k}} \psi(u_a) = \sum_{U_a(K)/U_{a,k}U_{2a}(K)} \psi'(u'_a) \tag{3.11}$$

where $\psi'(u'_a) = \sum_{U_{a,k}U_{2a}(K)/U_{a,k}} \psi(u'_a u)$, which is left invariant by $U_{a,l}$. Since $U_a(K)/U_{2a}(K)$ is abelian, left and right cosets of $U_{a,k}U_{2a}(K)$ in $U_a(K)$ coincide and we can rewrite (3.11) as

$$\sum_{U_{a,l}U_{2a}(K) \backslash U_a(K)} \ \sum_{U_{a,k}U_{2a}(K) \backslash U_{a,l}U_{2a}(K)} \psi'(u_2 u_1).$$

We claim that $U_{a,k}U_{2a}(K) \subset U_{a,l}U_{2a}(K)$ is a proper subgroup of p-power index. As in the previous case, this will finish the proof.

To see that $U_{a,k}U_{2a}(K) \subsetneq U_{a,l}U_{2a}(K)$, we show that $U_{a,k}U_{2a,2l} \subsetneq U_{a,l}$. Since $l \in \Gamma'_a$, we may pick $u \in U_a(K)$ such that $\varphi_a(u) = l$ and $\varphi_a(u) = \max \varphi_a(uU_{2a}(K))$. It follows that $u \in U_{a,l} - U_{a,k}U_{2a,2l}$. The index of $U_{a,k}U_{2a}(K)$ in $U_{a,l}U_{2a}(K)$ equals the index of $U_{a,k}/U_{2a,2k}$ in $U_{a,l}/U_{2a,2l}$. The group $U_a(K)/U_{2a}(K)$ is isomorphic to the additive group of a finite-dimensional K-vector space, and for any $r \in \Gamma'_a$, $U_{a,r}/U_{2a,2r}$ corresponds to an $\mathcal{O}$-lattice under this isomorphism (by II.4.3.7 and II.4.3.5 with $k = r, l = 2r \in \Gamma'_{2a}$). Thus the index of $U_{a,k}U_{2a}(K)$ in $U_{a,l}U_{2a}(K)$ is a p-power. □

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