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GENERIC DIFFERENTIABILITY OF ORDER-BOUNDED CONVEX OPERATORS

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Abstract

We give sufficient conditions for order-bounded convex operators to be generically differentiable (Gâteaux or Fréchet). When the range space is a countably order-complete Banach lattice, these conditions are also necessary. In particular, every order-bounded convex operator from an Asplund space into such a lattice is generically Fréchet differentiable, if and only if the lattice has weakly-compact order intervals, if and only if the lattice has strongly-exposed order intervals. Applications are given which indicate how such results relate to optimization theory.

1. Introduction

Convex analysis plays a central role in the study of optimality conditions and in non-linear analysis. Vector-valued convex operators occur naturally in a variety of settings. This was illustrated in [1], [2] and we give further examples in Section 4 below. There has also been considerable interest in the differentiability properties of non-linear operators, both for theoretical and applied reasons. If derivatives are known to exist sufficiently often (almost everywhere or on a dense G_δ subset) then one can often reduce the problem being studied to a more tractable differentiable problem. Moreover, convex operators are the most accessible class of non-linear operators, and as such demand study even if one is more directly interested in other, say Lipschitz, operators.

In our previous papers [1], [2] we studied the existence of subgradients for continuous convex operators, and gave various results on the generic differentiability of continuous convex operators. Kirov [4], [5] has continued this study,

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primarily by the use of generalized monotone operators. In [5] he observes that much more can be said if the operators are required to be order-bounded rather than merely convex. In this paper, we adapt the techniques of [1] and [3] to establish differentiability results for order-bounded convex operators between ordered Banach spaces. We also show that when the range space is an order-complete Banach lattice, our conditions are both necessary and sufficient. These results considerably extend various theorems given in [5].

We commence by recalling necessary facts and notations. The reader is directed to [1] and [7] for further details. For simplicity we restrict ourselves to Banach space. Let X be a Banach space and let Y be a (partially) ordered Banach space with closed *normal* positive cone S. We denote the induced order by $\leqslant$ or $\leqslant_s$. (Recall that S is normal if and only if there is an equivalent renorming with $0 \leqslant_s y \leqslant_s x$ implying $\|y\| \leqslant \|x\|$.) As elsewhere we adjoin an abstract "∞" to Y and S and consider mappings f between X and $Y \cup \{\infty\}$, written $\dot{Y}$. Then $f: X \to \dot{Y}$ is (S-) *convex* if for $0 \leqslant t \leqslant 1$ and x_1, x_2 in $\operatorname{dom} f := \{x \in X: f(x) \in Y\}$ one has

$$f(tx_1 + (1-t)x_2) \leqslant_s tf(x_1) + (1-t)f(x_2). \tag{1.1}$$

We will say that f is *order-bounded at* $\bar{x}$ *in* $\operatorname{dom} f$ if one can find a neighbourhood N of zero and some $y \in Y$ such that

$$\bar{x} + N \subset \{x \in X: f(x) \leqslant_s y\}. \tag{1.2}$$

Obviously such an $\bar{x}$ lies in int(dom f). Moreover, when f is convex and order-bounded at some $\bar{x}$, it is actually order-bounded throughout int(dom f). We will call such a mapping (*locally*) *order-bounded*. Since the cone is normal, order-bounded convex maps are continuous; but the converse obtains only when int(S) is non-empty. In general, even such nice convex mappings as the absolute value on a Banach lattice are not order-bounded.

Let us also recall that the cone S is *Daniell* if every positive decreasing net converges. When Y is a Banach lattice this is equivalent to the norm being order-continuous, [7, Theorem 5.11]. We make one new definition. We will say that an order interval $[0, x] := \{y \in Y: 0 \leqslant_s y \leqslant_s x\}$ is *strongly exposed* (*by* ϕ in $[0, x]^+ := \{g \in Y^*: g(y) \geqslant 0$ for $y \in [0, x]\}$) if, for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 \leqslant_s y \leqslant_s x \text{ and } \phi(y) \leqslant \delta \text{ implies } \|y\| < \varepsilon. \tag{1.3}$$

If we may only assert that

$$0 \leqslant_s y \leqslant_s x \text{ and } \phi(y) = 0 \text{ implies } y = 0 \tag{1.4}$$

we say that the interval is *exposed* (*by* ϕ).

Finally, a Banach space X is an *Asplund space*, respectively a *weak Asplund space*, if every extended real-valued convex function on X is *generically* Fréchet, respectively Gâteaux, differentiable throughout the interior of its domain. (A set is generic if it contains a dense G_δ.) Asplund spaces include reflexive spaces and separable dual spaces; weak Asplund spaces include all weakly compact generated spaces and so all separable spaces. (See [1] and references therein.)

2. Sufficient conditions for generic differentiablity

Our central result is:

THEOREM 2.1. *Let X be a Banach space, let Y be an ordered Banach space whose cone S is closed and normal, and let f: $X \to \dot{Y}$ be order-bounded and S-convex. Suppose S is Daniell.*

a] *If X is an Asplund space and order intervals in Y are strongly exposed, then f is generically Fréchet differentiable throughout the interior of its domain.*

b] *If X is a weak Asplund space and order intervals in Y are exposed, then f is generically Gâteaux differentiable throughout the interior of its domain.*

PROOF. Let $\bar{x}$ in $\operatorname{int}(\operatorname{dom} f)$ be given. Select y in Y and a ball N around zero such that (1.2) holds.

Let $x \in \bar{x} + N$. Then, as f is convex,

$$y - f(\bar{x}) \geqslant f(\bar{x} + x) - f(\bar{x}) \geqslant f(\bar{x}) - f(\bar{x} - x) \geqslant f(\bar{x}) - y,$$

and $f(\bar{x} + N)$ lies in an order interval, $[a, b]$. Again by convexity, for x in $\bar{x} + \frac{1}{2}N$ and h in $\frac{1}{2}N$ we have

$$f(x) - f(x - h) \leqslant \frac{f(x + th) - f(x)}{t} \leqslant \frac{f(x + sh) - f(x)}{s}$$

for $0 < t \leqslant s \leqslant 1$. Since $f(x) - f(x - h) \geqslant a - b$, and as S is Daniell, the directional minorant

$$\nabla f(x; h) := \inf_{t > 0} \frac{f(x + th) - f(x)}{t}$$

exists for x in $\bar{x} + \frac{1}{2}N$ and h in X. Moreover, $\nabla f(x; \cdot)$ is convex and finite and, again since S is Daniell,

$$\nabla f(x; h) = \lim_{t \downarrow 0} \frac{f(x + th) - f(x)}{t}. \tag{2.1}$$

a] Now, let ϕ strongly expose $[0, b - a]$. Since f is S-convex with $f(\bar{x} + N) \subset [a, b]$ while $\phi \in [0, b - a]^+$, ϕf is convex on $\bar{x} + N$. Since X is Asplund, there is a dense G_δ subset, G, in $\bar{x} + N$ such that ϕf is Fréchet differentiable at points of G. We show (much as in [1]) that f is actually Fréchet differentiable on G. Let x lie in G. First observe that, for $0 < t < 1$

$$0 \leqslant \frac{f(x + th) - f(x)}{t} - \nabla f(x; h) \leqslant 2(b - a), \tag{2.2}$$

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for $x \in \bar{x} + \frac{1}{2}N$ and $h \in \frac{1}{2}N$. Also $\nabla\phi f(x; h) = \phi\nabla f(x; h)$ and, as $\nabla\phi f(x; \cdot)$ is linear, we have

$$0 \leqslant \nabla f(x; h) + \nabla f(x; -h) \leqslant 2(b - a) \tag{2.3}$$

and

$$\phi(\nabla f(x; h) + \nabla f(x; -h)) = 0. \tag{2.4}$$

Since ϕ exposes $[0, b - a]$, (2.3) and (2.4) show that $\nabla f(x; \cdot)$ is linear, being both sublinear and homogeneous. This, in conjunction with (2.1), shows that f is linearly Gâteaux differentiable at x. To complete the argument let $\varepsilon > 0$ be given and choose $\delta > 0$ to satisfy (1.3) with $x := 2(b - a)$. Then, as ϕf is Fréchet at x, we may find $\gamma > 0$ so that when h lies in $\frac{1}{2}N$

$$\frac{\phi f(x + th) - \phi f(x)}{t} - \nabla\phi f(x; h) \leqslant \delta$$

for $0 < t < \gamma$. Since (2.2) holds, we have

$$\left\| \frac{f(x + th) - f(x)}{t} - \nabla f(x; h) \right\| \leqslant \varepsilon$$

if $0 < t < \gamma$ and $h \in \frac{1}{2}N$. As $\nabla f(x; \cdot)$ is linear and continuous we are done.

b] This follows as in the first part of the previous proof.

Conditions for a cone to be Daniell were discussed in detail in [1]. Conditions for exposed intervals are as follows:

PROPOSITION 2.1. *Let Y be a Banach space partially ordered by a normal closed cone S.*

a] *Order intervals in Y are exposed if*

(i) *S has separable order intervals*; *or* (ii) *S has a base*; *or* (iii) *Y has an equivalent strictly convex renorm which is S-monotone* ($0 \leqslant y \leqslant x$ *implies* $\|y\| \leqslant \|x\|$).

b] *Order intervals in Y are strongly exposed if*

(i) *S has norm compact intervals*; *or* (ii) *S has a bounded base*; *or*

(iii) *Y has equivalent locally uniformly convex renorm which is S-monotone.*

PROOF. Let x in S be fixed with $x \neq 0$.

a] (i) The cone generated by the order interval $[0, x]$ is separable and so has a base, B, [1] and as the space is locally convex we may separate 0 and B to produce an exposing functional. This also establishes (ii). In case (iii) we argue that the unique tangent, ϕ, to the renormed strictly convex ball $N := \{y \in Y: \|y\| \leqslant \|x\|\}$ exposes x in N and, by monotonicity, exposes x in $[0, x]$. But then ϕ exposes $[0, x]$ as well.

b] (i) Since $[0, x]$ is exposed by a] (i) and compact (every sequence has a convergent subsequence) it is strongly exposed; indeed, otherwise we have $\varepsilon > 0$ and $\phi(x_n)$ tending to 0 for $\|x_n\| \geqslant \varepsilon$ and $0 \leqslant x_n \leqslant x$. Since (x_n) has a convergent

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subsequence in norm, this is impossible. (ii) was established in [1]. (iii) Now ϕ strongly exposes the renormed locally uniformly convex ball at x and so strongly exposed 0 in $[0, x]$.

If the domain is not Asplund or weakly Asplund, or if the operator is not order-bounded, the examples given in [1] show that Theorem 2.1 will generally fail.

We continue by studying the case in which Y is a lattice.

3. Lattice characterizations

We suppose now that Y is a Banach lattice (a complete normed vector lattice whose norm satisfies $\|y\| \leqslant \|x\|$ whenever $|y| \leqslant |x|$). The key result is:

PROPOSITION 3.1. *Let Y be a Banach lattice. Then the following are equivalent:*
i] *Y has a lattice equivalent locally uniformly convex Banach lattice renorming.*
ii] *Order intervals in Y are strongly exposed.*
iii] *Order intervals in Y are weakly compact.*
iv] *The lattice cone in Y is Daniell.*

PROOF. i] $\Rightarrow$ ii]. Since strong exposure is preserved by lattice isomorphisms, this follows from b] (iii) of Proposition 2.1. ii] $\Rightarrow$ iii]. If Y possesses a non-weakly compact order interval then one can construct a lattice orthogonal norm one sequence (x_n) in Y with $0 \leqslant x_n \leqslant x_0$ for all n, [7, p. 94]. Now

$$s_n := \sum_{k=1}^{n} x_k = \bigvee_{k=1}^{n} x_k \leqslant x_0 \quad \text{since} \quad x_k \wedge x_j = 0 \text{ for } k \neq j.$$

Hence, for any positive ϕ in Y^*, $(\phi(s_n))$ is isotone and majorized. Thus $\phi(x_n)$ tends to zero. Since Y is a Banach lattice (x_n) is weakly convergent to 0. This certainly means that $[0, x_0]$ is not strongly exposed, as each x_n is norm one. iii] $\Rightarrow$ iv]. This implication holds for any partial order [1]. iv] $\Rightarrow$ i]. Since Y is a Daniell Banach lattice, Y is order continuous and we apply the Davis-Ghoussoub-Lindenstrauss renorming theorem [3] to complete the hard step. (The theorem guarantees a locally uniformly convex lattice equivalent renorm for an order continuous Banach lattice.)

As observed in [6, p. 28], it is also equivalent to assume that Y has a lattice-equivalent Kadec norm. Note also that every σ-finite $L_\infty(\mu, E)$ has a lattice-equivalent strictly-convex lattice renorming. Simply let $E := \bigcup_{n=1}^{\infty} E_n$ where $\mu(E_n) \leqslant 1$, and let $\|\cdot\|$ be given by $\|f\| := \|f\|_\infty + \Sigma_{n=1}^{\infty} 2^{-n} \|f \,|\, E_n\|_2$.

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Also, in $L_p(\mu)$, $1 \leqslant p < \infty$, (with the standard ordering), it is easy to exhibit the strongly exposing functional for $[0, \bar{x}]$. We have $\phi := \bar{x}^{p-1} \in L_q(\mu)$ $(q + p = pq)$ and $0 \leqslant y \leqslant \bar{x}$ implies $\phi(y) = \int \bar{x}^{p-1} y \, d\mu \geqslant \|y\|^p$.

THEOREM 3.1. *Let Y be a countably order-complete Banach lattice. Then the following are equivalent.*

i] *Order intervals in Y are strongly exposed.*

ii] *Order intervals in Y are weakly compact.*

iii] *Suppose that f: $X \to \dot{Y}$ is convex and order-bounded while X is an Asplund space. Then f is generically Fréchet differentiable.*

iv] *Suppose that f: $X \to \dot{Y}$ is convex and order-bounded while X is a weak Asplund space. Then f is generically Gâteaux differentiable.*

v] *Suppose that f: $\mathbf{R} \to Y$ is convex and order-bounded. Then f is generically Gâteaux differentiable.*

vi] *Y contains no Banach sub-lattice isomorphic to $l_\infty(\mathbf{N})$.*

PROOF. i] $\Leftrightarrow$ ii] follows from Proposition 3.1. ii] $\Leftrightarrow$ iii]. Since the cone is normal and Daniell, Theorem 2.1 a] now applies. ii] $\Leftrightarrow$ iv] follows similarly from part b] of the theorem. Clearly iii] implies v] and iv] implies v]. To complete the circle we establish that v] implies vi] and vi] implies ii]. v] $\Rightarrow$ vi]. Suppose that Y contains a lattice copy of $1_\infty(\mathbf{N})$. There is no loss in assuming $Y = l_\infty(\mathbf{N})$. Then let $\{r_n: n \in \mathbf{N}\}$ be chosen dense in $[-1, 1]$. Let f: $\mathbf{R} \to l_\infty(\mathbf{N})$ be defined (as in [4]) by

$$f(r) := \sup_{n \in \mathbf{N}} |r - r_n|.$$

Clearly, f is convex and order-bounded. Moreover, if $|r| < 1$, f is not Gâteaux differentiable at r. Indeed, since $\{r_n: n \in \mathbf{N}\}$ is dense in $[-1, 1]$ we may calculate that

$$\limsup_{\varepsilon \to 0^+} \frac{f(r + \varepsilon) + f(r - \varepsilon) - 2f(r)}{\varepsilon} = 2,$$

and so f is nowhere Gâteaux differentiable on $(-1, 1)$. (Note that, nonetheless, f has a unique linear subgradient whenever $r \notin \{r_n: n \in \mathbf{N}\}$.)

vi] $\Rightarrow$ ii]. Since Y is countably order-complete this follows from [7, Theorem 5.14].

The equivalences fail if Y is not countably order-complete. Indeed, $f(x) := |x|$ on $X := Y := C[0, 1]$ is nowhere Gâteaux differentiable on $N := \{x \in X: \|x - \bar{x}\| < \frac{1}{2}\}$ where $\bar{x}(t) := 1 - 2t$ for $0 \leqslant t \leqslant 1$, [1]. This is not entirely obvious, but follows after some routine but tedious calculations.

Kirov's Corollaries in [5] regarding Fréchet differentiability or order-bounded convex operators (established by entirely different methods) are all special cases of Theorem 3.1, sometimes with redundant hypotheses. He requires X to be a reflexive Banach space and Y to be a Banach lattice such that either a) intervals are norm compact, or b) intervals in Y *and* Y^* are weakly compact, or c) intervals in Y are weakly compact and f has only compact subgradients.

4. Applications

a] We consider the following vector convex program (VCP):

$$h(u) := \inf_s f(x) \text{ subject to } g(x) \leqslant_k u. \tag{4.1}$$

We assume that $f\colon X \to \dot{Y}$ is S-convex and that $g\colon X \to \dot{U}$ is K-convex. We suppose that int K is non-empty and that Slater's condition holds: there exists $\hat{x}$ in dom f with $g(\hat{x}) \in -\operatorname{int} K$. We also suppose that (Y, S) is a Banach lattice with weakly compact order intervals, and so is order-complete.

Then, as in [1], [2], h defines another S-convex mapping; which is actually locally order-bounded as a consequence of Slater's condition. (More general constraint qualifications ensure continuity but not order-boundedness.) Thus, if we assume that $h(0)$ is finite, h is order-bounded and convex on a neighbourhood of zero. In particular, Theorems 2.1 and 3.1 apply to h and give conditions for h to be generically differentiable. As explained in [1], if h is differentiable at u with Gâteaux derivative T, then $-T$ is the unique Lagrange multiplier for (VCP). In fact, if h is Fréchet differentiable at u we may conclude that the subgradient of h is norm-to-norm upper semi-continuous at u, [1].

b] Suppose now that $f := A$ and $g := B$ are continuous linear mappings. Then (VCP) becomes a form of the abstract Farkas lemma. Such inequality systems are central to the study of positive operators [7].

As outlined in a] the differentiability points of $h(u) := \inf_s\{Ax | Bx \leqslant_k u\}$ correspond to unique Lagrange multipliers. In this case $T = \nabla h(u)$ if and only if T is the unique linear operator solution to

$$Tv \leqslant_s h(v), \qquad \forall v \in U \tag{4.2}$$

and

$$Tu = h(u). \tag{4.3}$$

This in turn means that T is the unique solution in $L(U, Y)$ to

$$TB = A,\ T(K) \subset -S,\ Tu = h(u). \tag{4.4}$$

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c] Let f: $X \times T \in \mathbf{R}$ be convex in $x \in X$ and measurable in $t \in T$. Suppose that X is an Asplund space and that one can find $k \in L_p(T)$ $(1 \leqslant p < \infty)$ such that

$$f(x,t) \leqslant k(t) \tag{4.5}$$

if $\|x - x_0\| < \varepsilon$, for some $\varepsilon > 0$, $x_0 \in X$. We define a convex operator F: $X \to L_p(T)$ by $F(x)(t) := f(x,t)$. Then (4.5) guarantees that F is locally order-bounded. Theorem 3.1 applies and we may conclude that generically F is Fréchet differentiable.

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