ON TORSION-FREE HYPERCENTRAL GROUPS WITH ALL SUBGROUPS SUBNORMAL

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There is no example known of a non-nilpotent, torsion-free group which has all of its subgroups subnormal. It was proved in [**3**] that a torsion-free solvable group with all of its proper subgroups subnormal and nilpotent is itself nilpotent, but that seems to be the only published result in this area which is concerned specifically with torsion-free groups. Possibly the extra hypothesis that the group be hypercentral is sufficient to ensure nilpotency, though this is certainly not the case for groups with torsion, as was shown in [**7**]. The groups exhibited in that paper were seen to have hypercentral length $\omega + 1$, and we know from [**8**] that further restricting the hypercentral length can lead to some positive results. Here we shall prove the following theorem.

THEOREM. *A torsion-free, hypercentral group of length at most ω in which all subgroups are subnormal is nilpotent.*

Proof. Let G be a group which satisfies the hypotheses of the theorem. Then G is solvable [**1**] and we suppose that G is of minimal derived length subject to being non-nilpotent. Then G' is nilpotent; so the isolator I of G' in G is also nilpotent [**4**, Lemma 6.33]. Let T/I' be the torsion subgroup of I/I'. If G/T is nilpotent then G acts nilpotently on I/T and hence on I (see Proposition 3 in Chapter 3 of [**6**] for the case where I is finitely generated, the proof in the general case is identical). It follows that G is nilpotent, a contradiction. By minimality, G is therefore metabelian; so there is a normal abelian subgroup A of G such that G/A is abelian. Now, for any integer n, a torsion-free solvable n-Engel group is nilpotent [**4**, Theorem 7.36], and we use this fact to construct a sequence of pairs of elements (a_i, x_i) as follows.

Choose $a_1 \in A$, $x_1 \in G$ such that $[a_1, x_1] \neq 1$. Then, for some integer n_1, $x_1 \in Z_{n_1} = Z_{n_1}(G)$. Now choose $b_2 \in A$, $x_2 \in G$ such that $[b_{2,n_1+1}x_2] \neq 1$ and write $a_2 = [b_{2,n_1-1}x_2]$. So $a_2 \in G_{n_1} = \gamma_{n_1}(G)$, and $[a_{2,2}x_2] \neq 1$. Let n_2 be an integer such that $x_2 \in Z_{n_2}$. Then, since $[G_{n_1}, AZ_{n_1}] = 1$, we see that no non-zero power of x_2 belongs to AZ_{n_1} [**2**, Lemma 4.8]. For any integer $i > 1$, suppose elements $a_1, \ldots, a_i$ of A and $x_1, \ldots, x_i$ of G and integers $n_1, \ldots, n_i$ have been chosen such that, for each $j = 2, \ldots, i$, $[a_j, {}_jx_j] \neq 1$, $a_j \in G_{n_{j-1}}$ and $x_j \in Z_{n_j}$. Now let $a_{i+1} \in G_{n_i}$, $x_{i+1} \in G$ be such that $[a_{i+1,i+1}x_{i+1}] \neq 1$, and choose n_{i+1} such that $x_{i+1} \in Z_{n_{i+1}}$. Then x_{i+1} has infinite order modulo AZ_{n_i}, since $[G_{n_i}, AZ_{n_i}] = 1$. In particular, $n_i < n_{i+1}$ for each $i \geqslant 1$. Let $H = \langle a_i, x_i : i = 1, 2, \ldots \rangle$. Then H is non-nilpotent and, for each i and for all $j > i$, $[a_j, x_i] = 1$ (since $a_j \in G_{n_{j-1}} \leqslant G_{n_i}$). Further, the elements x_iA, $i = 1, 2, \ldots$ form a basis for the free abelian group XA/A, where $X = \langle x_1, x_2, \ldots \rangle$. For each $i = 1, 2, \ldots$, write $H_i = \langle a_1, \ldots, a_i, x_1, \ldots, x_i \rangle$ and $B_i = H_i \cap A$. Then, for any positive integer l_i, $H_i/B_i^{l_i}$ is finitely generated finite-by-abelian and thus central-by-finite (see [**4**, Theorem 4.32], for example). So there is a non-trivial element y_i of $\langle x_i \rangle$ such that $[B_i, \langle y_i \rangle] \leqslant B_i^{l_i}$ and $[\langle x_j \rangle, \langle y_i \rangle] \leqslant B_i^{l_i}$ for all $j < i$ (where $x_0 = 1$). We choose the integers l_i

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as follows: Let $l_1=2$ and suppose that, for some $i\geq 1$, $l_1,\ldots,l_i$ have been chosen. Let k_{i+1} be a non-zero integer such that $B_{i+1}^{k_{i+1}}$ is torsion-free modulo $\langle B_1^{l_1},\ldots,B_i^{l_i}\rangle$ (such an integer exists since B_{i+1} is finitely generated). Set $l_{i+1}=2^{i+1}k_{i+1}$. Now write $k_1=1$ and $D=\langle B_i^{k_i}:i=1,2,\ldots\rangle$, $C=\langle B_i^{l_i}:i=1,2,\ldots\rangle$, $Y=\langle y_1,y_2,\ldots\rangle$. Then D is free abelian, by the choice of the integers k_i, and $C\leq D$. Let i, j be any positive integers. If $j<i$ then $y_j\in H_i$ and so $B_i^{\langle y_j\rangle}=B_i$, while if $j\geq i$ then $[B_i,\langle y_j\rangle]\leq[B_j,\langle y_j\rangle]\leq B_j^{l_j}\leq C$. It follows that D and C are each normalised by Y. Let $J=CY$. We shall show that J is residually of finite rank—it will follow by Theorem 2 of [**8**] that J is nilpotent. However, this implies that H is nilpotent since H is the isolator of J in H. This contradiction will complete the proof of the theorem. So suppose h is a non-trivial element of J. We must show that there is a normal subgroup N of J, not containing h, such that J/N has finite rank. Since Y' is contained in C, we see that J/C is abelian. Now any abelian group M is residually of finite rank, for if $a\in M$ and N is a maximal subgroup of M subject to $a\notin N$ then M/N must be a p-group, for some prime p, and consideration of basic subgroups (see [**5**, 4.3.12]) shows that M/N has rank 1. This means that we need consider only the case where h is an element of C. Since $C\leq D$, there is a positive integer n such that $h\notin D^{2^n}$. Let $E=D^{2^n}\cap C$, a normal subgroup of J. Then, for each $j\geq n$, $B_j^{l_j}\leq E$, and so E has finite index in C. Also, for each $j\geq n$, $Y_{i,j}=[\langle y_i\rangle,\langle y_j\rangle]\leq B_j^{l_j}$ if $i\leq j$, and $Y_{i,j}\leq B_i^{l_i}$ if $i>j$—in either case $Y_{i,j}\leq E$. Further, $[B_i^{l_i},\langle y_i\rangle]\leq E$ for all i. So, writing Y^* for the subgroup generated by $y_n,y_{n+1},\ldots$, we have $[J,Y^*]\leq E$. Let $N=E(Y^*)^J=EY^*$. Since the elements y_i $(i=1,2,\ldots)$ form a basis for Y modulo A, it follows that $h\notin N$. But J/N is finitely generated nilpotent and so has finite rank. This concludes the proof.

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