*Proceedings of the Edinburgh Mathematical Society* (2011) **54**, 467–504
DOI:10.1017/S0013091510000015

# MODULE STRUCTURE ON LIE POWERS AND NATURAL COALGEBRA-SPLIT SUB-HOPF ALGEBRAS OF TENSOR ALGEBRAS

J. Y. LI[1], F. C. LEI[2] AND J. WU[3]

[1] *Institute of Mathematics and Physics, Shijiazhuang Railway Institute, Shijiazhuang 050043, People's Republic of China* (yanjinglee@163.com)
[2] *School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, People's Republic of China* (fclei@dlut.edu.cn)
[3] *Department of Mathematics, National University of Singapore, Block S17, 10 Lower Kent Ridge Road, Singapore 119076* (matwuj@nus.edu.sg)

(Received 17 December 2009)

*Abstract* We investigate the functors from modules to modules that occur as the summands of tensor powers and the functors from modules to Hopf algebras that occur as natural coalgebra summands of tensor algebras. The main results provide some explicit natural coalgebra summands of tensor algebras. As a consequence, we obtain some decompositions of Lie powers over the general linear groups.

*Keywords:* tensor algebras; Lie powers; coalgebras; sub-Hopf algebras; $T_n$-projective functors; natural coalgebra decompositions

2010 *Mathematics subject classification:* Primary 55P35; 16T15; 17B60
Secondary 20C30; 55P65

## 1. Introduction

The algebraic question of determining functorial coalgebra decompositions of tensor algebras arose from homotopy theory as follows. The classical results of Cohen *et al*. [**6**] on the exponents of the homotopy groups of spheres and Moore spaces were obtained by studying decompositions of the loop spaces of Moore spaces. Decompositions of the loop space functor $\Omega$ from $p$-local simply connected co-$H$-spaces to spaces have been investigated in [**20**–**22**, **24**, **26**]. By means of decompositions of the loop space functor, one gets natural decompositions $\Omega X \simeq \bar{A}(X) \times \bar{B}(X)$ for some homotopy functors $\bar{A}$ and $\bar{B}$ on $p$-local simply connected co-$H$-spaces $X$. Such decompositions may lose some information for an individual space $X$ in the sense that the functor $\bar{A}$ may be indecomposable but the space $\bar{A}(X)$ may have further decompositions. However, functorial decompositions have the good property that one can freely change the co-$H$-spaces $X$ in the decomposition formulae because they are functorial. Also, there are examples of spaces $X$ such as the Hopf invariant one complexes in [**11**] with the property that the space $\bar{A}(X)$ is indecomposable for a certain functor $\bar{A}$. A fundamental question concerning functorial decompositions is how to determine the homology of the factors $\bar{A}(X)$ and $\bar{B}(X)$, which can be reduced to a purely algebraic question as follows.

Let $V$ be any module over a field $\boldsymbol{k} = \mathbb{Z}/p$ and let $T(V)$ be the tensor algebra on $V$. Then $T(V)$ becomes a Hopf algebra by taking $V$ to be primitive. By forgetting the algebra structure on $T(V)$, we have the functor $T$ from modules to coalgebras. Let

$$T(V) \cong A(V) \otimes B(V) \tag{1.1}$$

be any natural coalgebra decomposition of $T(V)$ for some functors $A$ and $B$ from (ungraded) modules to coalgebras. From [**22**, Theorem 1.3], the functors $A$ and $B$ can be canonically extended as functors from graded modules to graded coalgebras and the above decomposition formula holds for any graded module $V$. Then from [**21**, Theorem 1.1], the functors $A$ and $B$ induce functors $\bar{A}$ and $\bar{B}$ from co-$H$-spaces to spaces and a natural decomposition $\Omega X \simeq \bar{A}(X) \times \bar{B}(X)$ with the property that there exist filtrations on the mod $p$ homology $H_*(\bar{A}(X))$ and $H_*(\bar{B}(X))$ such that isomorphisms $E^0H_*(\bar{A}(X)) \cong A(\Sigma^{-1}\bar{H}_*(X))$ and $E^0H_*(\bar{B}(X)) \cong B(\Sigma^{-1}\bar{H}_*(X))$ are obtained on the associated graded modules, where $\Sigma^{-1}$ is the desuspension of a graded module. In short, any coalgebra decomposition of the functor $T$ as in formula (1.1) induces a natural decomposition of the loops on $p$-local simply connected co-$H$-spaces in which the homology of its factors can be determined by their corresponding algebraic functors.

The functors $A$ and $B$ in decomposition (1.1) are complementary to each other and so it suffices to understand one of them as a coalgebra summand of the functor $T$. There exist some important coalgebra summands of $T$ in [**22**, Theorem 6.5] that give a functorial version of the Poincaré–Birkhoff–Witt Theorem. One such functor is the functor $A^{\min}$, which is the smallest natural coalgebra summand of $T(V)$ containing $V$. The coalgebra complement of the functor $A^{\min}$, denoted by $B^{\max}$, has the property that $B^{\max}(V)$ can be chosen as a sub-Hopf algebra of $T(V)$. However, the determination of $A^{\min}(V)$ and $B^{\max}(V)$ seems beyond current technology. As a consequence, the homology of the geometric realizations $\bar{A}^{\min}(X)$ and $\bar{B}^{\max}(X)$ remains unknown. It is therefore important to find coalgebra summands $B$ of $T$ with the explicit information on $B(V)$, because in such a case the homology of the geometric realization $\bar{B}(X)$ can be understood.

The purpose of this paper is to provide some explicit coalgebra summands $B$ of $T$. We are interested in the special cases where $B$ can be chosen as a sub-Hopf algebra of $T$. This will give a relatively large coalgebra summand of $T$ because any subalgebra of a tensor algebra is a tensor algebra, and so the complementary functor $A$ of $B$ as in decomposition (1.1) becomes relatively small. Let $L_n(V)$ be the $n$th free Lie power on $V$; namely, $L_n(V)$ is the homogenous component of the free Lie algebra $L(V)$ on $V$ of tensor length $n$. Our main result is as follows.

**Theorem 1.1.** *Let the ground ring be a field of characteristic $p$. Let $\{m_i\}_{i\in I}$ be a finite or infinite set of positive integers prime to $p$ with each $m_i > 1$. Then the sub-Hopf algebra of $T(V)$ generated by*

$$L_{m_ip^r}(V) \quad \text{for } i \in I,\ r \geqslant 0,$$

*is a natural coalgebra summand of* $T(V)$*. In particular, the sub-Hopf algebra* $B(V)$ *of* $T(V)$ *generated by*

$$L_n(V) \quad \textit{for } n \text{ not a } \textit{power of } p$$

*is a natural coalgebra summand of* $T(V)$*.*

By the maximum property of the functor $B^{\max}$, the sub-Hopf algebras in the theorem are all contained in $B^{\max}$. According to [**22**, Proposition 11.1] as well as [**4**], the indecomposable elements in $B^{\max}$ do not have tensor length $p$ for $p > 2$ and so the sub-Hopf algebra $B(V)$ coincides with $B^{\max}(V)$ up to tensor length $p^2 - 1$. For the case $p = 2$, the sub-Hopf algebra $B(V)$ coincides with $B^{\max}(V)$ up to tensor length 7 according to the computations in [**25**]. Our sub-Hopf algebra $B(V)$ is strictly smaller than $B^{\max}(V)$ for a general module $V$.

An application of Theorem 1.1 to homotopy theory is given in [**31**]. Let the functors $B$ and $B^{\max}$ be extended to functors from graded modules to graded modules in the sense of [**22**]. Then the sub-Hopf algebra $B(V)$ coincides with $B^{\max}(V)$ for graded modules $V$ of dimension less than or equal to $p - 1$ with $V_{\text{even}} = 0$ according to [**31**, Theorem 1.1]. From this, we obtain EHP fibrations for the spaces $\bar{A}^{\min}(X)$ for $(p-1)$-cell co-$H$-spaces $X$ [**31**, Theorem 1.5]. These fibrations help in understanding the homotopy groups of co-$H$-spaces.

There is a canonical connection between coalgebra decompositions of $T$ and the decompositions of the Lie powers $L_n(V)$ as modules over the general linear groups by restricting decomposition (1.1) to the primitives. The decompositions of Lie powers have been actively studied in the recent development of representation theory [**2**–**4**, **9**, **10**]. Thus, the study of coalgebra decompositions of the functor $T$ helps to establish closer relations between homotopy theory and representation theory.

The paper is organized as follows. In § 2, we investigate the sub-quotient functors of the tensor power functors $T_n\colon V \mapsto T_n(V) = V^{\otimes n}$ from modules to modules. These special functors are of course closely related to the tensor representation of the symmetric groups and the finite-dimensional polynomial representations of the general linear groups (by evaluating on a fixed module $V$). They are also related to modules over the Schur algebras and modules over the Steenrod algebra [**15**–**17**]. In this section, we introduce exact functors $\gamma_n(\cdot)$ from the category of functors from modules to modules to the category of modules over the symmetric groups which are variations of the classical Schur functor [**1**, **13**, **19**]. In geometry, the summands of the tensor power functors $T_n$ are closely related to decompositions of self-smash products [**23**, **29**].

In § 3, we investigate the subfunctors of the Lie powers $L_n$ that occur as the summands of the functor $T_n$, which we call $T_n$-projective subfunctors of $L_n$. According to Theorem 3.9, these functors are closely related to the summands of the Lie powers $L_n(V)$ that occur as summands of $V^{\otimes n}$ studied in [**3**, **9**, **10**].

We give a coalgebra decomposition of $T$ called the 'block decomposition' in § 4. According to Theorem 4.3, this is a coalgebra decomposition of $T$ in the form

$$T \cong \bigotimes_{i=1}^{\infty} C^{m_i},$$

where $\{m_i\}$ is the set of all positive integers prime to $p$ and the primitives of $C^{m_i}$ are exactly given by the primitives of the tensor algebra $T$ with tensor length $m_i p^r$ for $r \geqslant 0$. In other words, the primitives of the tensor algebra $T$ with tensor length $mp^r$ for $r \geqslant 0$ for each $m$ prime to $p$ can be blocked into a coalgebra summand $C^m$ of $T$.

The proof of Theorem 1.1 is given in § 5. In § 6, we give some applications of our decomposition theorem to Lie powers by restricting to the primitives.

## 2. The structure on the tensor powers

### 2.1. Tensor algebras

Let $V$ be any module and let

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$$

be the *tensor algebra* generated by $V$, where $V^{\otimes n} = \boldsymbol{k}$ and the multiplication on $T(V)$ is given by the formal tensor product of monomials. The tensor algebra admits the universal property that, for any associated algebra $A$ and any linear map $f\colon V \to A$, there exists a unique algebra map $\tilde{f}\colon T(V) \to A$ such that $\tilde{f}|_V = f$. In particular, the linear map

$$V \to T(V) \otimes T(V), \quad x \mapsto x \otimes 1 + 1 \otimes x$$

extends uniquely to an algebra map $\psi\colon T(V) \to T(V) \otimes T(V)$ and so $T(V)$ has the canonical Hopf algebra structure with the multiplication given by the formal tensor product of monomials and the comultiplication given by $\psi$. We refer to [**18**] as a classical reference for Hopf algebras and quasi-Hopf algebras. The comultiplication $\psi$ is coassociative and cocommutative. The module $T(V)$ with the comultiplication $\psi\colon T(V) \to T(V) \otimes T(V)$ is called a *shuffle coalgebra* as its graded dual

$$T^*(V) = \bigoplus_{n=0}^{\infty} (V^{\otimes n})^* \cong \bigoplus_{n=0}^{\infty} (V^*)^{\otimes n}$$

is the usual shuffle algebra under the multiplication $\psi^*\colon T^*(V) \otimes T^*(V) \to T^*(V)$.

We are interested in the functor $T\colon V \mapsto T(V)$. There are three variations on this functor:

- the functor $T^{\mathrm{H}}\colon V \mapsto T(V)$ from modules to Hopf algebras;
- the functor $T^{\mathrm{C}}\colon V \mapsto T(V)$ from modules to coalgebras by forgetting the multiplication;
- the functor $T^{\mathrm{M}}\colon V \mapsto T(V)$ from modules to modules by forgetting both the multiplication and comultiplication.

As notation, the functor $T$ refers to one of $T^{\mathrm{H}}$, $T^{\mathrm{C}}$ or $T^{\mathrm{M}}$ if the working category is clear.

By taking tensor length, the functor $T^{\mathrm{M}}$ admits a natural decomposition

$$T^{\mathrm{M}} \cong \bigoplus_{n=0}^{\infty} T_n, \tag{2.1}$$

https://doi.org/10.1017/S0013091510000015 Published online by Cambridge University Press

where $T_n(V) = V^{\otimes n}$ with $T_0(V) = \boldsymbol{k}$. Thus, the functors $T^{\mathrm{H}}$, $T^{\mathrm{C}}$ and $T^{\mathrm{M}}$ are graded functors. From the well-known property (see, for example, [**12**, Lemma 3.8]) that

$$\mathrm{Hom}(T_n, T_m) = \begin{cases} 0 & \text{if } n \neq m, \\ \boldsymbol{k}(\Sigma_n) & \text{if } n = m, \end{cases} \tag{2.2}$$

the decomposition of $T^{\mathrm{M}}$ is in fact an orthogonal decomposition. A direct consequence is that the comultiplication $\psi$ (as a natural transformation) is uniquely determined by the multiplication on $T(V)$ for its Hopf structure.

**Proposition 2.1.** *Let $\Delta_V\colon T^{\mathrm{M}}(V) \to T^{\mathrm{M}}(V) \otimes T^{\mathrm{M}}(V)$ be a natural transformation such that $T^{\mathrm{M}}(V)$ with the usual multiplication together with the comultiplication given by $\Delta_V$ is a quasi-Hopf algebra for every $V$. Then $\Delta_V = \psi_V$ for all $V$.*

**Proof.** For every $V$, from the property that $\mathrm{Hom}(T_n, T_m) = 0$ for $n \neq m$, we have

$$\Delta_V(T_1(V)) \subseteq T_1(V) \otimes T_0(V) \oplus T_0(V) \otimes T_1(V)$$

and the counit

$$\epsilon_V\colon T(V) = \bigoplus_{n=0}^{\infty} T_n(V) \to T_0(V)$$

is the canonical projection given by sending each $T_i(V)$ to 0 for $i > 0$ and $\epsilon|_{T_0(V)} = \mathrm{id}$. Since both $\Delta_V$ and $\psi_V$ have counit uniquely given by $\epsilon_V$, we have

$$\Delta_V|_{T_1(V)} = \psi_V|_{T_1(V)}.$$

It follows that $\Delta_V = \psi_V$ because both are algebra maps with respect to the formal tensor product. $\square$

**Remark 2.2.** Given a module $V$, of course one could have many comultiplications on $T(V)$ such that $T(V)$ is Hopf. The proposition states that $\psi$ is the only possible comultiplication on $T(V)$ which is a natural transformation.

### 2.2. Subfunctors of the tensor algebra functor

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $A, B\colon \mathcal{C} \to \mathcal{D}$ be functors. We call $A$ a *subfunctor* (*quotient functor*) of $B$ if there is a natural transformation $\phi\colon A \to B$ such that

$$\phi_X\colon A(X) \to B(X)$$

is injective (surjective) for every object $X \in \mathcal{C}$. A subfunctor (quotient functor) of $T$ refers to a subfunctor (quotient functor) of $T^{\mathrm{H}}$, $T^{\mathrm{C}}$ or $T^{\mathrm{M}}$. A subfunctor (quotient functor) of $T^{\mathrm{H}}$ is called a *sub-Hopf functor* (*quotient Hopf functor*) of $T$. Similarly we have a *sub-coalgebra functor* (*quotient coalgebra functor*) of $T$ and a *submodule functor* (*quotient module functor*) of $T$. A *graded subfunctor* (*graded quotient functor*) of $T$ refers to a subfunctor of $T^{\mathrm{H}}$, $T^{\mathrm{C}}$ or $T^{\mathrm{M}}$ as functors from modules to graded Hopf algebras, graded coalgebras or graded modules, respectively.

https://doi.org/10.1017/S0013091510000015 Published online by Cambridge University Press

**Proposition 2.3.** *Let $B$ be a functor from modules to modules. Suppose that*

(i) *there is a natural monomorphism $\phi_V\colon B(V) \to T(V)$ and*

(ii) *there is a natural epimorphism $\psi\colon T(V) \to B(V)$ for any module $V$.*

*Then there is a natural grading on $B(V)$ such that $\phi\colon B \to T$ and $\psi\colon T \to B$ are natural transformations of graded functors.*

**Proof.** From the hypothesis, $\phi$ induces a natural isomorphism

$$\phi\colon B(V) \xrightarrow{\cong} \mathrm{Im}(\phi \circ \psi\colon T(V) \to T(V)).$$

By the orthogonal property of $T$ as in (2.2), the composite

$$\phi \circ \psi\colon T \to T$$

is a natural transformation of graded functors. Thus, $\phi \circ \psi(T_n) \subseteq T_n$ and

$$\mathrm{Im}(\phi \circ \psi\colon T(V) \to T(V)) = \bigoplus_{n=0}^{\infty} \mathrm{Im}(\phi \circ \psi|_{T_n}\colon T_n(V) \to T_n(V)).$$

Let $B_n(V) = \phi_V^{-1}(\mathrm{Im}(\phi \circ \psi|_{T_n}\colon T_n(V) \to T_n(V)))$. Then

$$B = \bigoplus_{n=0}^{\infty} B_n$$

is a graded functor and $\phi\colon B \to T$ is a natural transformation of graded functors. Since

$$\phi(\psi(T_n)) = \mathrm{Im}(\phi \circ \psi|_{T_n}\colon T_n(V) \to T_n(V)),$$

we have $\psi(T_n) = B_n$ and so $\psi$ is also a natural transformation of graded functors, hence the result. □

**Corollary 2.4.** *Let $C$ be a sub-quotient functor of $T$. Suppose that $C$ is a natural summand of $T^{\mathrm{M}}$. Then $C$ is a graded sub-quotient functor of $T$.*

### 2.3. The associated symmetric group modules of the functors

Let $\bar{V}_n$ be the $n$-dimensional $\boldsymbol{k}$-module with a fixed choice of basis $\{x_1, \dots, x_n\}$. For each $1 \leqslant i \leqslant n$, define the linear transformation

$$d_i\colon \bar{V}_n \to \bar{V}_{n-1}$$

by setting

$$d_i(x_j) = \begin{cases} x_j & \text{if } j < i, \\ 0 & \text{if } j = i, \\ x_{j-1} & \text{if } j > i. \end{cases}$$

https://doi.org/10.1017/S0013091510000015 Published online by Cambridge University Press

The right $\boldsymbol{k}(\Sigma_n)$-action on $\bar{V}_n$ is given by

$$x_i \cdot \sigma = x_{\sigma(i)}$$

for $1 \leqslant i \leqslant n$ and $\sigma \in \Sigma_n$. Then, for each $1 \leqslant i \leqslant n$ and any $\sigma \in \Sigma_n$, clearly there exists a unique permutation $d_i\sigma \in \Sigma_{n-1}$ such that the diagram

$$\begin{tikzcd} \bar{V}_n \arrow[r, "\sigma", "x_j \mapsto x_{\sigma(j)}"'] \arrow[d, "d_i"] & \bar{V}_n \arrow[d, "d_{\sigma(i)}"] \\ \bar{V}_{n-1} \arrow[r, "d_i\sigma"] & \bar{V}_{n-1} \end{tikzcd} \tag{2.3}$$

commutes. Let $B$ be a functor from modules to modules. Then $B(\bar{V}_n)$ is a right $\boldsymbol{k}(\Sigma_n)$-module induced by the action of $\boldsymbol{k}(\Sigma_n)$ on $\bar{V}_n$. Define

$$\gamma_n(B) = \bigcap_{i=1}^{n} \operatorname{Ker}(B(d_i)\colon B(\bar{V}_n) \to B(\bar{V}_{n-1})). \tag{2.4}$$

By applying the functor $B$ to diagram (2.3), $\gamma_n(B)$ is a $\boldsymbol{k}(\Sigma_n)$-submodule of $B(\bar{V}_n)$. Let

$$\phi\colon B \to C$$

be a natural transformation of functors from modules to modules. Then clearly $\phi_{\bar{V}_n}$ induces a $\boldsymbol{k}(\Sigma_n)$-map

$$\gamma_n(\phi)\colon \gamma_n(B) \to \gamma_n(C).$$

**Proposition 2.5.** *Let*

$$A \overset{j}{\hookrightarrow} B \overset{p}{\twoheadrightarrow} C$$

*be a short exact sequence of functors from modules to modules. Then there is a short exact sequence of* $\boldsymbol{k}(\Sigma_n)$*-modules*

$$\gamma_n(A) \overset{\gamma_n(j)}{\hookrightarrow} \gamma_n(B) \overset{\gamma_n(p)}{\twoheadrightarrow} \gamma_n(C).$$

*Thus,* $\gamma_n(-)$ *is an exact functor from the category of functors from modules to modules to the category of* $\boldsymbol{k}(\Sigma_n)$*-modules.*

**Remark.** The exact functor $\gamma_n(-)$ is a variation of the Schur functor given in [**13**] in the following sense. Let $B$ is a sub-quotient functor of $T_n$ and let $V$ be a module with $m = \dim V \geqslant n$. Then $B(V)$ is a sub-quotient $\boldsymbol{k}(GL_m(\boldsymbol{k}))$-module of $V^{\otimes n}$. Let $\bar{V}_n$ embed into $V$ in the canonical way such that $V = \bar{V} \oplus V'$. In our definition, $\gamma_n(B) \subseteq B(\bar{V}) \subseteq B(V)$. According to [**9**, § 1.2, p. 71], $B(V) \mapsto \gamma_n(B)$ is the Schur functor.

**Proof.** Define the coface operation $d^i\colon \bar{V}_{n-1} \to \bar{V}_n$ by setting

$$d^i(x_j) = \begin{cases} x_j & \text{if } j < i, \\ x_{j+1} & \text{if } j \geqslant i, \end{cases}$$

https://doi.org/10.1017/S0013091510000015 Published online by Cambridge University Press

for $1 \leqslant i \leqslant n$. Then the sequence of modules $\{\bar{V}_{n+1}\}_{n\geqslant 0}$ with faces

$$d_1, \dots, d_n \colon \bar{V}_n \to \bar{V}_{n-1}$$

relabelled as $d_0, \dots, d_{n-1}$ and cofaces

$$d^1, \dots, d^n \colon \bar{V}_{n-1} \to \bar{V}_n$$

relabelled as $d^0, \dots, d^{n-1}$ by shifting indices down by 1 forms a bi-$\Delta$-group in the sense of [**30**, § 1.2]. By applying the functors to the bi-$\Delta$-group $\{\bar{V}_{n+1}\}_{n\geqslant 0}$, one gets a short exact sequence of bi-$\Delta$-groups

$$\{A(\bar{V}_{n+1})\}_{n\geqslant 0} \hookrightarrow \{B(\bar{V}_{n+1})\}_{n\geqslant 0} \twoheadrightarrow \{C(\bar{V}_{n+1})\}_{n\geqslant 0}.$$

The assertion then follows by [**30**, Proposition 1.2.10]. □

**Corollary 2.6.** *Let $\phi\colon A \to B$ be a natural transformation between functors from modules to modules. Suppose that*

$$\gamma_n(\phi)\colon \gamma_n(A) \to \gamma_n(B)$$

*is an isomorphism for each $n \geqslant 1$. Then*

$$\phi_V \colon A(V) \to B(V)$$

*is an isomorphism for any finite-dimensional module $V$. Thus, if both $A$ and $B$ preserve colimits, then $\phi$ is a natural equivalence.*

**Proof.** Let $C$ be the cokernel of $\phi$. Suppose that $C(V) \neq 0$ for some finite-dimensional module $V$. Let

$$n = \min\{k \mid C(V) \neq 0,\ \dim(V) = k\}.$$

Then $\gamma_n(C) = C(\bar{V}_n) \neq 0$. By Proposition 2.5, $\gamma_n(C) = 0$, which is a contradiction. Thus, $C(V) = 0$ for any finite-dimensional module $V$. Similarly, for $D$ the kernel of $\phi$, we have $D(V) = 0$ for any finite-dimensional module $V$, finishing the proof. □

### 2.4. $T_n$-projective functors

Consider the functor $T_n$. Let $\gamma_n = \gamma_n(T_n)$. Then $\gamma_n$ is the $\boldsymbol{k}$-submodule of $\bar{V}_n^{\otimes n}$ spanned by the monomials

$$x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$$

for $\sigma \in \Sigma_n$ with the right symmetric group action explicitly given by

$$(x_{i_1} \otimes \cdots \otimes x_{i_n}) \cdot \sigma = x_{\sigma(i_1)} \otimes \cdots \otimes x_{\sigma(i_n)} \tag{2.5}$$

for $\sigma \in \Sigma_n$ and the monomials $x_{i_1} \cdots x_{i_n} \in \gamma_n$. Observe that

$$\gamma_n \cong \boldsymbol{k}(\Sigma_n) \tag{2.6}$$

as a $\boldsymbol{k}(\Sigma_n)$-module.

https://doi.org/10.1017/S0013091510000015 Published online by Cambridge University Press

Let $V$ be any $\boldsymbol{k}$-module and let $a_1,\dots,a_n \in V$. We write $a_1\cdots a_n$ for the tensor product $a_1\otimes\cdots\otimes a_n \in V^{\otimes n}$ if there is no confusion. Let the symmetric group $\Sigma_n$ act on $V^{\otimes n}$ by permuting positions. More precisely, the left $\boldsymbol{k}(\Sigma_n)$-action on $V^{\otimes n}$ is given

$$\sigma\cdot(a_1\cdots a_n) = a_{\sigma(1)}\cdots a_{\sigma(n)} \tag{2.7}$$

for $\sigma\in\Sigma_n$ and the monomials $a_1\cdots a_n\in V^{\otimes n}$. Let $B$ be any functor from modules to modules. Define the functor $\gamma_n^B(-)$ by setting

$$\gamma_n^B(V) = \gamma_n(B)\otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \tag{2.8}$$

for any module $V$. Clearly,

$$\gamma_n^{T_n}(V) = \gamma_n\otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \cong T_n(V).$$

**Proposition 2.7.** *Let*

$$B \xhookrightarrow{j} T_n \xrightarrow{p} C$$

*be a short exact sequence of functors from modules to modules. Then the natural isomorphism $\gamma_n\otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n}\cong T_n(V)$ induces a natural commutative diagram of exact sequences*

$$\begin{tikzcd}
\gamma_n(B)\otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \arrow[r, "\gamma_n(j)\otimes\mathrm{id}"] \arrow[d] & \gamma_n\otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \arrow[r, two heads, "\gamma_n(p)\otimes\mathrm{id}"] \arrow[d, "\cong"'] & \gamma_n(C)\otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \arrow[d] \\
B(V) \arrow[r, hook] & T_n(V) \arrow[r, two heads] & C(V)
\end{tikzcd}$$

*with a natural exact sequence*

$$\mathrm{Tor}_1^{\boldsymbol{k}(\Sigma_n)}(\gamma_n(C),V^{\otimes n}) \hookrightarrow \gamma_n(B)\otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n}\to B(V)\to\gamma_n(C)\otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n}\twoheadrightarrow C(V).$$

*for any module $V$.*

**Proof.** By taking the image of $j_V$, we may consider $B(V)$ to be a submodule of $T_n(V)$ for any module $V$. Let

$$\theta\colon \gamma_n\otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n}\to T_n(V)$$

be the isomorphism and let $\Phi_V^B = \theta\circ(\gamma_n(j)\otimes\mathrm{id})$. Observe that the isomorphism $\theta$ is given by

$$\theta(x_1\cdots x_n\otimes a_1\cdots a_n) = a_1\cdots a_n$$

for $a_1,\dots,a_n\in V$. Let $a = a_1\cdots a_n\in V^{\otimes n}$ be any monomial with $a_j\in V$ for $1\leqslant j\leqslant n$ and let

$$\alpha = \sum_{\sigma\in\Sigma_n} k_\sigma x_{\sigma(1)}\cdots x_{\sigma(n)}\in\gamma_n(B).$$

Then

$$\begin{aligned}\Phi_V^B(\alpha\otimes a_1\cdots a_n) &= \sum_{\sigma\in\Sigma_n} k_\sigma x_{\sigma(1)}\cdots x_{\sigma(n)}\otimes a_1\cdots a_n\\ &= \sum_{\sigma\in\Sigma_n} k_\sigma (x_1\cdots x_n)\cdot\sigma\otimes a_1\cdots a_n\\ &= \sum_{\sigma\in\Sigma_n} k_\sigma x_1\cdots x_n\otimes\sigma\cdot(a_1\cdots a_n)\\ &= \sum_{\sigma\in\Sigma_n} k_\sigma x_1\cdots x_n\otimes a_{\sigma(1)}\cdots a_{\sigma(n)}\\ &= \sum_{\sigma\in\Sigma_n} k_\sigma a_{\sigma(1)}\cdots a_{\sigma(n)}\in V^{\otimes n}.\end{aligned}$$

Define a linear transformation $f_a\colon \bar V_n\to V$ by setting

$$f_a(x_i)=a_i$$

for $1\leqslant i\leqslant n$. Consider $T_n(f_a)=f_a^{\otimes n}\colon T_n(\bar V_n)\to T_n(V)$. Then

$$T_n(f_a)(\alpha)=f_a^{\otimes n}\biggl(\sum_{\sigma\in\Sigma_n} k_\sigma x_{\sigma(1)}\cdots x_{\sigma(n)}\biggr)=\Phi_V^B(\alpha\otimes a_1\cdots a_n).$$

From the commutative diagram

$$\begin{tikzcd} B(\bar V_n) \arrow[r, hook, "j_{\bar V_n}"] \arrow[d, "B(f_a)"] & T_n(\bar V_n) \arrow[d, "T_n(f_a)"] \\ B(V) \arrow[r, hook, "j_V"] & T_n(V) \end{tikzcd}$$

since

$$\alpha\in\gamma_n(B)\subseteq B(\bar V_n),$$

we have

$$\Phi_V^B(\alpha\otimes a_1\cdots a_n)=T_n(f_a)(\alpha)\in B(V). \tag{2.9}$$

It follows that

$$\mathrm{Im}(\Phi_V^B)\subseteq B(V)$$

for any module $V$. Thus, the left square in the statement of the proposition commutes.

By Proposition 2.5, there is a short exact sequence of $\boldsymbol{k}(\Sigma_n)$-modules

$$\gamma_n(B)\xhookrightarrow{\gamma_n(j)}\gamma_n\xrightarrow{\gamma_n(p)}\gamma_n(C).$$

Since $\gamma_n$ is a free $\boldsymbol{k}(\Sigma_n)$-module, there is an exact sequence

$$\mathrm{Tor}_1^{\boldsymbol{k}(\Sigma_n)}(\gamma_n(C),V^{\otimes n})\hookrightarrow\gamma_n(B)\otimes_{\boldsymbol{k}(\Sigma_n)}V^{\otimes n}\to\gamma_n\otimes_{\boldsymbol{k}(\Sigma_n)}V^{\otimes n}\twoheadrightarrow\gamma_n(C)\otimes_{\boldsymbol{k}(\Sigma_n)}V^{\otimes n}.$$

Hence, the right square in the statement of the proposition commutes, and the asserted exact sequence exists. □

**Example 2.8.** We give an example that the natural transformation

$$\gamma_n(B)\otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n}\to B(V)$$

could be neither an epimorphism nor a monomorphism for subfunctors $B$ of $T_n$. Let $\boldsymbol{k}$ be a field of characteristic 2. The Lie power $L_2(V)$ is the submodule of $V^{\otimes 2}$ spanned by $[a,b]=ab-ba$ for $a,b\in V$. The restricted Lie power $L_2^{\mathrm{res}}(V)$ is the submodule of $V^{\otimes 2}$ spanned by $a^2$, $[a,b]$ for $a,b\in V$. Then

$$\gamma_2(L_2)=\gamma_2(L_2^{\mathrm{res}})$$

is the one-dimensional submodule of $\boldsymbol{k}(\Sigma_2)$ generated by $1-\tau$. Since $\boldsymbol{k}$ is of characteristic 2, $\gamma_2(L_2)=\gamma_2(L_2^{\mathrm{res}})$ is the trivial $\boldsymbol{k}(\Sigma_2)$-module and so

$$\gamma_2(L_2^{\mathrm{res}})\otimes_{\boldsymbol{k}(\Sigma_2)} V^{\otimes 2}=S_2(V),$$

the two-fold symmetric product of $V$. The natural transformation

$$\gamma_2(L_2^{\mathrm{res}})\otimes_{\boldsymbol{k}(\Sigma_2)} V^{\otimes 2}\to L_2^{\mathrm{res}}(V)$$

is not an epimorphism for $V$ with $\dim V\geqslant 1$ because its image is given by $L_2(V)$. The kernel of this natural transformation is measured by

$$\mathrm{Tor}_1^{\boldsymbol{k}(\Sigma_2)}(\gamma_2/\mathrm{Lie}(2),V^{\otimes 2})\neq 0$$

for $V$ with $\dim V\geqslant 1$.

Let $B$ be a functor from modules to modules. The *dual functor* $B^*$ is defined as follows. For any finite-dimensional module $V$, define

$$B^*(V)=B(V^*)^*,$$

where $V^*=\mathrm{Hom}_{\boldsymbol{k}}(V,\boldsymbol{k})$ is the dual $\boldsymbol{k}$-module of $V$, and, for a general module $V$, let

$$B^*(V)=\mathrm{colim}_{V_\alpha} B^*(V_\alpha)$$

be the direct limit of the module $B^*(V_\alpha)$ subject to the direct system given by the diagram of all finite-dimensional submodules of $V$ with inclusions. Clearly, $T_n^*=T_n$.

**Proposition 2.9.** *Let $B$ be a functor from modules to modules. Then $\gamma_n(B^*)$ is the dual $\boldsymbol{k}(\Sigma_n)$-module of $\gamma_n(B)$ for each $n\geqslant 1$.*

**Proof.** For the basis $\{x_1,\dots,x_n\}$ for $\bar V_n$, let $\{x_1^*,\dots,x_n^*\}$ be the standard dual basis of $\bar V_n^*$. Let

$$\theta_n\colon \bar V_n\to\bar V_n^*$$

be the linear transformation such that $\theta_n(x_j)=x_j^*$ for $1\leqslant j\leqslant n$. Then it is routine to check that the composite

$$\gamma_n(B^*)\subseteq B^*(\bar V_n)=B(\bar V_n^*)^*\xrightarrow{B(\theta_n)^*}B(\bar V_n)^*\twoheadrightarrow\gamma_n(B)^*$$

is an isomorphism of $\boldsymbol{k}(\Sigma_n)$-modules. □

We call a direct sum of copies of $T_n$ a *free $T_n$-functor*. A functor $B$ from modules to modules is called *$T_n$-projective* if there exists a free $T_n$-functor $F$ together with natural transformations $s\colon B \to F$ and $r\colon F \to B$ such that $r \circ s\colon B \to B$ is a natural equivalence. In other words, a $T_n$-projective functor means a summand (or retract) of a free $T_n$-functor.

**Proposition 2.10.** *Let $B$ be a functor from modules to modules.*

(i) *If $B$ is a $T_n$-projective functor, then $\gamma_n(B)$ is a projective $\boldsymbol{k}(\Sigma_n)$-module and there is a natural isomorphism*

$$\gamma_n(B) \otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \cong B(V)$$

*for any module $V$.*

(ii) *If $B$ is a subfunctor of a direct sum of finite copies of $T_n$ with the property that $\gamma_n(B)$ is a projective $\boldsymbol{k}(\Sigma_n)$-module, then $B$ is a $T_n$-projective functor. Moreover, there is a natural equivalence $B \cong B^*$.*

(iii) *If $B$ is a quotient functor of a direct sum of finite copies of $T_n$ with the property that $\gamma_n(B)$ is a projective $\boldsymbol{k}(\Sigma_n)$-module, then $B$ is a $T_n$-projective functor. Moreover, there is a natural equivalence $B \cong B^*$.*

(iv) *Let $B$ be a sub-quotient functor of a direct sum of finite copies of $T_n$. Suppose that $B$ is a $T_n$-projective functor. Then $B$ is both projective and injective in the category of sub-quotient functors of direct sums of finite copies of $T_n$.*

**Proof.** The proof of assertion (i) is straightforward. Assertion (iii) follows from (ii) by considering the dual functor.

(ii) Let $B$ be a subfunctor of $F$, where $F$ is a finite direct sum of copies of $T_n$. Let $C = F/B$. Then there is a short exact sequence

$$\gamma_n(B) \hookrightarrow \gamma_n(F) \twoheadrightarrow \gamma_n(C).$$

Since $\gamma_n(B)$ is a finitely generated projective $\boldsymbol{k}(\Sigma_n)$-module, it is an injective $\boldsymbol{k}(\Sigma_n)$-module and so the above short exact sequence splits as $\boldsymbol{k}(\Sigma_n)$-modules. It follows that $\gamma_n(C)$ is a projective $\boldsymbol{k}(\Sigma_n)$-module because $\gamma_n(F)$ is a free $\boldsymbol{k}(\Sigma_n)$-module:

$$\mathrm{Tor}_1^{\boldsymbol{k}(\Sigma_n)}(\gamma_n(C), V^{\otimes n}) = 0.$$

Since $F$ is a direct sum of copies of the functor $T_n$, we can apply the exact sequence in Proposition 2.7. In particular, the natural transformation

$$\Phi_V^B\colon \gamma_n(B) \otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \to B(V) \tag{2.10}$$

is a natural monomorphism. The natural inclusion $B \hookrightarrow F$ induces an natural epimorphism $F = F^* \twoheadrightarrow B^*$. By Proposition 2.7, there is a natural epimorphism

$$\Phi_V^{B^*}\colon \gamma_n(B^*) \otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \to B^*(V). \tag{2.11}$$

By Proposition 2.9,

$$\gamma_n(B^*) \cong \gamma_n(B)^* \cong \gamma_n(B)$$

as $\boldsymbol{k}(\Sigma_n)$-modules because $\gamma_n(B)$ is a finitely generated $\boldsymbol{k}(\Sigma_n)$-projective module. Let $V$ be any finite-dimensional $\boldsymbol{k}$-module. From (2.10) and (2.11), we have

$$\dim B(V) = \dim B(V^*)^* = \dim B^*(V) \leqslant \dim(\gamma_n(B) \otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n}) \leqslant \dim B(V).$$

Thus, $\Phi^B_V$ and $\Phi^{B^*}_V$ are isomorphisms for any finite-dimensional module $V$. Since the functors $\gamma_n(B) \otimes_{\boldsymbol{k}(\Sigma_n)} (-)^{\otimes n}$, $B$ and $B^*$ preserve colimits, the natural transformations $\Phi^B$ and $\Phi^{B^*}$ are natural equivalences. The assertion now follows from the fact that $\gamma_n(B) \otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n}$ is a natural summand of $\gamma_n(F) \otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \cong F(V)$.

(iv) Let $\mathcal{C}$ be the category of sub-quotient functors of direct sums of copies of $T_n$. It suffices to show that $T_n$ is projective and injective in the $\mathcal{C}$. Let $B$ be an object in $\mathcal{C}$ with a natural epimorphism $q\colon B \twoheadrightarrow T_n$. It induces an epimorphism $\gamma_n(q)\colon \gamma_n(B) \to \gamma_n(T_n) = \gamma_n$. Since $\gamma_n$ is $\boldsymbol{k}(\Sigma_n)$-projective, there is a $\boldsymbol{k}(\Sigma_n)$-cross-section $s\colon \gamma_n(T_n) \to \gamma_n(B)$. Now the natural transformation

$$T_n(V) \cong \gamma_n(T_n) \otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \xrightarrow{s\otimes \mathrm{id}} \gamma_n(B) \otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \xrightarrow{\Phi^B_V} B(V)$$

is a cross-section to $q$ and so $T_n$ is projective in $\mathcal{C}$. Since $T_n \cong T_n^*$ is self-dual, $T_n$ is also injective in $\mathcal{C}$. The proof is finished. □

We remark that if $B$ is a sub-quotient functor of $T_n$ with the property that $\gamma_n(B)$ is $\boldsymbol{k}(\Sigma_n)$-projective, it is possible that $B$ is not $T_n$-projective. For instance, for the ground field $\boldsymbol{k}$ being of characteristic 2, the functor $B = L_2^{\mathrm{res}}/L_2$ has the property that $\gamma_2(B) = 0$ with $B$ not $T_2$-projective.

## 3. The structure on lie power functors

### 3.1. The Lie power functors and the symmetric group modules Lie($n$)

In this section, the ground ring is a field $\boldsymbol{k}$. Let $V$ be a module. The *free Lie algebra* $L(V)$ generated by $V$ is the smallest sub-Lie algebra of the tensor algebra $T(V)$ containing $V$, where the Lie structure on $T(V)$ is given by $[a, b] = ab - ba$. The functor $L$ admits a graded structure

$$L(V) = \bigoplus_{n=1}^{\infty} L_n(V),$$

where $L_n(V) = L(V) \cap T_n(V)$ is called the *nth Lie power* of $V$. By applying (2.4) to the functor $L_n$, we have the symmetric group module

$$\mathrm{Lie}(n) = \gamma_n(L_n).$$

Let $\bar{V}$ be the $n$-dimensional module with basis $\{x_1, \dots, x_n\}$ as in § 2.3. By definition,

$$\mathrm{Lie}(n) = L_n(\bar{V}_n) \cap \gamma_n$$

https://doi.org/10.1017/S0013091510000015 Published online by Cambridge University Press

is spanned by the homogenous Lie elements of length $n$ in which each $x_i$ occurs exactly once. By the Witt formula, $\mathrm{Lie}(n)$ is of dimension $(n-1)!$. By the antisymmetry and Jacobi identities, $\mathrm{Lie}(n)$ has a basis given by the elements

$$[[x_1, x_{\sigma(2)}], x_{\sigma(3)}, \dots, x_{\sigma(n)}]$$

for $\sigma \in \Sigma_{n-1}$ [**5**].

**Proposition 3.1.** *There is a natural short exact sequence*

$$\mathrm{Tor}_1^{\boldsymbol{k}(\Sigma_n)}(\gamma_n/\mathrm{Lie}(n), V^{\otimes n}) \hookrightarrow \mathrm{Lie}(n) \otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \twoheadrightarrow L_n(V)$$

*for any module $V$.*

**Proof.** By Proposition 2.7, it suffices to show that the natural transformation

$$\Phi_V^{L_n}\colon \mathrm{Lie}(n) \otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \to L_n(V)$$

is an epimorphism. Let $[[a_1, a_2], \dots a_n] \in L_n(V)$ with $a_1, \dots, a_n \in V$. Let

$$\alpha = [[x_1, x_2], \dots, x_n] \in \mathrm{Lie}(n).$$

There then exists a unique $k_\sigma \in \boldsymbol{k}$ such that

$$\alpha = [[x_1, x_2], \dots, x_n] = \sum_{\sigma \in \Sigma_n} k_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

Along the lines of the proof of Proposition 2.7, we have

$$\Phi_V^{L_n}(\alpha \otimes a_1 \cdots a_n) = \sum_{\sigma \in \Sigma_n} k_\sigma a_{\sigma(1)} \cdots a_{\sigma(n)} = [[a_1, a_2], \dots, a_n]. \tag{3.1}$$

The assertion follows from the fact that $L_n(V)$ is the $\boldsymbol{k}$-module spanned by the Lie elements $[[a_1, a_2], \dots, a_n]$ with $a_j \in V$. □

For any natural transformation $\phi\colon L_n \to L_n$, we have the $\boldsymbol{k}(\Sigma_n)$-linear map

$$\gamma_n(\phi)\colon \gamma_n(L_n) = \mathrm{Lie}(n) \to \gamma_n(L_n) = \mathrm{Lie}(n).$$

This defines a ring homomorphism $\gamma\colon \mathrm{End}(L_n) \to \mathrm{End}_{\boldsymbol{k}(\Sigma_n)}(\mathrm{Lie}(n))$.

**Proposition 3.2.** *If $n \neq m$, then $\mathrm{Hom}(L_n, L_m) = 0$. Moreover, the ring homomorphism*

$$\gamma\colon \mathrm{End}(L_n) \to \mathrm{End}_{\boldsymbol{k}(\Sigma_n)}(\mathrm{Lie}(n))$$

*is an isomorphism with a natural commutative diagram of functors*

$$\begin{tikzcd}
\mathrm{Lie}(n) \otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \arrow[r, "\gamma_n(\delta)\otimes \mathrm{id}"] \arrow[d, "\Phi_V^{L_n}"] & \mathrm{Lie}(n) \otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \arrow[d, "\Phi_V^{L_n}"] \\
L_n(V) \arrow[r, "\delta_V"] & L_n(V)
\end{tikzcd}$$

*for any natural transformation $\delta\colon L_n \to L_n$.*

**Proof.** Let $\phi\colon L_n \to L_m$ be a natural transformation. Let $\bar\beta_n\colon T_n \to L_n$ be the natural epimorphism defined by $\bar\beta_n(a_1\cdots a_n) = [[a_1,a_2],\dots,a_n]$ for any module $W$ and any monomial $a_1\cdots a_n \in T_n(W) = W^{\otimes n}$. Then the composite

$$T_n \xrightarrow{\bar\beta} L_n \xrightarrow{\phi} L_m \hookrightarrow T_m$$

is a natural transformation, which is zero as $\mathrm{Hom}(T_n, T_m) = 0$ for $n \neq m$. Thus, $\phi = 0$.

For the second statement, let $\delta\colon L_n \to L_n$ be a natural transformation. Let $V$ be any module. Consider $[[a_1,a_2],\dots,a_n] \in L_n(V)$ with $a_j \in V$. Let $f_a\colon \bar V_n \to V$ be the linear map with $f_a(x_j) = a_j$ for $1 \leqslant j \leqslant n$. Then there is a commutative diagram

$$\begin{tikzcd}
\gamma_n(L_n) \arrow[d, "\delta_{\bar V_n}"] & \subseteq & L_n(\bar V_n) \arrow[r, "L_n(f_a)"] \arrow[d, "\gamma_n(\delta)"] & L_n(V) \arrow[d, "\delta_V"] \\
\gamma_n(L_n) & \subseteq & L_n(\bar V_n) \arrow[r, "L_n(f_a)"] & L_n(V)
\end{tikzcd} \tag{3.2}$$

Thus,

$$\begin{aligned}
\delta_V \circ \Phi_V^{L_n}&([[x_1,x_2],\dots,x_n] \otimes a_1\cdots a_n) \\
&= \delta_V([[a_1,a_2],\dots,a_n]) && \text{(by (3.1))} \\
&= L_n(f_a)(\gamma_n(\delta)([[x_1,x_2],\dots,x_n])) && \text{(by (3.2))} \\
&= \Phi_V^{L_n}(\gamma_n(\delta)([[x_1,x_2],\dots,x_n]) \otimes a_1\cdots a_n) && \text{(by (2.9))}
\end{aligned}$$

and so the diagram in the statement commutes. It follows that the map

$$\gamma\colon\ \mathrm{End}(L_n) \to \mathrm{End}_{\boldsymbol{k}(\Sigma_n)}(\mathrm{Lie}(n))$$

is a monomorphism.

To show that $\gamma$ is an epimorphism, let $\theta\colon\ \mathrm{Lie}(n) \to \mathrm{Lie}(n)$ be any $\boldsymbol{k}(\Sigma_n)$-linear map. Since $\boldsymbol{k}(\Sigma_n)$ is a Fröbenius algebra, the free $\boldsymbol{k}(\Sigma_n)$-module $\gamma_n$ is injective and so there is a commutative diagram of exact sequences of $\boldsymbol{k}(\Sigma_n)$-modules

$$\begin{tikzcd}
\mathrm{Lie}(n) \arrow[r, hook] \arrow[d, "\theta"] & \gamma_n \arrow[r, two heads] \arrow[d, "\tilde\theta"] & \gamma_n/\mathrm{Lie}(n) \arrow[d, "\bar\theta"] \\
\mathrm{Lie}(n) \arrow[r, hook] & \gamma_n \arrow[r, two heads] & \gamma_n/\mathrm{Lie}(n)
\end{tikzcd}$$

It follows that there is a commutative diagram of short exact sequences of functors

$$\begin{tikzcd}
\mathrm{Tor}_1^{\boldsymbol{k}(\Sigma_n)}(\gamma_n/\mathrm{Lie}(n), V^{\otimes n}) \arrow[r, hook] \arrow[d, "\mathrm{Tor}(\bar\theta,\mathrm{id})"] & \mathrm{Lie}(n) \otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \arrow[r, two heads, "\Phi_V^{L_n}"] \arrow[d, "\theta\otimes\mathrm{id}"] & L_n(V) \arrow[d, "\delta"] \\
\mathrm{Tor}_1^{\boldsymbol{k}(\Sigma_n)}(\gamma_n/\mathrm{Lie}(n), V^{\otimes n}) \arrow[r, hook] & \mathrm{Lie}(n) \otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \arrow[r, two heads, "\Phi_V^{L_n}"] & L_n(V)
\end{tikzcd}$$

https://doi.org/10.1017/S0013091510000015 Published online by Cambridge University Press

for some natural transformation $\delta\colon L_n \to L_n$. By taking $V = \bar{V}_n$ and restricting to the submodule $\gamma_n \subseteq \bar{V}^{\otimes n}$, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Lie}(n) \arrow[r, "g", "\cong"'] \arrow[d, "\theta"] & \mathrm{Lie}(n) \otimes_{\boldsymbol{k}(\Sigma_n)} \gamma_n \arrow[r, "\Phi^{L_n}_{\bar{V}_n}|", "\cong"'] \arrow[d, "\theta\otimes\mathrm{id}"] & \mathrm{Lie}(n) \arrow[d, "\gamma_n(\delta)"] \\
\mathrm{Lie}(n) \arrow[r, "g", "\cong"'] & \mathrm{Lie}(n) \otimes_{\boldsymbol{k}(\Sigma_n)} \gamma_n \arrow[r, "\Phi^{L_n}_{\bar{V}_n}|", "\cong"'] & \mathrm{Lie}(n)
\end{tikzcd}$$

where $g(\alpha) = \alpha \otimes x_1 \cdots x_n$. Thus, $\theta = \gamma_n(\delta)$ because

$$\begin{aligned}
\Phi^{L_n}_{\bar{V}} \circ g([[x_{\sigma(1)}, x_{\sigma(2)}], \dots, x_{\sigma(n)}]) &= \Phi^{L_n}_{\bar{V}}([[x_1, x_2], \dots, x_n] \cdot \sigma \otimes x_1 \cdots x_n) \\
&= \Phi^{L_n}_{\bar{V}}([[x_1, x_2], \dots, x_n] \otimes \sigma \cdot (x_1 \cdots x_n)) \\
&= \Phi^{L_n}_{\bar{V}}([[x_1, x_2], \dots, x_n] \otimes x_{\sigma(1)} \cdots x_{\sigma(n)}) \\
&= [[x_{\sigma(1)}, x_{\sigma(2)}], \dots, x_{\sigma(n)}].
\end{aligned}$$

The proof is finished. □

**Corollary 3.3.** *There is a one-to-one correspondence, multiplicity preserving, between the decompositions of the functor $L_n$ and the decompositions of* $\mathrm{Lie}(n)$ *over* $\boldsymbol{k}(\Sigma_n)$.

### 3.2. The $T_n$-projective subfunctors of $L_n$

Let $Q$ be a subfunctor of $L_n$. Then $Q$ is a subfunctor of $T_n$ because $L_n$ is a subfunctor of $T_n$. By Proposition 2.10, the functor $Q$ is $T_n$-projective if and only if $\gamma_n(Q)$ is a $\boldsymbol{k}(\Sigma_n)$-projective module. From Corollary 3.3, we have the following.

**Proposition 3.4.** *There is a one-to-one correspondence, multiplicity preserving, between $T_n$-projective subfunctors of $L_n$ and $\boldsymbol{k}(\Sigma_n)$-projective submodules of* $\mathrm{Lie}(n)$ *given by $Q \mapsto \gamma_n(Q)$.*

According to [**22**, Lemma 6.2 and Theorem 7.4], there exists a subfunctor $L_n^{\max}$ of $L_n$ with $\mathrm{Lie}^{\max}(n) = \gamma_n(L_n^{\max})$ that has the following maximum properties:

- $\mathrm{Lie}^{\max}(n)$ is a $\boldsymbol{k}(\Sigma_n)$-projective submodule of $\mathrm{Lie}(n)$ and
- any $\boldsymbol{k}(\Sigma_n)$-projective submodule of $\mathrm{Lie}(n)$ is isomorphic to a summand of $\mathrm{Lie}^{\max}(n)$ as a $\boldsymbol{k}(\Sigma_n)$-module.

From the stated maximum properties, $\mathrm{Lie}^{\max}(n)$ is unique up to isomorphisms of $\boldsymbol{k}(\Sigma_n)$-modules. By the above proposition, $L_n^{\max}$ is unique up to natural equivalences with the maximum properties that

- $L_n^{\max}$ is a $T_n$-projective subfunctor of $L_n$ and
- any $T_n$-projective subfunctor of $L_n$ is isomorphic to a summand of $L_n^{\max}$.

From Proposition 2.10, we have the following.

**Proposition 3.5.** *There is a natural isomorphism*

$$\Phi_V^{L_n^{\max}}\colon \mathrm{Lie}^{\max}(n)\otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n}\to L_n^{\max}(V)$$

*for any module $V$.*

### 3.3. The $\boldsymbol{k}(\mathrm{GL}(V))$-module $L_n(V)$

In this subsection, the ground field $\boldsymbol{k}$ is an infinite field of characteristic $p>0$ and $V$ is a fixed $\boldsymbol{k}$-module with the action of the general linear group $\mathrm{GL}(V)=\mathrm{GL}_m(\boldsymbol{k})$ from the right, where $m=\dim V$. Let $\mathrm{GL}(V)$ act on $T_n(V)=V^{\otimes n}$ through the diagonal, i.e.

$$(a_1\cdots a_n)\cdot g=(a_1g)\cdots(a_ng)$$

for $a_i\in V$ and $g\in\mathrm{GL}(V)$. Recall that the *Schur algebra* is defined by

$$S(V,n)=\mathrm{End}_{\boldsymbol{k}(\Sigma_n)}(V^{\otimes n}),$$

where the left action of $\Sigma_n$ on $V^{\otimes n}$ is given by permuting factors. By the classical Schur–Weyl duality, the group $\mathrm{GL}(V,n)$ generates the algebra $S(V,n)=\mathrm{End}_{\boldsymbol{k}(\Sigma_n)}(V^{\otimes n})$ and so there is an epimorphism of rings

$$\boldsymbol{k}(\mathrm{GL}(V))\to S(V,n).$$

Observe that if $M$ is a sub-quotient of a direct sum of copies of $V^{\otimes n}$, then the $\boldsymbol{k}(\mathrm{GL}(V))$-action factors through its quotient algebra $S(V,n)$. Thus, if $M$ and $N$ are sub-quotients of direct sums of copies of $V^{\otimes n}$, then

$$\mathrm{Hom}_{\boldsymbol{k}(\mathrm{GL}(V))}(M,N)=\mathrm{Hom}_{S(V,n)}(M,N).$$

Recall from [**13**] that the category of $\boldsymbol{k}(\mathrm{GL}(V))$-modules that are sub-quotients of direct sums of copies of $V^{\otimes n}$ is equivalent to the category of modules over the Schur algebra $S(V,n)$, which is denoted by $\mathrm{Mod}(S(V,n))$.

Let $B$ be a sub-quotient of a direct sum of copies of $T_n$. The action of $\mathrm{GL}(V)$ on $V$ induces an action on $B(V)$ via the functor $B$. Thus, $B(V)$ is a module over $\boldsymbol{k}(\mathrm{GL}(V))$. Since $B(V)$ is a sub-quotient of a direct sum of copies of $V^{\otimes n}$, $B(V)$ is an object in $\mathrm{Mod}(S(V,n))$. Thus, we have a functor

$$\begin{aligned}&\Theta\colon B\mapsto B(V),\\ &\qquad \mathrm{Hom}(A,B)\to\mathrm{Hom}_{\boldsymbol{k}(\mathrm{GL}(V))}(A(V),B(V))=\mathrm{Hom}_{S(V,n)}(A(V),B(V))\end{aligned}$$

from the category of sub-quotients of direct sums of copies of $T_n$ to $\mathrm{Mod}(S(V,n))$.

**Lemma 3.6.** *Let $B$ be a sub-quotient of a free $T_n$-functor and let $V$ be a module with $\dim(V)\geqslant n$. Then $B=0$ if and only if $B(V)=0$.*

**Proof.** If $B = 0$, clearly $B(V) = 0$. Assume that $B(V) = 0$. Let $B = \tilde{B}/B'$ with $B' \hookrightarrow \tilde{B} \hookrightarrow F$, where $F$ is a direct sum of copies of $T_n$. It is routine to check that $\gamma_j(T_n) = 0$ for $j > n$. Thus, $\gamma_j(F) = 0$ for $j > n$ and so

$$\gamma_j(B') = \gamma_j(\tilde{B}) = \gamma_j(B) = 0$$

for $j > n$. Since $B(V) = 0$, we have $B(\bar{V}_n) = 0$ because $\dim V \geqslant \dim \bar{V}_n = n$. Thus, $\gamma_j(B) = 0$ for $j \leqslant n$. The assertion follows by Corollary 2.6. □

**Corollary 3.7.** *Let $A$ and $B$ be sub-quotients of free $T_n$-functors and let $V$ be a module with* $\dim(V) \geqslant n$. *Then*

$$\Theta\colon \operatorname{Hom}(A, B) \to \operatorname{Hom}_{\boldsymbol{k}(\mathrm{GL}(V))}(A(V), B(V)) = \operatorname{Hom}_{S(V,n)}(A(V), B(V))$$

*is a monomorphism.*

**Proof.** Let $f\colon A \to B$ be a natural transformation such that $f_V\colon A(V) \to B(V)$ is 0. Let $C = \operatorname{Im}(f\colon A \to B)$. Then $C(V) = 0$. Thus, $C = 0$ and hence the result. □

A direct sum of finite copies of $T_n$ is called a *finite free $T_n$-functor*.

**Proposition 3.8.** *Let $B$ be a sub-quotient of a finite free $T_n$-functor and let $A$ be a quotient functor of a finite free $T_n$-functor. Suppose that $\dim V \geqslant n$. Then the homomorphism*

$$\Theta_{A,B}\colon \operatorname{Hom}(A, B) \to \operatorname{Hom}_{\boldsymbol{k}(\mathrm{GL}(V))}(A(V), B(V)) = \operatorname{Hom}_{S(V,n)}(A(V), B(V))$$

*is an isomorphism.*

**Proof.** By Schur–Weyl duality, the monomorphism

$$\Theta_{T_n,T_n}\colon \operatorname{Hom}(T_n, T_n) \to \operatorname{Hom}_{\boldsymbol{k}(\mathrm{GL}(V))}(T_n(V), T_n(V))$$

is an epimorphism and so $\Theta_{A,B}$ is an isomorphism when $A$ and $B$ are free $T_n$-functors. According to [**8**, p. 94], $V^{\otimes n}$ is projective over $S(V, n)$. Let $A$ be a free $T_n$-functor. By tracking the exact sequence form

$$\Theta_{A,-}\colon \operatorname{Hom}(A, \cdot) \to \operatorname{Hom}_{S(V,n)}(A(V), -)$$

together with the fact that $\Theta_{A,B}$ is always a monomorphism, we have

$$\Theta_{A,B}\colon \operatorname{Hom}(A, B) \to \operatorname{Hom}_{S(V,n)}(A(V), B(V))$$

for any sub-quotient $B$ of a free $T_n$-functor. Let $A$ be a quotient of a free functor $F$ with an epimorphism $\phi\colon F \to A$. Let $C = \operatorname{Ker}(\phi)$. From the commutative diagram of exact sequences

$$\begin{tikzcd}
\operatorname{Hom}(A,B) \arrow[r, hook, "\phi^*"] \arrow[d, "\Theta_{A,B}"] & \operatorname{Hom}(F,B) \arrow[r] \arrow[d, "\cong\ \Theta_{F,B}"] & \operatorname{Hom}(C,B) \arrow[d, "\Theta_{C,B}"] \\
\operatorname{Hom}_{S(V,n)}(A(V),B(V)) \arrow[r, hook, "\phi^*"] & \operatorname{Hom}_{S(V,n)}(F(V),B(V)) \arrow[r] & \operatorname{Hom}_{S(V,n)}(C(V),B(V))
\end{tikzcd}$$

the monomorphism $\Theta_{A,B}$ is an epimorphism, proving the proposition. □

A $\boldsymbol{k}(\mathrm{GL}(V))$-submodule $M$ of $L_n(V)$ is called *$T_n$-projective* if $M$ is isomorphic to a summand of a direct sum of $T_n(V)$'s as modules over $\boldsymbol{k}(\mathrm{GL}(V))$. Let $\bar{\beta}_n\colon T_n \to L_n$ be the natural epimorphism defined by $\bar{\beta}_n(a_1\cdots a_n) = [[a_1, a_2], \dots, a_n]$ for any module $W$ and any monomial $a_1\cdots a_n \in T_n(W) = W^{\otimes n}$. Let $\beta_n$ be the composite

$$T_n \xrightarrow{\bar{\beta}_n} L_n \xhookrightarrow{i} T_n\ .$$

**Theorem 3.9.** *Suppose that* $\dim V \geqslant n$. *Then*

(i) *the ring homomorphism*

$$\Theta_{L_n,L_n}\colon \mathrm{Hom}(L_n, L_n) \to \mathrm{Hom}_{\boldsymbol{k}(\mathrm{GL}(V))}(L_n(V), L_n(V))$$

*is an isomorphism,*

(ii) *there is a one-to-one correspondence, multiplicity preserving, between summands of the functor $L_n$ and $\boldsymbol{k}(\mathrm{GL}(V))$-summands of $L_n(V)$,*

(iii) *there is a one-to-one correspondence, multiplicity preserving, between $T_n$-projective subfunctors of $L_n$ and $T_n$-projective $\boldsymbol{k}(\mathrm{GL}(V))$-submodules of $L_n(V)$,*

(iv) *for the functor $L_n^{\max}$, the module $L_n^{\max}(V)$ is the maximum $T_n$-projective submodule of $L_n(V)$ in the sense that any $T_n$-projective $\boldsymbol{k}(\mathrm{GL}(V))$-submodule of $L_n(V)$ is isomorphic to a summand of $L_n^{\max}(V)$,*

(v) *for any choice of the functor $L_n^{\max}$, the socle $\mathrm{Soc}(L_n^{\max}(V))$ is uniquely determined by*

$$\bar{\beta}_n(\mathrm{Soc}(V^{\otimes n})) = \beta_n(\mathrm{Soc}(V^{\otimes n})),$$

(vi) *for any choice of the functor $L_n^{\max}$, the head $\mathrm{Hd}(L_n^{\max}(V))$ is uniquely determined by*

$$\beta_n(\mathrm{Hd}(V^{\otimes n})).$$

**Remark.** By assertion (v), the functor $L_n^{\max}$ and the module $L_n^{\max}(V)$ are determined by evaluating the map $\bar{\beta}_n$ or $\beta_n$ on simple $\boldsymbol{k}(\mathrm{GL}(V))$-submodules of $V^{\otimes n}$.

**Proof.** Since $L_n$ is a quotient functor of $T_n$, assertion (i) is a direct consequence of Proposition 3.8. Assertion (ii) follows from (i) immediately. Assertion (iv) is a direct consequence of (iii). The proof of assertion (vi) is similar to that of assertion (v).

For proving assertion (iii), let $M$ be a $T_n$-projective $\boldsymbol{k}(\mathrm{GL}(V))$-submodule of $L_n(V)$. According to [**8**, p. 94], $T_n(V)$ is an injective module over $S(V, n)$ and so is $M$ because $M$ is a summand of a direct sum of finite copies of $T_n$. Thus, the inclusion

$$j\colon M \hookrightarrow L_n(V) \hookrightarrow T_n(V)$$

admits a $\boldsymbol{k}(\mathrm{GL}(V))$-retraction $r\colon T_n(V) \to M$. The composite

$$e = j \circ r\colon T_n(V) \to T_n(V)$$

https://doi.org/10.1017/S0013091510000015 Published online by Cambridge University Press

is an idempotent in

$$\mathrm{End}_{\boldsymbol{k}(\mathrm{GL}(V))}(T_n(V)).$$

The natural transformation $\alpha = \Theta^e\colon T_n \to T_n$ is an idempotent. Let $B = \mathrm{Im}(\alpha)$. Then $B$ is a $T_n$-projective subfunctor of $L_n$ with $B(V) = M$ and hence assertion (iii).

(v) Let

$$V^{\otimes n} = \bigoplus_{i \in I} P_i$$

be a decomposition over $S(V, n)$ such that each $P_i$ is indecomposable. The map $\bar{\beta}_n\colon V^{\otimes n} \to L_n(V)$ induces a map

$$\bar{\beta}_n\colon \mathrm{Soc}(V^{\otimes n}) = \bigoplus_{i \in I} \mathrm{Soc}(P_i) \to \mathrm{Soc}(L_n(V)).$$

Note that each indecomposable $S(V, n)$-summand of $V^{\otimes n}$ has a unique socle (see, for example, [**13**, (6.4b)]). Thus, there exists $I' \subseteq I$ such that

$$P = \bigoplus_{i \in I'} P_i$$

has the property that

$$\bar{\beta}_n|\colon \mathrm{Soc}(P) \to \bar{\beta}_n(\mathrm{Soc}(V^{\otimes n}))$$

is an isomorphism. It follows that

$$\bar{\beta}_n|\colon P \to L_n(V)$$

is a monomorphism because its restriction to the socle is a monomorphism. Since $P$ is an injective $S(V, n)$-module, the map $\bar{\beta}_n|_P$ has a retraction. Thus, the $T_n$-projective $S(V, n)$-module $P$ is isomorphic to a $S(V, n)$-summand of $L_n(V)$. From the maximum property of $L^{\max}(n)$, $P$ is isomorphic to a $S(V, n)$-summand of $L_n^{\max}(V)$. In particular,

$$\bar{\beta}_n(\mathrm{Soc}(V^{\otimes n})) = \bar{\beta}_n|(\mathrm{Soc}(P)) \subseteq \mathrm{Soc}(L_n^{\max}(V)).$$

On the other hand, since $L_n^{\max}(V)$ is $S(V, n)$-projective, the inclusion

$$j\colon L_n^{\max}(V) \hookrightarrow L_n(V)$$

admits a $S(V, n)$-lifting $\tilde{j}\colon L_n^{\max}(V) \to V^{\otimes n}$ such that $j = \bar{\beta} \circ \tilde{j}$. Thus,

$$\mathrm{Soc}(L_n^{\max}(V)) \subseteq \bar{\beta}_n(\mathrm{Soc}(V^{\otimes n})).$$

Note that the inclusion $i\colon L_n(V) \hookrightarrow T_n(V)$ induces a monomorphism $i|\colon \mathrm{Soc}(L_n(V)) \hookrightarrow \mathrm{Soc}(T_n(V))$. Thus,

$$\bar{\beta}_n(\mathrm{Soc}(V^{\otimes n})) = \beta_n(\mathrm{Soc}(V^{\otimes n}))$$

and hence the result follows. □

We remark that if $\dim V < n$, then assertions (iii) and (iv) are not true by the following example.

**Example 3.10.** Let $\boldsymbol{k}$ be of characteristic 3 and let $V$ be a two-dimensional module with basis $\{u, v\}$. Then the canonical map

$$f\colon L_2(V) \otimes V \to L_3(V)[a_1, a_2] \otimes a_3 \mapsto [[a_1, a_2], a_3]$$

is an isomorphism of modules over $\boldsymbol{k}(\mathrm{GL}_2(\boldsymbol{k}))$. Since $L_2(V)$ is a $\boldsymbol{k}(\mathrm{GL}(V))$-summand of $V^{\otimes 2}$, $L_2(V) \otimes V$ is $T_3$-projective. Thus, $L_3(V)$ is $T_3$-projective. On the other hand, it is easy to see that the functor $L_3^{\max} = 0$ and so $L_3^{\max}(V) = 0$.

In this case, $L_3(V)$ is not an injective $S(V, 3)$-module. In fact, the inclusion $L_3(V) \hookrightarrow V^{\otimes 3}$ does not have an $S(V, 3)$-retraction by inspecting the Steenrod module structure on $V^{\otimes 3}$. Also it is easy to check that $L_3(V)$ is not a projective $S(V, 3)$-module.

We call $M \subseteq L_n(V)$ *functorial $T_n$-projective* if there exists a $T_n$-projective subfunctor $Q$ of $L_n$ such that $M = Q(V)$. (Note that here we require that $Q(V)$ is strictly equal to $M$ rather than just isomorphic to $M$.)

**Proposition 3.11.** *Assume that the ground field $\boldsymbol{k}$ has infinitely many elements. Let $V$ be any $\boldsymbol{k}$-module and let $M$ be a $\boldsymbol{k}(\mathrm{GL}(V))$-submodule of $L_n(V)$. Then $M$ is functorial $T_n$-projective if and only if $M$ satisfies the following two conditions:*

(i) *there exists a $\boldsymbol{k}(\mathrm{GL}(V))$-linear map $r\colon V^{\otimes n} \to M$ such that $r|_M$ is the identity;*

(ii) *the inclusion $M \hookrightarrow L_n(V)$ admits the following lifting:*

$$\begin{tikzcd} & V^{\otimes n} \arrow[d, "\bar{\beta}_n"] \\ M \arrow[ur, dotted] \arrow[r, hook] & L_n(V) \end{tikzcd}$$

*as modules over $\boldsymbol{k}(\mathrm{GL}(V))$.*

**Proof.** Suppose that $M$ is functorial $T_n$-projective. Let $Q$ be a subfunctor of $L_n$ with $Q(V) = M$. The inclusion

$$Q \hookrightarrow L_n \hookrightarrow T_n$$

admits a natural retraction because $\gamma_n(Q)$ is injective. By evaluating at $V$, condition (i) is satisfied. Since $\gamma_n(Q)$ is projective, there a natural lifting $\tilde{j}\colon Q \to T_n$ such that $\beta_n \circ \tilde{j}$ is the inclusion of $Q$ in $L_n$ and so condition (ii) is satisfied by evaluating at $V$.

Conversely, suppose that $M$ satisfies conditions (i) and (ii). Let $j\colon M \hookrightarrow L_n(V)$ and $L_n(V) \hookrightarrow V^{\otimes n}$ be the inclusions. Let $\tilde{j}\colon M \to V^{\otimes n}$ be a $\boldsymbol{k}(\mathrm{GL}(V))$-map such that $\beta_n \circ \tilde{j} = j$. By the Schur–Weyl duality, the map

$$\boldsymbol{k}(\Sigma_n) = \mathrm{Hom}(T_n, T_n) \to \mathrm{End}_{\boldsymbol{k}(\mathrm{GL}(V))}(V^{\otimes n})$$

is an epimorphism. There exists a natural transformation $\alpha\colon T_n \to T_n$ such that $\alpha_V = \tilde{j} \circ r$. Let $\theta = \beta_n \circ \alpha$. Consider the colimit of the sequence

$$T_n \xrightarrow{\theta} T_n \xrightarrow{\theta} T_n \to \cdots .$$

There exists $k \gg 0$ such that

$$Q = \mathrm{Im}(\theta^k) \to \mathrm{colim}_\theta\, T_n$$

is an isomorphism because, by taking $\gamma_n(-)$ to the above sequence, the submodules $\mathrm{Im}(\gamma_n(\theta^t))$ of $\gamma_n(T_n)$ are monotone decreasing in dimension:

$$\dim \mathrm{Im}(\gamma_n(\theta)) \geqslant \dim \mathrm{Im}(\gamma_n(\theta^2)) \geqslant \dim \mathrm{Im}(\gamma_n(\theta^3)) \geqslant \cdots .$$

Since $Q = \beta_n(\alpha\theta^{k-1}(T_n))$, $Q$ is a $T_n$-projective subfunctor of $L_n$. By evaluating at $V$, we check that $Q(V) = M$. Since

$$\begin{aligned}
\theta_V \circ \theta_V &= \beta_n \circ \alpha_V \circ \beta_n \circ \alpha_V \\
&= i \circ \bar{\beta}_n \circ \tilde{j} \circ r \circ i \circ \bar{\beta}_n \circ \tilde{j} \circ r \\
&= i \circ j \circ r \circ i \circ j \circ r \\
&= i \circ j \circ r \\
&= \theta_V,
\end{aligned}$$

the map $\theta_V$ is an idempotent. It follows that

$$Q(V) = \mathrm{Im}(\theta_V) = i \circ j \circ r(V^{\otimes n}) = M$$

and hence the result. □

## 4. Coalgebra structure on tensor algebras

In this section, the tensor algebra $T(V)$ admits the comultiplication

$$\psi\colon T(V) \to T(V) \otimes T(V)$$

described in § 2.1.

### 4.1. Changing ground-rings

Some results in representation theory help us to change the ground ring. Let $\mathbb{Z}_{(p)}$ be the $p$-local integers. By the modular representation theory of symmetric groups (see, for example, [**7**, Exercise 6.16, p. 142]), any idempotent in $(\mathbb{Z}/p)(\Sigma_n)$ lifts to an idempotent in $\mathbb{Z}_{(p)}(\Sigma_n)$. It is well known [**14**] that any irreducible module $M$ over $\mathbb{Z}/p(\Sigma_n)$ is absolutely irreducible, that is, for any extension field $\boldsymbol{k}$, $M \otimes \boldsymbol{k}$ is irreducible over $\boldsymbol{k}(\Sigma_n)$. Thus, there is a one-to-one correspondence between idempotents in $\mathbb{Z}/p(\Sigma_n)$ and idempotents in $\boldsymbol{k}(\Sigma_n)$.

Let $R$ be any commutative ring with identity. Consider $T\colon V \mapsto T(V)$ as the functor from projective $R$-modules to coalgebras over $R$. Denote by $\mathrm{coalg}^R(T,T)$ the set of natural coalgebra self-transformations of $T$. Let $\boldsymbol{k}$ be any field of characteristic $p$. We have canonical functions

$$R\colon \mathrm{coalg}^{\mathbb{Z}_{(p)}}(T,T) \to \mathrm{coalg}^{\mathbb{Z}/p}(T,T)$$

by reducing mod-$p$ and

$$K\colon \mathrm{coalg}^{\mathbb{Z}/p}(T,T) \to \mathrm{coalg}^{\boldsymbol{k}}(T,T)$$

by tensoring with $\boldsymbol{k}$ over $\mathbb{Z}/p$. By [**22**, Corollary 6.9], there is a one-to-one correspondence between natural indecomposable retracts of $T$ over $\boldsymbol{k}$ and indecomposable $\boldsymbol{k}(\Sigma_n)$-projective submodules of $\mathrm{Lie}(n)$ for $n \geqslant 1$. Thus, we have the following.

**Proposition 4.1.** *The functions $R$ and $K$ have the following properties.*

- *The map $R\colon \mathrm{coalg}^{\mathbb{Z}_{(p)}}(T,T) \to \mathrm{coalg}^{\mathbb{Z}/p}(T,T)$ induces a one-to-one correspondence betweens idempotents. Thus, every natural coalgebra decomposition of $T$ over $\mathbb{Z}/p$ lifts to a natural coalgebra decomposition over $\mathbb{Z}_{(p)}$.*
- *The map $K\colon \mathrm{coalg}^{\mathbb{Z}_p}(T,T) \to \mathrm{coalg}^{\boldsymbol{k}}(T,T)$ induces a one-to-one correspondence between idempotents. Thus, natural coalgebra decompositions of $T$ depend only on the characteristic of the ground field.*

By this proposition, we can freely change between ground fields with the same characteristic and lift natural coalgebra decompositions to the $p$-local integers if necessary.

### 4.2. Block decompositions

Henceforth in this section, the ground field $\boldsymbol{k}$ is algebraically closed with $\mathrm{char}(\boldsymbol{k}) = p$. For any coalgebra $C$, let $PC$ be the set of the primitives of $C$. If $C$ is a functor from modules to coalgebras, then $PC$ is a functor from modules to modules. Recall from Corollary 2.4 that if $C$ is a sub-quotient coalgebra functor of $T^{\mathrm{C}}$ such that $C$ is a natural summand of $T^{\mathrm{M}}$, then $C$ is graded and so we have the homogenous functors $C_n$ and $P_nC = PC \cap C_n$ for each $n$. For the case $C = T$, $PT(V) = L^{\mathrm{res}}(V)$ is the free restricted Lie algebra generated by $V$ and $P_nT = L^{\mathrm{res}}_n$ for each $n$.

For natural transformations $f, g\colon T \to T$, the *convolution product* $f * g$ is defined by the composite

$$T(V) \xrightarrow{\psi} T(V) \otimes T(V) \xrightarrow{f\otimes g} T(V) \otimes T(V) \xrightarrow{\text{multi.}} T(V).$$

If $f$ and $g$ are natural coalgebra transformations, clearly $f * g$ is also a natural coalgebra transformation. For any element $\zeta \in \boldsymbol{k}$, define $\lambda_\zeta\colon T(V) \to T(V)$ by setting

$$\lambda_\zeta(a_1 \cdots a_n) = \zeta^n a_1 \cdots a_n \tag{4.1}$$

for $a_1, \ldots, a_n \in V$. In other words, $\lambda_\zeta\colon T(V) \to T(V)$ is the (unique) Hopf map such that $\lambda_\zeta(a) = \zeta a$ for $a \in V$. Let $\chi\colon T(V) \to T(V)$ be the conjugation of the Hopf algebra $T(V)$, namely $\chi$ is the anti-homomorphism such that $\chi(a) = -a$ for $a \in V$. More precisely,

$$\chi(a_1 \cdots a_n) = (-1)^n a_n a_{n-1} \cdots a_1 \tag{4.2}$$

https://doi.org/10.1017/S0013091510000015 Published online by Cambridge University Press

for $a_1, \dots, a_n \in V$. For any element $\zeta \in \boldsymbol{k}$, we have the natural coalgebra transformation

$$\theta_\zeta = \lambda_\zeta * \chi\colon T(V) \to T(V). \tag{4.3}$$

If $\alpha \in P_n T(V)$, then

$$\theta_\zeta(\alpha) = (\zeta^n - 1)\alpha \tag{4.4}$$

by the definition of convolution product because $\psi(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha$. For general monomials in $T_n(V)$, it is straightforward to check that we have the formula

$$\theta_\zeta(a_1 \cdots a_n) = \sum_{\substack{\sigma(1)<\cdots<\sigma(k)\\ \sigma(k+1)<\cdots<\sigma(n)\\ \sigma\in\Sigma_n\\ 0\leqslant k\leqslant n}} (\zeta^k + (-1)^{n-k}) a_{\sigma(1)} \cdots a_{\sigma(k)} a_{\sigma(k+1)} \cdots a_{\sigma(n)}. \tag{4.5}$$

The maps $\theta_\zeta$ are useful for obtaining natural coalgebra decompositions of $T(V)$.

**Theorem 4.2.** *Let the ground ring $\boldsymbol{k}$ be a field of characteristic $p$. Then there exists a natural coalgebra decomposition*

$$T(V) \cong C(V) \otimes D(V)$$

*for any module $V$ with the property that*

$$PC_n = \begin{cases} 0 & \text{if } n \text{ is not a power of } p,\\ P_n T & \text{if } n = p^r \text{ for some } r. \end{cases}$$

**Remark.** From the decomposition, we have $P_n D = 0$ if $n$ is a power of $p$ and $P_n D = P_n T$ if $n$ is not a power of $p$. The theorem allows one to give a decomposition that puts all primitives of tensor length a power of $p$ into one coalgebra factor and all the remaining primitives into the other coalgebra factor.

**Proof.** Let $\{m_1 < m_2 < m_3 < \cdots\}$ be the set of all positive integers prime to $p$ excluding 1 and let $\zeta_{m_i}$ be a primitive $m_i$th root of 1. We shall construct by induction a sequence of sub-coalgebra functors $C(k)$ of $T$, with the inclusion denoted by $j_k\colon C(k) \hookrightarrow T$, and a sequence of quotient coalgebra functors $q_k\colon T \to E(k)$ with the following properties:

(i) $C(k+1)$ is a subfunctor $C(k)$ for each $k \geqslant 0$;

(ii) there exists a coalgebra natural transformation $q'_k\colon E(k) \to E(k+1)$ such that $q_{k+1} = q'_k \circ q_k$ for each $k \geqslant 0$;

(iii) the composite $q_k \circ j_k\colon C(k) \to E(k)$ is a natural isomorphism;

(iv) $P_n C(k) = 0$ if $n$ is divisible by one of $m_1, m_2, \dots, m_k$;

(v) $P_n C(k) = P_n T$ if $n$ is not divisible by any of $m_1, m_2, \dots, m_k$.

Let $C(0) = E(0) = T$ and let $i_0 = q_0 = \mathrm{id}$. The construction of $C(1)$ and $E(1)$ is as follows. Let $E(1) = \operatorname{colim}_{\theta_{\zeta_{m_1}}} T$ be the colimit of the sequence of coalgebra natural transformations

$$T \xrightarrow{\theta_{\zeta_{m_1}}} T \xrightarrow{\theta_{\zeta_{m_1}}} T \to \cdots.$$

Let $q_1 \colon T \to E(1)$ be the map to its colimit. By [**22**, Theorem 4.5], there exists a sub-coalgebra functor $C(1)$ of $T$, with the inclusion denoted by $j_1 \colon C(1) \to T$, such that $q_1 \circ i_1$ is a natural isomorphism. From Equation (4.4),

$$\theta_{\zeta_{m_1}} \colon P_nT \to P_nT$$

is zero if $m_1|n$ and an isomorphism if $m_1 \nmid n$. Thus, $P_nE(1) = \operatorname{colim}_{\theta_{\zeta_{m_1}}} P_nT = 0$ if $m_1|n$ and

$$q_1 \colon P_nT \to P_nE(1)$$

is an isomorphism if $m_1 \nmid n$. Since $C(1) \cong E(1)$, conditions (iv) and (v) hold. Now suppose that we have constructed $C(j)$ and $E(j)$ satisfying conditions (i)–(v) for $j \leqslant k$. Let $f \colon T \to T$ be the composite

$$T \xrightarrow{q_k} E_n \xrightarrow[\cong]{(q_k \circ j_k)^{-1}} C(k) \xrightarrow{j_k} T \xrightarrow{\theta_{\zeta_{m_{k+1}}}} T$$

and let $E(k+1) = \operatorname{colim}_f T$. Let $q_{k+1} \colon T \to E(k+1)$ be the canonical map to its colimit. Notice that

$$q_{k+1} \circ f = q_{k+1} \colon T \to E(k+1).$$

Let $q'_k = q_{k+1} \circ j_k \circ (q_k \circ j_k)^{-1}$. Then $q_{k+1} = q'_k \circ q_k$ and so condition (ii) is satisfied. Since $f$ factors through the subfunctor $C(k)$, there exists a subfunctor $C(k+1)$ of $C(k)$, with the inclusion into $T$ denoted by $j_{k+1}$, such that $q_{k+1} \circ j_{k+1}$ is a natural isomorphism. Hence, we have conditions (i) and (iii). Let $\alpha \in P_nT(V)$. Then

$$f(\alpha) = \theta_{\zeta_{m_{k+1}}}((j_k \circ (q_k \circ j_k)^{-1} \circ q_k)(\alpha)) = (\zeta^n_{m_{k+1}} - 1)((j_k \circ (q_k \circ j_k)^{-1} \circ q_k)(\alpha)).$$

Thus, $f(\alpha) = 0$ if $n$ is divisible by one of $m_1, \dots, m_{k+1}$ and

$$f \colon P_nT \to P_nT$$

is an isomorphism if $n$ is not divisible by any of $m_1, \dots, m_{k+1}$. It follows that $P_nE(k+1) = 0$ if $n$ is divisible by one of $m_1, \dots, m_{k+1}$ and

$$q_{k+1} \colon P_nT \to P_nE(k+1)$$

is an isomorphism if $n$ is not divisible by any of $m_1, \dots, m_{k+1}$. Since $C(k+1) \cong E(k+1)$, we have conditions (iv) and (v). The induction is finished.

Now let

$$C = \bigcap_{k=0}^{\infty} C(k)$$

be the intersection of the subfunctors $C(k)$ of $T$ and let $E(\infty)$ be the colimit of the sequence

$$T \xrightarrow{q_1} E(1) \xrightarrow{q_2'} E(2) \xrightarrow{q_3'} E(3) \longrightarrow \cdots .$$

From condition (iii), each $C(k)$ is coalgebra retract of $T$ and so each $C(k)$ is a functor from modules to coassociative and cocommutative quasi-Hopf algebras with the multiplication on $C(k)$ given by

$$C(k) \otimes C(k) \hookrightarrow T \otimes TT \xrightarrow{\text{multi}} C(k),$$

where we use the definition of a quasi-Hopf algebra given in [**18**]. By conditions (i)–(iii), $C(k+1)$ is a coalgebra retract of $C(k)$ and so there is a natural coalgebra decomposition

$$C(k) \cong C(k+1) \otimes C'(k)$$

by [**22**, Lemma 5.3]. From conditions (iv) and (v), $P_nC(k+1) = P_nC(k)$ for $n < m_{k+1}$ and so $P_nC'(k) = 0$ for $n < m_{k+1}$. It follows that

$$C'(k)_n = 0$$

for $0 < n < m_{k+1}$. Thus,

$$C(k+1)_n = C(k)_n$$

for $n < m_{k+1}$ and from conditions (i)–(iii),

$$q_k : E(k)_n \to E(k+1)_n$$

is an isomorphism for $n < m_{k+1}$. Notice that the integers $m_k \to \infty$ as $k \to \infty$. Let $n$ be a fixed positive integer. For the integers $k$ with $m_k > n$, we have $C_n = C(k)_n$ and

$$E(k)_n \underset{\cong}{\xrightarrow{q_k'}} E(k+1)_n \underset{\cong}{\xrightarrow{q_{k+1}'}} E(k+2)_n \dashrightarrow \cdot$$

It follows that the composite

$$C_n = C(k)_n \xhookrightarrow{j_k} T_n \xrightarrow{q_k} E(k)_n \longrightarrow E(\infty)_n$$

is an isomorphism. Thus, the composite

$$C \hookrightarrow T \to E(\infty)$$

is an isomorphism and so $C$ is a coalgebra retract of $T$. This gives a natural coalgebra decomposition

$$T \cong C \otimes D$$

for some coalgebra retract $D$ of $T$. From conditions (iv) and (v), we have $P_nC = 0$ if $n$ is not a power of $p$ and $PC_{p^r} = PT_{p^r}$ for $r \geqslant 0$. The proof is finished. □

https://doi.org/10.1017/S0013091510000015 Published online by Cambridge University Press

**Theorem 4.3 (Block Decomposition Theorem).** *Let $\boldsymbol{k}$ be a field of characteristic $p$. Let $\{m_i\}_{i\geqslant 0}$ be the set of all positive integers prime to $p$ with the order that $m_0 = 1 < m_1 < m_2 < \cdots$. Then there exist natural coalgebra retracts $C^{m_i}$ of $T$ with a natural coalgebra decomposition*

$$T(V) \cong \bigotimes_{i=0}^{\infty} C^{m_i}(V)$$

*such that*

$$P_n C^{m_i} = \begin{cases} P_n T & \text{if } n = m_i p^r \text{ for some } r \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** We shall show by induction that there exist natural coalgebra retracts $C^{m_i}$ of $T$, for $0 \leqslant i \leqslant k$, with a natural coalgebra decomposition

$$T(V) \cong \left( \bigotimes_{i=0}^{k} C^{m_i}(V) \right) \otimes D^k(V) \tag{4.6}$$

for some natural coalgebra retract $D^k$ of $T$ such that

$$P_n C^{m_i} = \begin{cases} P_n T & \text{if } n = m_i p^r \text{ for some } r \geqslant 0, \\ 0 & \text{otherwise,} \end{cases}$$

for $0 \leqslant i \leqslant k$. The statement holds for $k = 0$ by Theorem 4.2, where $C^{m_0}$ is the natural coalgebra retract $C$ of $T$ given in Theorem 4.2. Suppose that the statement holds for $k$. Along the lines of the proof of Theorem 4.2, using $\{\theta_{\zeta_{m_i}}\}$ for $i \geqslant k+2$, there is a natural coalgebra decomposition

$$D^k(V) \cong C^{m_{k+1}}(V) \otimes D^{k+1}(V).$$

In brief, we first construct $E^{m_{k+1}}(1) = \operatorname{colim}_g T$ as the colimit of the map $g$ given by the composite

$$T \twoheadrightarrow D^k \hookrightarrow T \xrightarrow{\theta_{\zeta_{m_{k+2}}}} T$$

and then take an inductive construction along the lines of the proof of Theorem 4.2 by pre-composing with $T \twoheadrightarrow D^k \hookrightarrow T$. This gives a monotone decreasing sequence of natural coalgebra retracts $C^{m_{k+1}}(i)$ of $D^k$ for $i = 1, 2, \ldots$ and the resulting natural coalgebra retract $C^{m_{k+1}} = \bigcap_{i=1}^{\infty} C^{m_{k+1}}(i)$ of $D^k$ has the property, on the level of primitives, that $P_n C^{m_{k+1}} = 0$ if $n$ is divisible by one of $m_{k+2}, m_{k+3}, \ldots$ and

$$P_n C^{m_{k+1}} = P_n D^k$$

if $n$ is not divisible by any $m_i$ with $i \geqslant k+2$. Together with the fact that $P_n D = 0$ if $n = m_i p^r$ for some $0 \leqslant i \leqslant k$ and $r \geqslant 0$ and $P_n D = P_n T$ otherwise, we have $P_n C^{m_{k+1}} = P_n T$ if $n = m_{k+1} p^r$ for some $r \geqslant 0$ and $P_n C^{m_{k+1}} = 0$ otherwise. The induction is finished.

https://doi.org/10.1017/S0013091510000015 Published online by Cambridge University Press

Now decomposition (4.6) induces a commutative diagram

$$\begin{tikzcd} T \cong \left(\bigotimes_{i=0}^{k} C^{m_i}\right) \otimes D^k \cong \left(\bigotimes_{i=0}^{k+1} C^{m_i}\right) \otimes D^{k+1} \arrow[r, "q_{k+1}", "\text{proj.}"'] \arrow[dr, "q_k"] & \bigotimes_{i=0}^{k+1} C^{m_i} \arrow[d, "\text{proj.}"] \\ & \bigotimes_{i=0}^{k} C^{m_i} \end{tikzcd}$$

that induces a natural coalgebra transformation

$$T \xrightarrow{q} \bigotimes_{i=0}^{\infty} C^{m_i},$$

which is an isomorphism because, for each $n$, $D^k_n = 0$ for sufficiently large $k \gg 0$. This finishes the proof. $\square$

## 5. Proof of Theorem 1.1

In this section, the ground ring is a field $\boldsymbol{k}$ of characteristic $p$.

**Lemma 5.1.** *Let $Q$ be a $T_n$-projective subfunctor of $L^{\mathrm{res}}_n$. Then $Q$ is a subfunctor of $L_n$ and the sub Hopf algebra $T(Q(V))$ of $T(V)$ generated by $Q(V)$ is a natural coalgebra retract of $T(V)$.*

**Proof.** It is easy to see that $\gamma_n(L^{\mathrm{res}}_n) = \gamma_n(L_n) = \mathrm{Lie}(n)$. By Proposition 3.1, the image of the natural transformation

$$\Phi_V^{L^{\mathrm{res}}_n} : \gamma_n(L^{\mathrm{res}}_n) \otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \to L^{\mathrm{res}}_n(V)$$

is $L_n(V)$. Since $Q$ is $T_n$-projective,

$$\Phi_V^{Q} \colon \gamma_n(Q) \otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \to Q(V)$$

is an isomorphism by Proposition 2.10 (i). From the commutative diagram

$$\begin{tikzcd} \gamma_n(Q) \otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \arrow[r] \arrow[d, "\cong"', "\Phi_V^Q"] & \gamma_n(L^{\mathrm{res}}_n) \otimes_{\boldsymbol{k}(\Sigma_n)} V^{\otimes n} \arrow[d, "\Phi_V^{L^{\mathrm{res}}_n}"] \\ Q(V) \arrow[r, hook] & L^{\mathrm{res}}_n(V) \end{tikzcd}$$

we have $Q \subseteq L_n$. By Proposition 2.10 (iv), there is a natural linear transformation

$$r_V \colon T_n(V) = V^{\otimes n} \to Q(V)$$

with $r_V|_{Q(V)} = \mathrm{id}_{Q(V)}$. Let

$$H_n\colon T(V) \to T(V^{\otimes n})$$

be the algebraic James–Hopf map induced by taking the homology of the geometric James–Hopf map. Then there is a commutative diagram

$$\begin{tikzcd}
T(Q(V)) \arrow[r, hook] \arrow[dr, equal] & T(L_n(V)) \arrow[r, hook] \arrow[dr, "T(j_V)"] & T(V) \arrow[d, "H_n"] \\
& T(Q(V)) & T(V^{\otimes n}) \arrow[l, "T(r_V)"']
\end{tikzcd}$$

where the maps in the top row are the inclusions of sub-Hopf algebras, $j_V$ is the canonical inclusion and the right triangle commutes by the geometric realization theorem in [**28**, Theorem 1.1]. Thus, the sub-Hopf algebra $T(Q(V))$ of $T(V)$ admits a natural coalgebra retraction and hence the result. □

A natural sub-Hopf algebra $B(V)$ of $T(V)$ is called *coalgebra-split* if the inclusion $B(V) \to T(V)$ admits a natural coalgebra retraction. For a Hopf algebra $A$, denote by $QA$ the set of indecomposable elements of $A$. Let $IA$ be the augmentation ideal of $A$. If $B(V)$ is a natural sub-Hopf algebra of $T(V)$, then there is a natural epimorphism $IB(V) \to QB(V)$. Let $Q_nB(V)$ be the quotient of $B_n(V) = IB(V) \cap T_n(V)$ in $QB(V)$.

**Theorem 5.2.** *Let $B(V)$ be a natural sub-Hopf algebra of $T(V)$. Then the following statements are equivalent:*

(i) *$B(V)$ is a natural coalgebra-split sub-Hopf algebra of $T(V)$;*

(ii) *there is a natural linear transformation $r\colon T(V) \to B(V)$ such that $r|_{B(V)}$ is the identity;*

(iii) *each $Q_nB$ is naturally equivalent to a $T_n$-projective subfunctor of $L_n$;*

(iv) *each $Q_nB$ is a $T_n$-projective functor.*

**Proof.** (i) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (iv) are obvious. By [**22**, Theorem 8.6], (ii) $\Longrightarrow$ (i). Thus, (i) $\Longrightarrow$ (ii). From the proof of [**22**, Theorem 8.8], (ii) $\Longrightarrow$ (iii).

(iv) $\Longrightarrow$ (ii). Since $B(V)$ is a sub-Hopf algebra of primitively generated Hopf algebra $T(V)$, $B(V)$ is primitively generated and so

$$r_n\colon P_nB(V) = B(V) \cap L_n^{\mathrm{res}}(V) \to Q_nB(V)$$

is a natural epimorphism, where $L^{\mathrm{res}}(V) = PT(V)$ is the free restricted Lie algebra generated by $V$. Since $Q_nB$ is $T_n$-projective, the map $r_n$ admits a natural cross-section $s_n\colon Q_nB(V) \hookrightarrow P_nB(V)$ by Proposition 2.10 (iv).

Now we show that the inclusion $B(V) \to T(V)$ admits a natural linear retraction. By identifying $Q_nB(V)$ with $s_n(Q_nB(V))$, we have

$$B(V) = T\biggl(\bigoplus_{k=1}^{\infty} Q_kB(V)\biggr) \subseteq T(V).$$

https://doi.org/10.1017/S0013091510000015 Published online by Cambridge University Press

Since each $Q_kB$ is a retract of the functor $T_k$,

$$Q_{i_1}B \otimes \cdots \otimes Q_{i_t}B$$

is a retract of $T_{i_1+i_2+\cdots+i_t}$ for any sequence $(i_1, \dots, i_t)$. Note that $\{Q_iB(V) \mid i \geqslant 1\}$ are algebraically independent. Thus, the summation

$$\sum_{i_1+i_2+\cdots+i_t=q} Q_{i_1}B(V) \otimes Q_{i_2}B(V) \otimes \cdots \otimes Q_{i_t}B(V) \subseteq T_q(V) = V^{\otimes q}$$

is a direct sum. From the fact that

$$\bigoplus_{i_1+i_2+\cdots+i_t=q} Q_{i_1}B \otimes Q_{i_2}B \otimes \cdots \otimes Q_{i_t}B$$

is $T_n$-projective, there is natural linear retraction

$$V^{\otimes q} \to \bigoplus_{i_1+i_2+\cdots+i_t=q} Q_{i_1}B(V) \otimes Q_{i_2}B(V) \otimes \cdots \otimes Q_{i_t}B(V)$$

for any $q \geqslant 1$. Hence, the inclusion $B(V) \to T(V)$ admits a natural linear retraction. □

**Proof of Theorem 1.1.** Let $B(V)$ be the sub-Hopf algebra of $T(V)$ generated by

$$L_{m_ip^r}(V) \quad \text{for } i \in I,\ r \geqslant 0.$$

Let $\{n_j\}_{j \geqslant 1} = \{m_ip^r \mid i \geqslant 1,\ r \geqslant 0\}$ with

$$n_1 = m_1 < n_2 < \cdots .$$

That is, we rewrite the integers $m_ip^r$ in order. Let $B[k](V)$ be the sub-Hopf algebra of $V$ generated by $L_{n_j}(V)$ for $1 \leqslant j \leqslant k$. By Theorem 5.2, it suffices to show that $Q_nB$ is $T_n$-projective for $n \geqslant 1$. Let $n$ be a fixed positive integer. Choose $k$ such that $n_k \geqslant n$. Then the inclusion $B[k] \hookrightarrow B$ induces an isomorphism

$$Q_nB[k] \cong Q_nB$$

because $B[k]$ and $B$ has the same set of generators in tensor length $\leqslant n$. Thus, it suffices to prove the following statement.

For each $k \geqslant 1$, $B[k]$ is coalgebra-split.

The proof of this statement is given by induction on $k$. The statement holds for $k = 1$ by Lemma 5.1 because $L_{m_1}$ is $T_{m_1}$-projective by [**22**, Corollary 6.7] from the assumption that $m_i$ is prime to $p$. Suppose that $B[k-1]$ is coalgebra-split. Thus, there is a coalgebra natural transformation $r\colon T \to B[k-1]$ such that $r|_{B[k-1]}$ is the identity map. Let $n_k = m_ip^r$ for some $i$ and $r$. Let $C^{m_i}$ be the natural coalgebra retract in Theorem 4.3 with a natural coalgebra retraction $r_C\colon T \to C^{m_i}$. Define $f\colon T \to T$ to be the composite

$$T \xrightarrow{r_C} C^{m_i} \hookrightarrow T \xrightarrow{r} B[k-1] \hookrightarrow T. \tag{5.1}$$

https://doi.org/10.1017/S0013091510000015 Published online by Cambridge University Press

Let $\tilde{E} = \mathrm{colim}_f\, T$ be the colimit with the canonical map

$$q\colon T \to \tilde{E}.$$

As in the proof of Theorem 4.2, there exists a coalgebra subfunctor $\tilde{C}$ of $C^{m_i}$ such that

$$q|_{\tilde{C}}\colon \tilde{C} \to \tilde{E}$$

is an isomorphism by [**22**, Theorem 4.5] with a natural coalgebra decomposition

$$C^{m_i} \cong \tilde{C} \otimes \tilde{D}. \tag{5.2}$$

According to [**22**, Lemma 5.3], the subfunctor $\tilde{D}$ of $C^{m_i}$ can be chosen as the cotensor product $\boldsymbol{k} \,\square_{\tilde{\mathrm{E}}}\, C^{m_i}$ under the coalgebra map

$$q|_{C^{m_i}}: C^{m_i} \to \tilde{E}$$

and so there is a left exact sequence

$$P_n\tilde{D} \hookrightarrow P_n C^{m_i} \xrightarrow{P_n q|_{C^{m_i}}} \tilde{E} \tag{5.3}$$

for any $n$.

By restricting the map $f$ as the composite in (5.1) to the primitives, we have the map

$$P_n f\colon P_n T \xrightarrow{P_n r_C} P_n C^{m_i} \hookrightarrow P_n T \xrightarrow{P_n r} P_n B[k-1] \hookrightarrow P_n T.$$

If $n \neq m_i p^t$ for $t \geqslant 0$, then $P_n f = 0$ because $P_n C^{m_i} = 0$. If $n = m_i p^t$ for some $t \geqslant 0$ with $n < n_k$, then $P_n f$ is the identity map because $P_n C^{m_i} = P_n T$ and $P_n B[k-1] = P_n T = L_n^{\mathrm{res}}$ as the sub-Hopf algebra $B[k-1]$ contains $L_{m_i p^s}$ for $s \geqslant 0$. Thus,

$$P_n\tilde{C} = P_n C^{m_i}$$

for $n < n_k$. From decomposition (5.2), we have

$$P_n C^{m_i} = P_n\tilde{C} \oplus P_n\tilde{D} \tag{5.4}$$

for all $n$ and so $P_n\tilde{D} = 0$ for $n < n_k$. It follows that $\tilde{D}_n = 0$ for $0 < n < n_k$ and

$$\tilde{D}_{n_k} = P_{n_k}\tilde{D}. \tag{5.5}$$

Now consider the case $P_n f$ for $n = n_k = m_i p^r$. Since $P_n C^{m_i} = P_n T$, $P_n r_C = \mathrm{id}$ and so $P_n f \circ P_n f = P_n f$ with

$$P_n f(\alpha) = P_n r(\alpha)$$

for $\alpha \in P_n T$. Thus, the composite

$$\mathrm{Im}(P_n f) = P_n B[k-1] \xrightarrow{P_n q|_{B[k-1]}} P_n\tilde{E} = \mathrm{colim}_{P_n f}\, P_n T$$

https://doi.org/10.1017/S0013091510000015 Published online by Cambridge University Press

is an isomorphism. From the exact sequence (5.3), we have

$$P_n\tilde{D} = \operatorname{Ker}(P_n r\colon P_nC^{m_i} = P_nT \to P_nB[k-1]).$$

Let $j\colon P_nB[k-1] \hookrightarrow P_nC^{m_i} = P_nT$ be the inclusion. From the commutative diagram

$$\begin{tikzcd}
& P_n\tilde{D} = \operatorname{Ker}(P_n r) \arrow[d, hook] \arrow[dr, "\cong"] & \\
P_n\tilde{C} \arrow[r, hook, "i"] \arrow[dr, "\cong"'] & P_nC^{m_i} \arrow[r, two heads] \arrow[d, two heads, "P_n r"] & \operatorname{Coker}(i) \\
& P_nB[k-1] &
\end{tikzcd}$$

the summation $P_nB[k-1] + P_n\tilde{D}$ in $P_nT = P_nC^{m_i}$ is a direct sum and there is a decomposition

$$P_nC^{m_i} = P_nT = P_nB[k-1] \oplus P_n\tilde{D}.$$

From definition of $B[k]$, $PB[k](V)$ is the restricted sub-Lie algebra of $L^{\mathrm{res}}(V) = PT(V)$ generated by $L_{n_i}(V)$ for $1 \leqslant i \leqslant k$. It follows that $P_nB[k] = L_n^{\mathrm{res}} = P_nT = P_nC^{m_i}$ and

$$Q_nB[k] \cong P_nB[k]/P_nB[k-1] = P_nC^{m_i}/P_nB[k-1] \cong P_n\tilde{D}.$$

From decomposition (5.2), $\tilde{D}_n$ is a natural summand of $C_n^{\min}$. Since $C^{\min}$ is a coalgebra retract of $T$, $C_n^{\min}$ is a natural summand of $T_n$. Thus, $\tilde{D}_n$ is $T_n$-projective. By identity (5.5), $P_n\tilde{D}$ is $T_n$-projective. Thus, $Q_nB[k]$ is $T_n$-projective. By Theorem 5.2, $B[k]$ is coalgebra-split. The induction is finished and hence the result. □

By inspecting the proof, we obtain the following slightly stronger statement.

**Theorem 5.3.** *Let $\mathcal{M} = \{m_i\}_{i\in I}$ be a finite or infinite set of positive integers prime to $p$ with each $m_i > 1$. Let $f\colon I \to \{0,1,2,\dots\} \cup \{\infty\}$ be a function. Then the sub-Hopf algebra $B^{\mathcal{M},f}(V)$ of $T(V)$ generated by*

$$L_{m_ip^r}(V) \quad \text{for } i \in I,\ 0 \leqslant r < f(i),$$

*is natural coalgebra-split.*

## 6. Decompositions of Lie powers

Let $m = kp^r$ with $k \not\equiv 0 \bmod p$ and $k > 1$. According to [**22**, Theorem 10.7], the functor $L_{kp^r}$ admits the following functorial decomposition:

$$L_{kp^r} = L'_{kp^r} \oplus L_p(L'_{kp^{r-1}}) \oplus \cdots \oplus L_{p^r}(L'_k)$$

for each $r \geqslant 0$ starting with $L'_k = L_k$, where each $L'_{kp^r}$ is a summand of $T_{kp^r}$, which is called $T_{kp^r}$-projective in our terminology. By evaluating on $V$, one gets the decomposition

https://doi.org/10.1017/S0013091510000015 Published online by Cambridge University Press

of the $\boldsymbol{k}(\mathrm{GL}(V))$-module $L_{kp^r}(V)$ given in [**3**, Theorem 4.4] by using an approach different from representation theory, where $L'_{kp^r}(V)$ was denoted by $B_{kp^r}$ in [**3**]. From Theorem 5.3, we can obtain various new decompositions of $L_{kp^r}$, and therefore, by evaluating on $V$, new decompositions of the $\boldsymbol{k}(\mathrm{GL}(V))$-module $L_{kp^r}(V)$.

Let $\mathcal{M} = \{m_i\}_{i\in I}$ be a finite or infinite set of positive integers prime to $p$ with each $m_i > 1$. Let $f\colon I \to \{0,1,2,\dots\} \cup \{\infty\}$ be a function. Let $B^{\mathcal{M},f}(V)$ be the sub-Hopf algebra of $T(V)$ generated by

$$L_{m_ip^r}(V) \quad \text{for } i \in I,\ 0 \leqslant r < f(i).$$

According to Theorem 5.3, $B^{\mathcal{M},f}$ is coalgebra-split and so $Q_nB^{\mathcal{M},f}$ is $T_n$-projective by Theorem 5.2. Since $B^{\mathcal{M},f}(V)$ is a sub-Hopf algebra of the primitively generated Hopf algebra $T(V)$, it is primitively generated by [**18**, Proposition 6.13] and so there is a natural epimorphism

$$\phi_n\colon P_nB^{\mathcal{M},f} \twoheadrightarrow Q_nB^{\mathcal{M},f}.$$

From Proposition 2.10 (iv), the map $\phi_n$ admits a natural cross-section because $Q_nB^{\mathcal{M},f}$ is $T_n$-projective. Thus, there is a subfunctor $D_n^{\mathcal{M},f}$ of $P_nB^{\mathcal{M},f}$ such that

$$\phi_n|\colon D_n^{\mathcal{M},f} \to Q_nB^{\mathcal{M},f}$$

is a natural isomorphism. Since $D_n^{\mathcal{M},f}$ is $T_n$-projective, we obtain

$$D_n^{\mathcal{M},f} \subseteq P_nB^{\mathcal{M},f} \cap L_n$$

along the lines of the proof of Lemma 5.1. Thus, $D_n^{\mathcal{M},f}$ is a $T_n$-projective subfunctor of $L_n$. From the fact that $D^{\mathcal{M},f} \cong Q_nB^{\mathcal{M},f}$ and $B^{\mathcal{M},f}$ is isomorphic to the tensor algebra generated by $Q_nB^{\mathcal{M},f}$ with $n \geqslant 1$, the inclusion

$$\bigoplus_{n=1}^{\infty} D_n \hookrightarrow B^{\mathcal{M},f}$$

induces a natural isomorphism

$$T\biggl(\bigoplus_{n=1}^{\infty} D_n\biggr) \cong B^{\mathcal{M},f}. \tag{6.1}$$

Since the algebra $B^{\mathcal{M},f}$ is generated by $L_{m_ip^r}(V)$ for $m_i \in \mathcal{M}$ and $0 \leqslant r < f(i)$, we have

$$D_n^{\mathcal{M},f} = 0 \text{ if } n \neq m_ip^r \text{ for some } m_i \in \mathcal{M} \text{ and some } 0 \leqslant r < f(i). \tag{6.2}$$

Let $\{m_ip^r \mid m_i \in \mathcal{M},\ 0 \leqslant r < f(i)\} = \{n_1, n_2, \dots\}$ with $n_1 < n_2 < \cdots$ and let $\alpha$ be the cardinality of the set $\{m_ip^r \mid m_i \in \mathcal{M},\ 0 \leqslant r < f(i)\}$. Then decomposition (6.1) becomes

$$T\biggl(\bigoplus_{i=1}^{\alpha} D_{n_i}\biggr) \cong B^{\mathcal{M},f} \tag{6.3}$$

https://doi.org/10.1017/S0013091510000015 Published online by Cambridge University Press

and so we obtain a natural isomorphism

$$PT\biggl(\bigoplus_{i=1}^{\alpha} D_{n_i}\biggr) = L^{\mathrm{res}}\biggl(\bigoplus_{i=1}^{\alpha} D_{n_i}\biggr) \cong PB^{\mathcal{M},f} = B^{\mathcal{M},f} \cap L^{\mathrm{res}}. \tag{6.4}$$

According to Proposition 4.1, the sub-Hopf algebra $B^{\mathcal{M},f}$ is also a natural coalgebra retract of $T$ if we change the ground ring $R$ to the $p$-local integers. Notice that $PT = L$ and $PB^{\mathcal{M},f} = B^{\mathcal{M},f} \cap L$ when $R = \mathbb{Z}_{(p)}$. By changing the ground ring back to $\mathbb{Z}/p$ and then extending it to $\boldsymbol{k}$, we have the following decomposition:

$$L\biggl(\bigoplus_{i=1}^{\alpha} D_{n_i}\biggr) \cong B^{\mathcal{M},f} \cap L. \tag{6.5}$$

We shall apply ideas from the Hilton–Milnor Theorem to determine $B^{\mathcal{M},f} \cap L_n$. Recall the terminology of a *basic product* from [**27**, p. 512]. Let $x_1, \dots, x_k$ be letters. A monomial means a formal product $w = x_{i_1} x_{i_2} \cdots x_{i_n}$ with $1 \leqslant i_1, \dots, i_t \leqslant k$, where the word length $n$ is called the *weight* of $w$. We define the *basic products* of weight $n$ by induction on $n$ and, for each such product, a non-negative integer $r(w)$, called its *rank*. These are to be linearly ordered, in such a way that $w_1 < w_2$ if the weight of $w_1$ is less than the weight of $w_2$. The *serial number* $s(w)$ is the number of basic products $\leqslant w$ in terms of this ordering. The basic products of weight 1 are the letters $x_1, \dots, x_k$ with the order $x_1 < x_2 < \cdots < x_k$. Set $r(x_i) = 0$ and $s(x_i) = i$. Suppose that the basic products of weight less than $n$ have been defined and linearly ordered in such a way that $w_1 < w_2$ if the weight of $w_1$ is less than that of $w_2$, and suppose that the rank $r(w)$ of such a product has been defined. Then the basic products of weight $n$ are all monomials $w_1 w_2$ of weight $n$, for which $w_1$ and $w_2$ are basic products, $w_2 < w_1$ and $r(w_1) \leqslant s(w_2)$. Give these an arbitrary linear order, and define $r(w_1 w_2) = s(w_2)$.

Let $\mathcal{W}_k$ be the set of all basic products on the letters $x_1, \dots, x_k$ by forgetting the ordering. Then

$$\mathcal{W}_k \subseteq \mathcal{W}_{k+1}$$

for each $k$. Let

$$\mathcal{W}_\infty = \bigcup_{k=1}^{\infty} \mathcal{W}_k.$$

The elements in $\mathcal{W}$ are called basic products on the sequence of the letters $x_i$ for $i \geqslant 1$. For each basic product $w = x_{i_1} \cdots x_{i_t} \in \mathcal{W}_\alpha$, define

$$w(D^{\mathcal{M},f}) = D^{\mathcal{M},f}_{n_{i_1}} \otimes \cdots \otimes D^{\mathcal{M},f}_{n_{i_t}} \tag{6.6}$$

with the *tensor length with respect to* $D^{\mathcal{M},f}$

$$d(w) = n_{i_1} + n_{i_2} + \cdots + n_{i_t}$$

and the natural transformation

$$\phi_w \colon w(D^{\mathcal{M},f})(V) \to T\biggl(\bigoplus_{i=1}^{\alpha} D_{n_i}(V)\biggr) \cong B^{\mathcal{M},f}(V)$$

given by

$$\phi_w(z_1 \otimes z_2 \otimes \cdots \otimes z_t) = [[z_1, z_2], \ldots, z_t]$$

for $z_j \in D^{\mathcal{M},f}_{n_{i_j}}(V)$. Then the map $\phi_w$ extends uniquely to a natural transformation of Hopf algebras

$$T\phi_w \colon T(w(D^{\mathcal{M},f})) \to T\biggl(\bigoplus_{i=1}^{\alpha} D_{n_i}(V)\biggr) \cong B^{\mathcal{M},f}(V)$$

by the universal property of tensor algebras. Now by taking homology as in the Hilton–Milnor Theorem [**27**, Theorem 6.7], we have the natural isomorphism of coalgebras

$$\theta\colon \bigotimes_w T(w(D^{\mathcal{M},f})) \xrightarrow{\cong} T\biggl(\bigoplus_{i=1}^{\alpha} D_{n_i}\biggr) \cong B^{\mathcal{M},f}, \tag{6.7}$$

where $w$ runs over all basic products in $\mathcal{W}_\alpha$, the tensor product is linearly ordered and the natural transformation $\theta$ is given by the ordered product of $T\phi_w$, which is well defined because the tensor length $d(w)$ tends to $\infty$ as the weight of $w$ tends to $\infty$. By restricting to Lie powers, we have the decomposition

$$\theta|\colon \bigotimes_w L(w(D^{\mathcal{M},f})) \xrightarrow{\cong} L\biggl(\bigoplus_{i=1}^{\alpha} D_{n_i}\biggr) \cong B^{\mathcal{M},f} \cap L. \tag{6.8}$$

By taking tensor length, we obtain the following decomposition theorem.

**Theorem 6.1.** *Let $\mathcal{M} = \{m_i\}_{i\in I}$ be a finite or infinite set of positive integers prime to $p$ with each $m_i > 1$ and let $f\colon I \to \{0, 1, 2, \ldots\} \cup \{\infty\}$ be a function. Then there exists a $T_{m_ip^r}$-projective subfunctor $D^{\mathcal{M},f}_{m_ip^r}$ of $L_{m_ip^r}$ for each $m_i \in \mathcal{M}$ and $0 \leqslant r < f(i)$ such that*

$$L_{m_ip^r} = \bigoplus_{d(w)|m_ip^r} L_{m_ip^r/d(w)}(w(D^{\mathcal{M},f}))$$

*for $m_i \in \mathcal{M}$ and $0 \leqslant r < f(i)$, where $w$ runs over basic products with $d(w)|m_ip^r$.*

**Remarks.**

- Since each $D_{n_i}$ is $T_{n_i}$-projective, the tensor product $w(D^{\mathcal{M},f})$ is $T_{d(w)}$-projective. If $m_ip^r/d(w)$ is prime to $p$, then the Lie power $L_{m_ip^r/d(w)}(w(D^{\mathcal{M},f}))$ is $T_{m_ip^r}$-projective. Thus, the non-$T_{m_ip^r}$-projective summands of $L_{m_ip^r}$ occur in the factors $L_{m_ip^r/d(w)}(w(D^{\mathcal{M},f}))$ with $m_ip^r/d(w) \equiv 0 \bmod p$.
- The multiplicity of each factor $L_{m_ip^r/d(w)}(w(D^{\mathcal{M},f}))$ can be determined as follows. Let $w$ be a basic product involving the letters $x_{j_1}, \ldots, x_{j_k}$ such that $x_{j_i}$ occurs $l_i$ times and $d(w)|m_ip^r$. According to [**27**, (6.4), p. 514], the multiplicity of the factor $L_{m_ip^r/d(w)}(w(D^{\mathcal{M},f}))$ is given by the formula

$$\frac{1}{l}\sum_{d|l_0} \mu(d)\frac{(l/d)!}{(l_1/d)!\cdots(l_k/d)!},$$

where $\mu$ is the Möbius function, $l_0$ is the greatest common divisor of $l_1, \ldots, l_k$ and $l = l_1 + \cdots + l_k$.

**Example 6.2.** Let $\boldsymbol{k}$ be of characteristic 2. Let $\mathcal{M} = \{m_1 = 3\}$ and let $f(1) = 3$. Then we have the natural coalgebra-split sub-Hopf algebra

$$B^{\mathcal{M},f}(V) = \langle L_3(V), L_6(V), L_{12}(V)\rangle$$

of $T(V)$ with $D_3^{\mathcal{M},f} = L_3$, $D_6^{\mathcal{M},f} \cong L'_6 = L_6/L_2(L_3)$ and

$$D_{12}^{\mathcal{M},f} \cong L_{12}/([L'_6, L'_6] \oplus [[L'_6, L_3], L_3] \oplus L_4(L_3)).$$

From Theorem 6.1, we have the decomposition

$$\begin{aligned} L_{12} &= D_{12} \oplus L(D_3 \oplus D_6) \cap L_{12} \\ &= D_{12} \oplus L_4(D_3) \oplus L_2(D_6) \oplus [[D_6, D_3], D_3] \\ &\cong D_{12} \oplus L_4(L_3) \oplus L_2(L'_6) \oplus [[L'_6, L_3], L_3]. \end{aligned}$$

By comparing this with [**22**, Theorem 10.7] or [**3**, Theorem 4.4], the $T_{12}$-projective summand

$$[[L'_6, L_3], L_3] \cong L'_6 \otimes L_3 \otimes L_3$$

can be recognized in our decomposition for $L_{12}$.

Let $\mathcal{M}$ be the set of all positive integers $m_i$ with $m_i$ prime to $p$ and $m_i > 1$ and let $f(i) = \infty$ for all $i$. Then we have

$$\{m_i p^r \mid m_i \in \mathcal{M} r \geqslant 0\} = \mathbb{N} \setminus \{1, p, p^2, p^3, \dots\}.$$

Let

$$\bar{D}_n = D_n^{\mathcal{M},f}$$

for $n$ not a power of $p$. As a special case of Theorem 6.1, we have the following.

**Corollary 6.3.** *There exists a $T_n$-projective subfunctor $\bar{D}_n$ of $L_n$ for each $n$ not a power of $p$ such that*

$$L_m = \bigoplus_{d(w)|m} L_{m/d(w)}(w(\bar{D}))$$

*for any $m$ not a power of $p$, where $w$ runs over basic products with $d(w)|m$.*

**Acknowledgements.** J.Y.L. was partly supported by the Academic Research Fund from Shijiazhuang Railway Institute. The research of F.C.L. is supported in part by a key grant (no. 10931005) of the NSFC. F.C.L. and J.W. are partly supported by Grant no. 11028104 of the NSFC. The research of J.W. is supported in part by the AcRF Tier 1 (WBS nos R-146-000-101-112 and R-146-000-137-112) of MOE of Singapore.

## References


1. K. Akin, D. A. Buchsbaum and J. Weyman, Schur functors and Schur complexes, *Adv. Math.* **44** (1982), 207–278.
2. R. M. Bryant and M. Johnson, Lie powers and Witt vectors, *J. Algebraic Combin.* **28** (2008), 169–187.
3. R. M. Bryant and M. Schocker, The decomposition of Lie powers, *Proc. Lond. Math. Soc.* **93** (2006), 175–196.
4. R. M. Bryant and R. Stöhr, Lie powers in prime degree, *Q. J. Math.* **56** (2005), 473–48.
5. F. Cohen, On combinatorial group theory in homotopy, *Contemp. Math.* **188** (1995), 57–63.
6. F. R. Cohen, J. C. Moore and J. A. Neisendorfer, Torsion in homotopy groups, *Annals Math. (2)* **109** (1979), 121–168.
7. C. W. Curtis and I. Reiner, Methods of representation theory, *Pure Applied Mathematics*, Volume I (Wiley, 1981).
8. S. Donkin, *The q-Schur algebra*, London Mathematical Society Lecture Notes Series, Volume 253 (Cambridge University Press, 1998).
9. S. Donkin and K. Erdmann, Tilting modules, symmetric functions, and the module structure of the free Lie algebra, *J. Alg.* **203** (1998), 69–90.
10. K. Erdmann and M. Schocker, Modular Lie powers and the Solomon descent algebra, *Math. Z.* **253** (2006), 295–313.
11. J. Grbić, P. S. Selick and J. Wu, The decomposition of the loop space of the mod 2 Moore space, *Alg. Geom. Topology* **8** (2008), 945–951.
12. J. Grbić and J. Wu, Natural transformations of tensor algebras and representations of combinatorial groups, *Alg. Geom. Topology* **6** (2006), 2189–2228.
13. J. A. Green, *Polynomial representation of* $\mathrm{GL}_n$, Lecture Notes in Mathematics, Volume 830 (Springer, 1980).
14. G. James and A. Kerber, *The representation theory of the symmetric groups*, Encyclopedia of Mathematics and Its Applications, Volume 16 (Cambridge University Press, 1981).
15. N. Kuhn, Generic representation theory of the finite general linear groups and the Steenrod algebra, I, *Am. J. Math.* **116** (1994), 327–360.
16. N. Kuhn, Generic representation theory of the finite general linear groups and the Steenrod algebra, II, *K-Theory* **8** (1994), 395–428.
17. N. Kuhn, Generic representation theory of the finite general linear groups and the Steenrod algebra, III, *K-Theory* **9** (1995), 273–303.
18. J. Milnor and J. Moore, On the structure of Hopf algebras, *Annals Math.* **81** (1965), 211–264.
19. I. Schur, Über eine Klasse von Matrizen, die sich einer gegebenen Matrix zourden lassen, in *Gesammelte Abhandlungen I*, pp. 1–70 (Springer, 1973).
20. P. S. Selick, S. D. Theriault and J. Wu, Functorial decompositions of looped coassociative co-$H$ spaces, *Can. J. Math.* **58** (2006), 877–896.
21. P. S. Selick, S. D. Theriault and J. Wu, Functorial homotopy decompositions of looped co-$H$-spaces, *Math. Z.* **267** (2011), 139–153.
22. P. S. Selick and J. Wu, *On natural decompositions of loop suspensions and natural coalgebra decompositions of tensor algebras*, Memoirs of the American Mathematical Society, Volume 148, Number 701 (American Mathematical Society, Providence, RI, 2000).
23. P. S. Selick and J. Wu, On functorial decompositions of self-smash products, *Manuscr. Math.* **111** (2003), 435–457.
24. P. S. Selick and J. Wu, The functor $A^{\min}$ on $p$-local spaces, *Math. Z.* **253** (2006), 435–451.


https://doi.org/10.1017/S0013091510000015 Published online by Cambridge University Press

25. P. S. Selick and J. Wu, Some calculations of $\mathrm{Lie}^{\max}(n)$ for low $n$, *J. Pure Appl. Alg.* **212** (2008), 2570–2580.
26. S. D. Theriault, Homotopy decompositions involving the loops of coassociative Co-$H$ spaces, *Can. J. Math.* **55** (2003), 181–203.
27. G. W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Mathematics (Springer, 1978).
28. J. Wu, On combinatorial calculations of the James–Hopf maps, *Topology* **37** (1998), 1011–1023.
29. J. Wu, *Homotopy theory of the suspensions of the projective plane*, Memoirs of the American Mathematical Society, Volume 162, Number 769 (American Mathematical Society, Providence, RI, 2003).
30. J. Wu, *On maps from loop suspensions to loop spaces and the shuffle relations on the Cohen groups*, Memoirs of the American Mathematical Society, Volume 180, Number 851 (American Mathematical Society, Providence, RI, 2006).
31. J. Wu, The functor $A^{\min}$ for $(p-1)$-cell complexes and EHP sequences, *Israel J. Math.* **178** (2010), 349–391.