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ON A CLASS OF MULTIVALUED MAPPINGS IN BANACH SPACES

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1. **Introduction.** A. Granas [4] has studied single-valued compact vector fields in Banach spaces. In [3], he extended the fixed point theorems of Roth, Boknenblust and Karlin to the case of multi-valued functions. Closely following [4], we give here some general theorems in a class of multi-valued functions in Banach spaces.

Let E be an arbitrary infinite dimensional Banach space and P the space E without the point 0. If x_0 is a point of E and r is a positive number, then we denote by $V(x_0, r)$ an open ball with center x_0 and radius r. If A is a subset of E, then $V(A, r) = \bigcup \{V(x, r) \mid x \in A\}$.

A mapping f defined on the set A and assigning to each $x \in A$ a nonempty set $f(x) \subset E$ is called upper semicontinuous, if the conditions $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $y_n \in F(x_n)$ imply $y \in F(x)$. *In what follows we consider only upper semicontinuous mappings and assume their values to be closed convex sets in E.* The notation $f: A \to E$ denotes an upper semicontinuous mapping defined on A whose every value $f(x)$ is a compact convex set in E.

2. **Compact mappings.** A multi-valued mapping $F: A \to E$ is compact, if the closure of the image $F(A) = \bigcup \{F(x) \mid x \in A\}$ is compact in E.

THEOREM 1. *Let A be a subset of E and $F: A \to E$ is a compact mapping. Then there is a sequence of compact mappings $F_m: A \to E_{n(m)} \subset E$, where $E_{n(m)}$ is a finite dimensional subspace of E, such that if $\varepsilon > 0$ there is a positive integer N such that*

$$F_m(x) \subset V(F(x), \varepsilon) \quad \text{and} \quad F(x) \subset V(F_m(x), \varepsilon) \quad \text{for each } x \in A \quad \text{and} \quad m \geq N.$$

Proof. Let $\varepsilon_n = 1/n$, $n = 1, 2, 3 \cdots$. Since the closure of $F(A)$ is compact, let N_m be a $\frac{1}{3}\varepsilon_m$-net in $F(A)$, and $E_{n(m)}$ be the finite dimensional subspace of E generated by N_m. For each $x \in A$, and each $m = 1, 2, 3, \ldots$, let $N(x, m) = \{y \in \bigcup_{k=1}^m N_k \mid d(y, F(x)) \leq \frac{1}{3}\varepsilon_m\}$, where $d(y, F(x))$ is the distance between the point y and the set $F(x)$, and define $F_n(x)$ to be the convex closure of $N(x, m)$. Let m be a fixed number. Let $x \in A$ and $y \in F(x)$. Since $\bigcup_{k=1}^m N_k$ is also a $\frac{1}{3}\varepsilon_m$-net in $F(A)$, there is a point $y_0 \in \bigcup_{k-1}^m N_k$ such that $\|y - y_0\| \leq \frac{1}{3}\varepsilon_m$. Hence we have $y \in V(F_m(x), \varepsilon_m)$. On the other hand, $N(x, m) \subset V(F(x), \varepsilon_m)$ and $V(F(x), \varepsilon_m)$ is convex, so that $F_m(x) \subset V(F(x), \varepsilon_m)$.

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For each $m=1, 2, 3, \ldots$, and each $x \in A$, $F_m(x)$ is contained in the convex closure of $F(A)$ which is compact so that $\overline{F_m(A)}$ is compact.

Suppose $\lim_{i\to\infty} x_i=x_0$, $x_i \in A$, $y_i \in F_n(x_i)$, $\lim_{i\to\infty} y_i=y_0$. We first prove that if $w_i \in N(x_i, n) \subset \bigcup_{k=1}^n N_k$ such that $\lim_{i\to\infty} w_i=w_0$, then $w_0 \in N(x_0, n)$. Let $z_i \in F(x_i)$ such that $\|z_i-w_i\| \leq \frac{1}{3}\varepsilon_n$ for $i=1, 2, 3, \ldots$. Since each z_i is a point of the convex closure of $F(A)$, we may assume that $\lim_{i\to\infty} z_i=z_0$. Then by the upper semicontinuity of F, we have $z_0 \in F(x_0)$. But $\lim_{i\to\infty} \|z_i-w_i\|=\|z_0-w_0\| \leq \frac{1}{3}\varepsilon_n$. So that $w_0 \in N(x_0, n)$.

Now corresponding to the sequence $\{x_i\}$, we have a sequence $\{N(x_i, n)\}_{i=1}^\infty$ of subsets of the finite set $\bigcup_{k=1}^n N_k$. Select a subsequence $\{N(x_{n_i}, n)\}$ such that $N(x_{n_i}, n)= N(x_{n_j}, n)$ for all i and j. Then it is easy to see that $N(x_{n_i}, n) \subset N(x_0, n)$ for all i. To see upper semicontinuity of F_n, we observe that $\lim_{i\to\infty} y_i=y_0$, $y_i \in F_n(x_i)$, $y_{n_i} \in$ convex closure of $N(x_{n_i}, n)$ and conclude that $y_0 \in F_n(x_0)$.

THEOREM 2. *Let A be a closed subset of $X \subset E$ and $F: A \to E$ is a compact mapping. Then there is an extension $\tilde{F}$ of F over X such that $\tilde{F}(x) \subset$ convex closure of $F(A)$.*

Proof. Since X is a stratifiable space, according to [2], F has an extension to an upper semicontinuous function $\bar{F}$ of F whose values lie as closed subsets in $F(A)$. We define $\tilde{F}(x)$ to be the convex closure of $\bar{F}(x)$ for each $x \in X$. Let $x_n \in X$, $\lim_{n\to\infty} x_n=x_0$ and $y_n \in \tilde{F}(x_n)$, $\lim_{n\to\infty} y_n=y_0$. Assume that $y_0 \notin \tilde{F}(x_0)$. Let ε be a positive number such that $y_0 \notin V(\tilde{F}(x_0), \varepsilon)$. Since $\lim_{n\to\infty} x_n=x_0$, $\bar{F}$ is upper semicontinuous and $\bar{F}(x_0) \subset \tilde{F}(x_0)$, there is an integer N such that

$$\bar{F}(x_n) \subset V(\bar{F}(x_0), \varepsilon) \subset V(\tilde{F}(x_0), \varepsilon) \quad \text{for } n \geq N.$$

But $V(\tilde{F}(x_0), \varepsilon)$ is convex and contains the closed set $\bar{F}(x_n)$, so we have $y_n \in \tilde{F}(x_n) \subset V(\tilde{F}(x_0), \varepsilon)$. This is a contradiction, $\tilde{F}$ is upper semicontinuous. Since, for each $x \in X$, $\tilde{F}(x) \subset$ convex closure of $F(A)$, the closure of $\tilde{F}(A)$ is compact.

Let $F: X \to E$ be a compact mapping, $X \subset E$. A point $x_0 \in X$ such that $x_0 \in F(x_0)$ is called a fixed point of F.

LEMMA 3. *If $F: X \to E$ is a compact mapping, then the set of fixed points of F is closed.*

The following theorem is an extension to the case of multi-valued functions of the well known theorem of Kakutani [6].

THEOREM 4. *Let X be a closed, bounded and convex subset of E. If $F: X \to X$ is a compact mapping, then F has a fixed point.*

Proof. By Theorem 1 there is a sequence of compact mappings

$$F_k: X \to E_{n(k)} \subset E$$

such that

$$F(x) \subset V\left(F_k(x), \frac{1}{k}\right) \quad \text{and} \quad F_k(x) \subset V\left(F(x), \frac{1}{k}\right) \quad \text{for each } x \in X.$$

Since $F_k(x) \subset X \cap E_{n(k)}$ for each $x \in X$, we may suppose without loss of generality that the partial mapping $F_k^* = F_k \mid X \cap E_{n(k)}$ is a compact mapping of $X \cap E_{n(k)}$ into itself, and hence by Kakutani's fixed point theorem [6], there is $x_k \in X$ such that $x_k \in F_k(x_k)$, for each $k = 1, 2, 3, \cdots$. Since $x_k \in V(F(x_k), 1/k)$, choose $y_k \in F(x_k)$ such that $\|x_k - y_k\| < 1/k$. But the closure of $F(x)$ is compact, so we may assume that $\lim_{k\to\infty} y_k = y_0 \in$ closure of $F(X) \subset X$. Then $\lim_{k\to\infty} (y_k - x_k) = 0$. Hence $\lim_{k\to\infty} x_k = y_0$. By the upper semicontinuity of F, we have $y_0 \in F(y_0)$.

3. **Compact vector fields.** A multi-valued mapping $f: X \to E$ is called a compact vector field on X, if it can be represented in the form:

$$f(x) = x - F(x) = \{x - y \in E \mid y \in F(x)\},$$

where F is a compact mapping on X.

THEOREM 5. *Let X be a closed subset of E and $f: X \to E$ be a compact vector field, $f(x) = x - F(x)$. Then $f(X)$ is closed.*

Proof. Let $z_n \in f(X)$, $n = 1, 2, \ldots$ such that $\lim_{n\to\infty} z_n = z_0$. Let $z_n = x_n - y_n$, $y_n \in F(x_n)$, $n = 1, 2, 3, \cdots$. Since the closure of $F(X)$ is compact, we may assume without loss of generality that $\lim_{n\to\infty} y_n = y^*$. Then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} (z_n + y_n) = z_0 + y^* \in X$. So by the upper semicontinuity of f, $z_0 \in f(z_0 + y^*) \subset f(X)$.

Let X and Y be subsets of E. Denote by $\mathscr{L}(X, Y)$ the set of compact vector fields on X into Y. Two elements $f_i \in \mathscr{L}(X, Y)$ $i = 0, 1$ are said to be homotopic in $\mathscr{L}(X, Y)$ if there is a compact mapping $H: X \times I \to E$ such that (1) $H(x, 0) = F_0(x)$, $H(x, 1) = F_1(x)$, where $f_i(x) = x - F_i(x)$, $i = 0, 1$ and (2) for each $t \in I$, $f_t \in \mathscr{L}(X, Y)$, where $f_t(x) = x - H(x, t)$.

LEMMA 6. *Any two compact vector fields $f_1, f_2 \in \mathscr{L}((X, E)$ are homotopic in $\mathscr{L}(X, E)$.*

Proof. Let $f_i(x) = x - F_i(x)$, $i = 1, 2$, and define

$$H(x, t) = tF_1(x) + (1-t)F_2(x) = \{ty_1 + (1-t)y_2 \mid y_1 \in F_1(x), i = 1, 2\}.$$

Then for each $(x, t) \in X \times I$, $H(x, t)$ is compact and convex, and contained in the convex closure of $F_1(X) \cup F_2(X)$.

Let $h(x, t) = x - H(x, t)$. $t \in X$ and $0 \leq t \leq 1$.

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THEOREM 7. *Let f_1 and f_2 be a compact vector fields on X into P. $f_i(x)=x-F_i(x)$. Suppose any one of the following conditions are satisfied*

(i) $0 \notin tf_1(x)+(1-t)f_2(x), \quad 0 \le t \le 1, \quad x \in X$

(ii) $x \notin tF_1(x)+(1-t)F_2(x), \quad 0 \le t \le 1, \quad x \in X.$

Then f_1 and f_2 are homotopic in $\mathscr{L}(X, P)$.

Proof. In fact, the two conditions are equivalent. For

$$tf_1(x)+(1-t)f_2(x) = x-[tF_1(x)+(1-t)F_2(x)], \quad x \in X, \quad 0 \le t \le 1.$$

If one of the conditions is satisfied, then we define $H: X \times I \to E$ by $H(x, t)= tF_1(x)+(1-t)F_2(x)$ for each $x \in X$ and $0 \le t \le 1$, and let $h(x, t)=x-H(x, t)$. Then h is a homotopy between f_1 and f_2 in $\mathscr{L}(X, P)$.

COROLLARY 1. *Suppose $f_1, f_2 \in \mathscr{L}(X, P)$ such that for each $x \in X$ $\|z_1-z_2\| \le \|z_1\|$ for all $z_1 \in f_1(x)$, $i=1, 2$. Then f_1 and f_2 are homotopic in $\mathscr{L}(X, P)$.*

Proof. Suppose $f_i(x)=x-F_i(x)$, $z_i=x-y_i$, $y_i \in F_i(x)$, $i=1, 2$. Then $\|z_1-z_2\|= \|y_1-y_2\| \le \|x-y_1\| = \|z_1\|$. Let $\varepsilon = \|z_1\|$. If $\|z_1-z_2\| < \varepsilon$ then, since $V(z_1, \varepsilon)$ is convex, $tz_1+(1-t)z_2 \in V(z_1, \varepsilon)$, and hence $tz_1+(1-t)z_2 \neq 0$.

If $\|z_1-z_2\|=\varepsilon$, and for some t, $0<t<1$, $tz_1+(1-t)z_2=0$, then we would have $\|z_1\|=\|z_1-(tz_1+(1-t)z_2)\|=(1-t)\|z_1-z_1\|<\|z_1-z_2\|=\|z_1\|=\varepsilon$. So we conclude that $tz_1+(1-t)z_2 \neq 0$ for $0 \le t \le 1$. Hence by Theorem 7, f_1 and f_2 are homotopic in $\mathscr{L}(X, P)$.

COROLLARY 2. *Suppose $f \in \mathscr{L}(X, P)$ is a compact vector field such that the distance ε from the point 0 to the set $f(X)$ is positive. Suppose $g \in \mathscr{L}(X, P)$ such that $g(x) \subset V(f(x), \varepsilon)$ for each $x \in X$. Then f and g are homotopic in $\mathscr{L}(X, P)$.*

Proof. $V(f(x), \varepsilon)$ is a convex set which does not contain 0. Hence $0 \notin tf(x)+ (1-t)g(x)$ for each $x \in X$ and $0 \le t \le 1$.

THEOREM 8. *Let X_0 be a closed subset of $X \subset E$ and $f_0, g_0 \in \mathscr{L}(X_0, P)$ such that f_0 and g_0 are homotopic in $\mathscr{L}(X_0, P)$. If there is an extension $f \in \mathscr{L}(X, P)$ of f_0 over X, then there is an extension $g \in \mathscr{L}(X, P)$ of g_0 such that f and g are homotopic in $\mathscr{L}(X, P)$.*

Proof. Let $f_0(x)=x-F_0(x)$, and $g_0(x)=x-G_0(x)$, and $f(x)=x-F(x)$. Since f_0 and g_0 are homotopic in $\mathscr{L}(X, P)$ there is a compact mapping $H_0: X_0 \times I \to E$ such that $x \notin H_0(x, t)$ for each $x \in X_0$ and $0 \le t \le 1$, and $H_0(x, 0)=F_0(x)$, $H_0(x, 1)= G_0(x)$ for $x \in X_0$.

Let $T_0 = X_0 \times I \cup X \times 0$ and define $H_0^*: T \to E$ by

$$H_0^*(x, t) = \begin{cases} F(x), & x \in X, \quad t = 0 \\ H_0(x, t), & x \in X_0, \quad 0 \leq t \leq 1. \end{cases}$$

The mapping H_0^* is compact on T_0 and hence by Theorem 2 it can be extended to a compact mapping $H^*: X \times I \to E$. Let $X_1 = \{x \in X \mid 0 \in x - H^*(x, t) \text{ for some } t\}$. Then X_1 is a closed subset of X disjoint from X_0. Let $u: X \to I$ be a Urysohn function such that $u(x_0) = 1$, $x_0 \in X_0$ and $u(x_1) = 0$ for $x_1 \in X_1$.

Now consider a mapping $H: X \times I \to E$ defined by $H(x, t) = H^*(x, u(x) \cdot t)$, for $x \in X$ and $0 \leq t \leq 1$. It is clear that H is a compact mapping and $x \notin H(x, t)$ for $x \in X$ and $0 \leq t \leq 1$.

If we define a mapping $h: X \times I \to P$ by $h(x, t) = x - H(x, t)$ and $g(x) = h(x, 1)$, we see that $g \in \mathscr{L}(X, P)$ is an extension of g_0 and f and g are homotopic in $\mathscr{L}(X, P)$.

4. **Essential and inessential compact vector fields.** Let X be a closed subset of E and U a component of the complement of X in E. An element $f \in \mathscr{L}(X, P)$ is said to be inessential, with respect to U, if there is an extension $\bar{f}$ in $\mathscr{L}(X \cup U, P)$ of f over $X \cup U$. Otherwise f is said to be essential.

THEOREM 9. *Let X be a closed subset of E. Let $A_0 \subset E \backslash X$ be a compact and convex subset of $E \backslash X$, and U a component of $E \backslash X$, and let $f \in \mathscr{L}(X, P)$ be defined by $f(x) = x - A_0$, $x \in X$. Then*

(1) *f is inessential with respect to U if $A_0 \cap U = \phi$.*
(2) *f is essential with respect to U if $A_0 \cap U \neq \phi$ and both X and U are bounded.*

Proof. (1) Define $\bar{f}(x) = x - A_0$ for each $x \in X \cup U$.

(2) $A_0 \cap U \neq \phi$ implies $A_0 \subset U$. Suppose, on the contrary, that f is inessential with respect to U. Then there is a compact mapping $F: X \cup U \to E$ such that $x \notin F(x)$ for each $x \in X \cup U$, and $F(x) = A_0$ for $x \in X$. Define

$$F^*(x) = \begin{cases} F(x) & \text{if } x \in X \cup U \\ A_0 & \text{if } x \in K \backslash (X \cup U), \end{cases}$$

where K is a closed ball which contains $X \cup U$ and $F(X \cup U)$. Evidently F^* is a compact mapping of K into K without a fixed point, which is a contradiction with Theorem 4.

THEOREM 10. *Let $f \in \mathscr{L}(S, P)$, S the unit sphere in E, $f(x) = x - F(x)$. Suppose F is a compact mapping of S into a finite dimensional subspace E_n of E. Let $f_0 = f \mid S_{n-1}$, where $S_{n-1} = S \cap E_n$. If f is essential with respect to the unit open ball V in E then f_0 is essential with respect to the unit open ball V_n in E_n.*

Proof. Let $f_0(x) = x - F_0(x)$. Suppose f_0 is inessential with respect to V_n. Then the mapping $F_0: S_{n-1} \to E_n$ can be extended to a compact mapping $G_0: K_n \to E_n$ such

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that $x \notin G_0(x)$ for each $x \in K_n$, where K_n is the closed unit ball in E_n. Let $T = S \cup K_n$ and define

$$G_0^*(x) = \begin{cases} F(x) & \text{if } x \in S \\ G_0(x) & \text{if } x \in K_n. \end{cases}$$

Then $G_0^*: T \to E_n$ is a compact mapping such that $x \notin G_0^*(x)$ for each $x \in T$. Since T is the closed subset of the closed unit ball K in E, by virtue of Theorem 2, the mapping G_0^* can be extended over K to a compact mapping $F^*: K \to$ convex closure of $G_0^*(T) \subset E_n$. Then for $x \in S \cup K_n$, we have $x \notin F^*(x) = G_0^*(x)$. If $x_0 \in K \backslash (S \cup K_n)$ such that $x_0 \in F^*(x_0)$, then $x_0 \in V_n \subset K_n$. This is impossible. Hence $x \notin F^*(x)$ for $x \in K$. So this would mean that f has an extension $\bar{f}$, $\bar{f}(x) = x - F^*(x)$, $\bar{f} \in \mathscr{L}(K, P)$ which is a contradiction.

THEOREM 11. *Let X be a closed subset of E. Suppose A_1 and A_2 are disjoint compact convex subsets of $E \backslash X$. Let $F_i(x) = A_i$ and $f_i(x) = x - F_i(x)$, $x \in X$, $i = 1, 2$.*

(1) *If the set X does not separate A_1 and A_2, then f_1 and f_2 are homotopic in $\mathscr{L}(X, P)$.*

(2) *If X is bounded and one of A_i is contained in a bounded component of $E \backslash X$ and f_1 and f_2 are homotopic in $\mathscr{L}(X, P)$ then X does not separate A_1 and A_2.*

Proof. (1) If A_1 and A_2 belong to the same component of $E \backslash X$, let $r: I \to E \backslash X$ be an arc from a point in A_1 to a point in A_2. Define $\bar{r}: I \to E \backslash X$ by

$$\bar{r}(t) = \begin{cases} A_i & \text{if } r(t) \in A_i \\ r(t) & \text{if otherwise.} \end{cases}$$

Then $\bar{r}$ is a compact mapping. Let $h(x, t) = x - \bar{r}(t)$, $x \in X$ and $0 \leq t \leq 1$. Then $h(x, 0) = f_1(x)$ and $h(x, 1) = f_2(x)$. $x \in X$.

(2) Suppose A_1 and A_2 belong to two different components U_1 and U_2 respectively, and suppose U_1 is bounded. Then the mapping f_2 has an extension $\bar{f}_2 \in \mathscr{L}(X \cup U_1. P)$, $\bar{f}_2(x) = x - F_2(x)$, $x \in X \cup U_1$. Since f_1 and f_2 are homotopic in $\mathscr{L}(X, P)$, by Theorem 8 we have an extension $f_1 \in \mathscr{L}(X \cup U_1, P)$ of f_1. But this is a contradiction to the second part of Theorem 9.

THEOREM 12. *Let S be the boundary of $V(x_0, \varepsilon)$, and $K = \overline{V(x_0, \varepsilon)}$. Suppose $f \in \mathscr{L}(K, E)$ such that for some $y_0 \in f(x_0)$ we have $y_0 \notin f(S)$. Let $f_0 \in \mathscr{L}(S, P)$ be defined by $f_0(x) = f(x) - y_0$. If f_0 is essential with respect to $V(x_0, \varepsilon)$, then there is a δ-neighborhood U of y_0 in E such that $U \subset f(K)$.*

Proof. Since $f_0(S)$ is closed, let $\delta = d(f_0(S), 0) > 0$. Let U be a δ-neighborhood of y_0 in E and let $y \in U$. Let $g_0 \in \mathscr{L}(S, E)$ be defined by $g_0(x) = f(x) - y$, $x \in S$. Then $f_0(x) \subset V(g_0(x), \delta)$ and $g_0(x) \subset V(f_0(x), \delta)$, $x \in S$. Hence $g_0 \in \mathscr{L}(S, P)$. Then by Corollary 2 of Theorem 7, f_0 and g_0 are homotopic in $\mathscr{L}(S, P)$. Since f_0 is essential so is g_0 by Theorem 8. From this we infer that the compact vector field $g:K\to E$ defined by $g(x)=f(x)-y$, being an extension of g_0 over K, has at least one point $x\in K$ such that $0\in g(x)=f(x)-y$, i.e., $y\in f(x)$.

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