The Equation $x^3 - u^3 = y^3 - v^3$ when x, y, u, v are rational.

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Diophantus (Heath, 1885, page 122) is said to have the porism—"The difference of two cubes can be expressed as the sum of two cubes."

The given numbers being a, b, we have

$$a^3 - b^3 = x^3 + y^3,$$
$$x(a^3 + b^3) = a(a^3 - 2b^3),$$
$$y(a^3 + b^3) = b(2a^3 - b^3).$$

It follows that, and we may take $a > b$,

$$a^3 + b^3 = \left\{\frac{a(a^3 + 2b^3)}{a^3 - b^3}\right\}^3 - \left\{\frac{b(2a^3 + b^3)}{a^3 - b^3}\right\}^3,$$

$$a^3 - b^3 = \left\{\frac{a(a^3 - 2b^3)}{a^3 + b^3}\right\}^3 - \left\{\frac{b(b^3 - 2a^3)}{a^3 + b^3}\right\}^3,$$

$$= \left\{\frac{b(2a^3 - b^3)}{a^3 + b^3}\right\}^3 - \left\{\frac{a(2b^3 - a^3)}{a^3 + b^3}\right\}^3.$$

It is briefly stated by Heath that we can always solve the problem in positive integers whether $a^3 > 2b^3$ or $< 2b^3$, but as this is not obvious, the following may make this clear. Let $a = bx$, where $1 < x^3 < 2$, then we have $a^3 - b^3 = A^3 - B^3$, where $A/B = b(2a^3 - b^3)/a(2b^3 - a^3) = (2x^3 - 1)/(2x - x^4)$. It is easy to see that $A/B > x^8$, and $A^3/B^3 > x^{24}$. If $A^3 > 2B^3$ we can now express $A^3 - B^3\ (= a^3 - b^3)$ as the sum of two cubes: but if $x^{24} < 2$ we can repeat the process until sooner or later, and generally the former, we get $a^3 - b^3 = C^3 - D^3$ where $C^3 > 2D^3$, and then $a^3 - b^3$ can be expressed as the sum of two cubes.

Algebraically, all are included in

$$a^3 + b^3 = \left\{\frac{a(a^3 + 2b^3)}{a^3 - b^3}\right\}^3 + \left\{\frac{b(b^3 + 2a^3)}{b^3 - a^3}\right\}^3.$$

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To find this result Poselger (1832) proceeds thus:—

Let $(a-x)^3+(mx+b)^3=a^3+b^3$,

then $x^3(m^3-1)+3x(m^2b+a)+3(mb^2-a^2)=0$.

Make $mb^2=a^2$, then

$$x\left(\frac{a^6}{b^6}-1\right)+3\left(\frac{a^4}{b^3}+a\right)=0,$$

and $$x=\frac{3b^3a}{b^3-a^3},\quad a-x=\frac{a(a^3+2b^3)}{a^3-b^3},\quad mx+b=-\frac{b(b^3+2a^3)}{a^3-b^3}.$$

But although Heath refers to Poselger, the same result by the same method was obtained by Euler, and published by him in 1754.

Euler in the paper referred to arrived at a formidable result by a tedious process, and then, attacking the problem differently, obtains for the equation $x^3-u^3=y^3-v^3$

$$x=n\{(3ac+3bc-ad+3bd)(3c^2+d^2)-(a^2+3b^2)^2\},$$
$$u=n\{(-3ac+3bc-ad-3bd)(3c^2+d^2)-(a^2+3b^2)^2\},$$
$$y=n\{(3c^2+d^2)^2+(3ac-3bc+ad+3bd)(a^2+3b^2)\},$$
$$v=n\{(3c^2+d^2)^2-(3ac+3bc-ad+3bd)(a^2+3b^2)\}.$$

and deduces values "magis speciales" by writing $c=0$, namely,

$$x=\quad n\{d^3(3b-a)-(a^2+3b^2)^2\},$$
$$u=-n\{d^3(3b+a)+(a^2+3b^2)^2\},$$
$$y=\quad n\{d^4+d(a+3b)(a^2+3b^2)\},$$
$$v=\quad n\{d^4+d(a-3b)(a^2+3b^2)\}.$$

But these values are not really less general, for if in the first set of equations we write $ad-3bc=\alpha$, $ac+bd=\beta$, $3c^2+d^2=\delta$, we get at once

$$x=\quad n\{(3\beta-\alpha)\delta^3-(\alpha^2+3\beta^2)^2\}/\delta^2,$$
$$u=-n\{(3\beta+\alpha)\delta^3+(\alpha^2+3\beta^2)^2\}/\delta^2,$$
$$y=\quad n\{(\delta^4+\delta(\alpha+3\beta)(\alpha^2+3\beta^2)\}/\delta^2,$$
$$v=\quad n\{(\delta^4+\delta(\alpha-3\beta)(\alpha^2+3\beta^2)\}/\delta^2.$$

The most direct solution seems as follows:—

We have $x^3-u^3=y^3-v^3$.

Let $x-u=X,\quad x+u=U,$

$y-v=Y,\quad y+v=V,$

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then $$X(X^2+3U^2) = Y(Y^2+3V^2),$$

whence $$X^2+3U^2=\mu Y,$$
$$Y^2+3V^2=\mu X, \;;$$
$$\therefore \quad (\mu Y - X^2)(\mu X - Y^2) = (3UV)^2,$$
$$= (XY - \mu K)^2,$$
$$\therefore \quad \mu = \frac{X^3+Y^3-2KXY}{XY-K^2}.$$
$$\therefore \quad 3U^2 = \mu Y - X^2 = \frac{X^3Y+Y^4-2KXY-X^3Y+X^2}{XY-K^2}$$
$$= \frac{(Y^2-KX)^2}{XY-K^2}.$$

Similarly $$3V^2 = \frac{(X^2-KY)^2}{XY-K^2}.$$

And $$XY-K^2=3t^2,$$

so that $$U = \frac{Y^2-KX}{3t}, \quad V = \frac{X^2-KY}{3t}.$$

Whence $$2x = \{Y^2+X(3t-K)\} \div 3t,$$
$$2y = \{X^2+Y(3t-K)\} \div 3t,$$
$$2u = \{Y^2-X(3t+K)\} \div 3t,$$
$$2v = \{X^2-Y(3t+K)\} \div 3t,$$

with the one condition $XY = K^2+3t^2$.

So we have
$$2x = \{(K^2+3t^2)^2+X^3(3t-K)\} \div 3tX^2,$$
$$2y = \{X^4+X(3t^2+K^2)(3t-K)\} \div 3tX^2,$$
$$2u = \{(K^2+3t^2)^2-X^3(3t+K)\} \div 3tX^2,$$
$$2v = \{X^4-X(3t^2+K^2)(3t+K)\} \div 3tX^2,$$

for the most general rational solution, where we can clearly interchange y and u, or x and u, v and y, etc.

It will be convenient now to write, noting that we are only regarding the ratios of x, y, u, v,
$$x = a^4 + a(b+3c)(b^2+3c^2),$$
$$u = a^3(b+3c)+(b^2+3c^2)^2,$$
$$y = a^4 + a(b-3c)(b^2+3c^2),$$
$$v = a^3(b-3c)+(b^2+3c^2)^2,$$

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and we find on writing $b^2+3c^2=\beta^2+3\gamma^2$, $b+3c=\beta+3\gamma$, that we get also

$$2\beta=3c-b,\ \ 2\gamma=c+b,$$

and a new solution

$$z=a^4-2ab\,(b^2+3c^2),$$
$$w=\ -2a^3b+(b^2+3c^2)^2.$$

We may further simplify these equations by writing $a=1$, and we get

$$x=1+(b+c)^3+(2c)^3,$$
$$u=b+3c+(b^2+3c^2)^2,$$
$$y=1+(b-c)^3-(2c)^3,$$
$$v=b-3c+(b^2+3c^2)^2,$$
$$z=1-(b+c)^3-(b\ -c)^3,$$
$$w=\ -2b+(b^2+3c^2)^2.$$

Now write $b+c=\mu$, $2c=\ -\nu$, $c-b=\lambda$, then $\lambda+\mu+\nu=0$, and

$$x=1+\mu^3-\nu^3,\qquad u=\mu-\nu+(\mu\nu+\nu\lambda+\lambda\mu)^2,$$
$$y=1+\nu^3-\lambda^3,\qquad v=\nu-\lambda+(\mu\nu+\nu\lambda+\lambda\mu)^2,$$
$$z=1+\lambda^3-\mu^3,\qquad w=\lambda-\mu+(\mu\nu+\nu\lambda+\lambda\mu)^2.$$

This form of the result is perhaps the best. The value of

$$x^3-u^3=y^3-v^3=z^3-w^3$$

is $\{1-(\mu\nu+\nu\lambda+\lambda\mu)^3\}\,\{1-(\mu-\nu)(\nu-\lambda)(\lambda-\mu)+(\mu\nu+\nu\lambda+\lambda\mu)^3\}$.

Two other methods of procedure may be worth noting. We may express x, y, u, v in the obviously convenient forms

$$x=1+\lambda t^3,\qquad y=pt+rt^4,$$
$$u=1+\mu t^3,\qquad v=qt+rt^4,$$

which, it may be remarked, are always possible for all values of x, y, u, v; and we may then equate the coefficients of powers of t in

$$(1+\lambda t^3)^3-(1+\mu t^3)^3=t^3\{(p+rt^3)^3+(q+rt^3)^3\}.$$

This gives

$$\frac{3\,(\lambda-\mu)}{p^3-q^3}=\frac{\lambda^2-\mu^2}{(p^2-q^2)\,r}=\frac{\lambda^3-\mu^3}{3\,(p-q)\,r^2}=1\ ;$$

whence

$$\frac{\lambda}{\mu}=\frac{p}{q}\left(\text{or }\frac{q}{p}\right)=k.$$

Taking the former we readily find

$$3k = p^2 + pq + q^2,$$
$$3\lambda = p(p^2 + pq + q^2),$$
$$3\mu = q(p^2 + pq + q^2),$$
$$3^2 r = (p^2 + pq + q^2)^2.$$

Writing $p = b + 3c$, $q = b - 3c$, and making any necessary reductions, we get the general results already obtained, although the assumptions made with regard to the separate coefficients do not necessitate their generality.

Another method is suggested by the solution of the equation

$$x^2 - u^2 = y^2 - v^2,$$

as given by the form

$$x = 1 + ab, \qquad u = a + b,$$
$$y = 1 - ab, \qquad v = a - b.$$

We consider the general problem

$$x^n - u^n = y^n - v^n,$$

and we assume

$$\lambda x = 1 + ab, \qquad \lambda u = a^{n-1} + b,$$
$$\lambda y = 1 + ac, \qquad \lambda v = a^{n-1} + c,$$

giving the equation

$$1 + n\,ab + \frac{n(n-1)}{1.2} a^2 b^2 + \ldots + nb^{n-1} a^{n-1} + a^n b^n$$
$$- a^{(n-1)n} - n\,a^{(n-1)^2} b - \ldots - n\,a^{n-1} b^{n-1} - b^n$$
$$= f(a, b) = f(a, c).$$

We see that the assumptions give definite rational values of a, b, c, λ for all values of x, y, u, v, so that the solution derived will be general in this case. In fact,

$$\frac{1 + ab}{x} = \frac{1 + ac}{y} = \frac{b + a^{n-1}}{u} = \frac{c + a^{n-1}}{v},$$

and each fraction is

$$\frac{a(b-c)}{x-y} = \frac{b-c}{u-v} = \frac{1-a^n}{x-au} = \frac{1-a^n}{y-av}$$
$$= \frac{b(1-a^n)}{u - xa^{n-1}} = \frac{c(1-a^n)}{v - ya^{n-1}},$$

whence a, b, c can always be found in terms of x, y, u, v.

The general equation connecting a, b, c is

$$n(b-c)(a-a^{(n-1)^2})$$
$$+\frac{n(n-1)}{1\ 2}(b^2-c^2)(a^2-a^{(n-1)(n-2)})$$
$$+\ldots+\frac{n(n-1)}{1.2}(b^{n-2}-c^{n-2})(a^{n-2}-a^{2(n-1)})$$
$$+\ldots+(b^n-c^n)(a^n-1)=0.$$

For $n=2$, $(b^2-c^2)(a^2-1)=0$;
for $n=3$, $3(b-c)(a-a^4)+(b^3-c^3)(a^3-1)=0$,

$\therefore\quad 3a=b^2+bc+c^2$, which gives

$$x=1+\frac{(b^2+bc+c^2)b}{3},\qquad y=1+\frac{(b^2+bc+c^2)c}{3},$$
$$u=\frac{(b^2+bc+c^2)^2}{3}+b,\qquad v=\frac{(b^2+bc+c^2)^2}{3}+c,$$

i.e. one of the general forms already obtained.

For $n=4$ we have

$$4(b-c)(a-a^9)+6(b^2-c^2)(a^2-a^6)$$
$$+(b^4-c^4)(a^4-1)=0.$$

For $n=5$

$$5(b-c)(a-a^{16}+10(b^2-c^2)(a^2-a^{12})$$
$$+10(b^3-c^3)(a^3-a^8)$$
$$+(b^5-c^5)(a^5-1)=0.$$

The case $n=4$ reduces to

$$4a(1+a^4)+6(b+c)a^2=(b+c)(b^2+c^2),$$

and if $b+c=2\mu a$, we have

$$4(1+a^4)+12\mu a^2=\mu\{4\mu^2a^2+(b-c)^2\}.$$

$$\therefore\qquad \frac{4(1+a^4)}{\mu}+(12-4\mu^2)a^2=(b-c)^2,$$

a diophantine equation of which $\mu=1$ is an obvious and useless solution.

Writing $\mu=1+t$, we have

$$4\{(1+a^4)(1+t)+a^2(1+t)^2(2-2t-t^2)\}$$
$$=4\{(1+a^2)^2+t(1+a^2)^2-3t^2a^2-4t^3a^2-t^4a^2\} = \text{a perfect square,}$$

of which

$$t=\frac{8(1+a^2)^2(1-18a^2+a^4)}{(1+14a^2+a^4)^2+64a^2(1+a^2)^2}$$

is a particular solution.

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This solution reduces to one found by Euler by an ingenious tentative method, but it gives very large values of x, y, u, v for quite small values of a: thus $a = 3$ gives

$$12231^4 - 10203^4 = 10381^4 - 2903^4$$

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From this identity the rather curious arithmetical result follows:

$$(2 \,.\, 5653)^2 + (25 \,.\, 37)^2 = 73 \,.\, 269 \,.\, 6553$$

$$(6553)^2 + (2 \,.\, 25 \,.\, 73)^2 = 37 \,.\, 269 \,.\, 5653.$$

The smallest set of numbers that satisfy the equation $x^4 - u^4 = y^4 - v^4$ is 158, 134, 133, 59: and on examining the values of a, b, c, λ, we get $a = 1/3$, $b = 173/153$, $c = 73/153$, $\lambda = 4/459$. If we take the order 134, 158, 59, 133, we get $a = 3$, $b = 173/17$, $c = 73/17$, $\lambda = 4/17$, but this solution is *not* one of the special solutions, for which $a = 3$ gives indeed only the large numbers given above.

The case when $n = 5$ seems hopeless, but I have not examined it.