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James-Hopf Invariants, Anick's Spaces, and the Double Loops on Odd Primary Moore Spaces

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Abstract. Using spaces introduced by Anick, we construct a decomposition into indecomposable factors of the double loop spaces of odd primary Moore spaces when the powers of the primes are greater than the first power. If n is greater than 1, this implies that the odd primary part of all the homotopy groups of the $2n+1$ dimensional sphere lifts to a mod p^r Moore space.

0 Introduction

Throughout this paper, p will be a fixed odd prime and all spaces will be localized at p. Let $P^m(p^r)$ be the mod p^r Moore space $S^{m-1}\bigcup_{p^r} e^m$ which is formed by attaching an m-cell to an $(m-1)$-sphere by a map of degree a power p^r of the prime. The fibre of the degree p^r map $p^r\colon S^m \to S^m$ will be denoted by $S^m\{p^r\}$. In [4], [5], maps $\partial_r\colon \Omega^2 S^{2n+1} \to S^{2n-1}$ were constructed with a strong relation to the double suspension $\Sigma^2\colon S^{2n-1} \to \Omega^2 S^{2n+1}$, namely, the compositions $\Sigma^2 \circ \partial_r\colon \Omega^2 S^{2n+1} \to S^{2n-1} \to \Omega^2 S^{2n+1}$ and $\partial_r \circ \Sigma^2\colon S^{2n-1} \to \Omega^2 S^{2n+1} \to S^{2n-1}$ are $\Omega^2(p^r)$ and p^r, respectively. Let $D(n,r)$ be the fibre of ∂_r.

In [11], a proof was given that, if $r \geq 2$, there is a homotopy equivalence

$$D(n,r) \times \Omega\Big[\prod_{k=1}^{\infty} S^{2p^k n-1}\{p^{r+1}\}\Big] \times \Omega^2\Sigma P(n,r) \to \Omega^2 P^{2n+1}(p^r) \tag{0.1}$$

where $P(n,r)$ is some infinite bouquet of mod p^r Moore spaces. In this paper we give another proof which is quite different from the proof in [11]. The proof given here is valid only for primes $p \geq 5$ while the proof in [11] is valid for $p \geq 3$. This happens because certain properties of Anick's spaces are known only for $p \geq 5$. It would not be surprising if this situation were remedied in the future.

Nonetheless, the proof given here has some advantages. It is a more straightforward attack on the problem and seems closer to a confrontation with the case $r = 1$. It is a striking fact that both proofs fail in the case $r = 1$. The failure here seems to be more enlightening, more closely related to the truth or falsity of the theorem in the case $r = 1$, whereas the failure for $r = 1$ in the first proof seems to be more a consequence of the method of attack. Of course, neither failure resolves the issue and there is as yet no reason to withdraw the conjecture that (0.1) is true for all $r \geq 1$.

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Already in [6], it was shown that there is a homotopy equivalence

$$T^{2n+1}(p^r) \times \Omega\Sigma P(n,r) \to \Omega P^{2n+1}(p^r) \tag{0.2}$$

where $T^{2n+1}(p^r)$ is the fibre of the map $\Sigma P(n,r) \to P^{2n+1}(p^r)$ and sits in a fibration sequence

$$\Omega^2 S^{2n+1} \to S^{2n-1} \times \Pi_{r+1} \to T^{2n+1}(p^r) \to \Omega S^{2n+1} \tag{0.3}$$

with

$$\Pi_{r+1} = \prod_{k=1}^{\infty} S^{2p^k n-1}\{p^{r+1}\}.$$

The map ∂_r is nothing but the composition of the first map in (0.3) with projection on the first factor. It follows immediately that there is a fibration

$$\Omega\Pi_{r+1} \to \Omega T^{2n+1}(p^r) \to D(n,r) \tag{0.4}$$

and hence that (0.1) is equivalent to the problem of constructing a retraction $\Omega T^{2n+1}(p^r) \to \Omega\Pi_{r+1}$. In this paper we will construct this retraction with the aid of James-Hopf invariants [7], [8]. The James-Hopf invariants do not give this retraction immediately. The James-Hopf invariants must be modified and restricted. After that, some lifting of maps is required. It is there that the proof breaks down for $r = 1$ in an essential way.

As described in [11], (0.1) implies that the natural map $P^{2n+1}(p^r) \to S^{2n+1}$ induces split epimorphisms on all homotopy groups in dimensions greater than $2n+1$ if r is $\geq$ the maximum of 2 and n.

I wish to thank Stephen Theriault for valuable e-mail tutorials on the contents of his thesis [13]. Without his help in Section 2 to deloop the maps η and ζ, the proof given here would have been strong enough to prove only the loop of (0.1).

1 James-Hopf Invariants

The James-Hopf invariants [7], [8] are natural maps $h_j \colon \Omega\Sigma X \to \Omega\Sigma X^{\wedge j}$ with $X^{\wedge j} = X \wedge \cdots \wedge X$ being the j-fold smash. These maps have the following homological property for field coefficients: if $x_1 \in \bar{H}_*(X), \dots, x_i \in \bar{H}_*(X)$, then

$$h_{j*}(x_1 \otimes \cdots \otimes x_i) = \begin{cases} 0 & \text{if } i < j, \\ x_1 \otimes \cdots \otimes x_i & \text{if } i = j, \\ \text{a decomposable element} & \text{if } i > j. \end{cases}$$

Throughout this paper, the coefficients will be Z/pZ and we will be concerned with the cases where $j = p^k$ and X is S^{2n} or $P^{2n}(p^r)$. In the second case, we will replace the James-Hopf invariant with a modified James-Hopf invariant $\bar{h}_j$ as follows.

Since $P^a(p^r) \wedge P^b(p^r) \simeq P^{a+b}(p^r) \vee P^{a+b-1}(p^r)$, it follows that, if $q \colon P^a(p^r) \to S^a$ is the natural map, there is a factorization of $\wedge^j q$ into $\big(P^{2n}(p^r)\big)^{\wedge j} \xrightarrow{\alpha} P^{2jn}(p^r) \xrightarrow{q} (S^{2n})^{\wedge j}$.

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Hence, there is a commutative diagram

$$\begin{array}{ccccc} \Omega P^{2n+1}(p^r) & \xrightarrow{h_j} & \Omega\Sigma\big(P^{2n}(p^r)\big)^{\wedge j} & \xrightarrow{\Omega\Sigma\alpha} & \Omega P^{2jn+1}(p^r) \\ \downarrow{\scriptstyle \Omega q} & & \downarrow{\scriptstyle \Omega\Sigma(\wedge^j q)} & & \downarrow{\scriptstyle \Omega q} \\ \Omega S^{2n+1} & \xrightarrow{h_j} & \Omega\Sigma(S^{2n})^{\wedge j} & \xrightarrow{=} & \Omega S^{2jn+1}. \end{array} \tag{1.1}$$

Let the modified James-Hopf invariant $\bar{h}_j = \Omega\Sigma\alpha \circ h_j \colon \Omega P^{2n+1}(p^r) \to \Omega P^{2jn+1}(p^r)$ be the composition in the top row of (1.1). If $u = u(a-1,r) \in H_{a-1}\big(P^a(p^r)\big)$ and $v = v(a,r) \in H_a\big(P^a(p^r)\big)$ are generators, then $\bar{h}_{j_*}\big(v(2n,r)^j\big) = v(2nj,r)$. Let β^r be the r-th Bockstein. Then $\beta^r v(a,r) = u(a-1,r)$ and $\beta^r\big(v(2n,r)^{p^k}\big) = ad^{p^k-1}\big(v(2n,r)\big)\big(u(2n-1,r)\big) = \tau_k\big(v(2n,r)\big)$ [4]. Hence, $\bar{h}_{p^k_*}\Big(\tau_k\big(v(2n,r)\big)\Big) = u(2p^k n - 1, r)$.

The space $T^{2n+1}(p^r)$ is defined to be the fibre of a map $\Sigma P(n,r) \to P^{2n+1}(p^r)$ [6]. This map is defined on a bouquet of mod p^r Moore spaces as a bouquet of mod p^r Whitehead products which, since S^{2n+1} is an H-space, map to zero when composed with the map $P^{2n+1}(p^r) \to S^{2n+1}$. It follows that we may form the commutative diagram below in which the rows and columns are fibration sequences:

$$\begin{array}{ccccccc} \Omega F^{2n+1}\{p^r\} & \longrightarrow & W^{2n+1}\{p^r\} & \longrightarrow & \Sigma P(n,r) & \longrightarrow & F^{2n+1}\{p^r\} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \downarrow \\ \Omega P^{2n+1}(p^r) & \longrightarrow & T^{2n+1}(p^r) & \longrightarrow & \Sigma P(n,r) & \longrightarrow & P^{2n+1}(p^r) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega S^{2n+1} & \longrightarrow & \Omega S^{2n+1} & \longrightarrow & * & \longrightarrow & \Omega S^{2n+1}. \end{array} \tag{1.2}$$

The result quoted in (0.2) gives a section $\sigma \colon T^{2n+1}(p^r) \to \Omega P^{2n+1}(p^r)$ and (1.2) implies that σ restricts to $\bar{\sigma} \colon W^{2n+1}\{p^r\} \to \Omega F^{2n+1}\{p^r\}$. At a certain point in the proof below we will restrict a further modification of the modified James-Hopf invariant $\bar{h}_{p^k}$ from $\Omega P^{2n+1}(p^r)$ to $T^{2n+1}(p^r)$ via the map σ. We will need to know the following: Suppose v_k and $\tau_k = \beta^r v_k$ are the respective images of $v(2n,r)^{p^k}$ and $\tau_k\big(v(2n,r)\big)$ in $H_*\big(T^{2n+1}(p^r)\big)$. These are primitive elements. Since $v(2n,r)^{p^k}$ is the primitive element of least length in its degree, it follows that $\bar{h}_{p^k_*} \circ \sigma_*$ sends v_k to $v(2p^k n, r)$ and τ_k to $u(2p^k n, r)$.

2 Theriault's Reconstruction of Anick's Spaces $\mathrm{BD}(n,r)$

In order to avoid proving our decomposition theorem for triple loops instead of double loops, we shall use spaces $\mathrm{BD}(n,r)$, defined for $r \geq 1$, which are candidates for classifying spaces for the spaces $D(n,r)$ (and, in fact, it is an elementary consequence of the product decomposition (0.1) that they are classifying spaces if $r \geq 2$). These spaces were introduced by Anick [1] for $p \geq 5$, further studied by Anick and Gray [2], and reconstructed for all $p \geq 3$ in the thesis of Theriault [13] and his subsequent paper [14]. (Theriault used the notation $T^{2n-1}_{\infty}\{p^r\}$ for the spaces we call $\mathrm{BD}(n,r)$. Anick and Gray have used variations of this notation involving the letter T. Because of a conflict with the notation of Cohen, Moore, and Neisendorfer, this notation will not be used here.) In this section we provide a brief summary of some of the work of Theriault and derive some consequences which we use in the proof of (0.1).

First, if $p \geq 3$ and $r \geq 1$, there are H-spaces $\mathrm{BD}(n,r)$, [2], [13], [14], and there are factorizations of the natural maps $\Omega P^{2n+1}(p^r) \to \Omega S^{2n+1}$ into

$$\Omega P^{2n+1}(p^r) \to \mathrm{BD}(n,r) \to \Omega S^{2n+1}\{p^r\} \to \Omega S^{2n+1} \tag{2.1}$$

where the second and third maps are H-maps. There is a fibration sequence

$$\Omega^2 S^{2n+1} \to S^{2n-1} \to \mathrm{BD}(n,r) \to \Omega S^{2n+1}$$

in which the first map has degree p^r on the bottom cell [1].

The first and second maps in (2.1) are mod p homology isomorphisms in dimensions $2n-1$ and $2n$. Accordingly, we shall denote the generators of both $H_*\big(\mathrm{BD}(n,r)\big)$ and $H_*(\Omega S^{2n+1}\{p^r\})$ in these dimensions by $u(n,r)$ and $v(n,r)$, respectively.

If $p \geq 5$, the H-spaces $\mathrm{BD}(n,r)$ are homotopy commutative, homotopy associative, have null homotopic p^r-th power maps, and the first map in (2.1) is an H-map [13], [14].

Second, if $p \geq 3$, the spaces $\mathrm{BD}(n,r)$ and the natural maps $\iota \colon P^{2n} \xrightarrow{\Sigma} \Omega P^{2n+1}(p^r) \to \mathrm{BD}(n,r)$ satisfy the universality property [14]: if X is a homotopy commutative and homotopy associative H-space and $f \colon P^{2n}(p^r) \to X$ is any map, then there is an extension to an H-map $\bar{f} \colon \mathrm{BD}(n,r) \to X$. The extension is unique up to homotopy.

Following a suggestion of Theriault we apply the universality property to the maps ζ and η uniquely defined for $s < r$ by the maps of cofibration sequences

$$\begin{array}{ccccc}
S^{2n-1} & \xrightarrow{p^s} & S^{2n-1} & \longrightarrow & P^{2n}(p^s) \\
\big\downarrow{\scriptstyle p^{r-s}} & & \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle \zeta} \\
S^{2n-1} & \xrightarrow{p^r} & S^{2n-1} & \longrightarrow & P^{2n}(p^r)
\end{array}
\qquad
\begin{array}{ccccc}
S^{2n-1} & \xrightarrow{p^r} & S^{2n-1} & \longrightarrow & P^{2n}(p^r) \\
\big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle p^{r-s}} & & \big\downarrow{\scriptstyle \eta} \\
S^{2n-1} & \xrightarrow{p^s} & S^{2n-1} & \longrightarrow & P^{2n}(p^s).
\end{array} \tag{2.2}$$

There are also maps ζ and η uniquely defined by maps of fibration sequences

$$\begin{array}{ccccc}
S^{2n+1}\{p^s\} & \longrightarrow & S^{2n+1} & \xrightarrow{p^s} & S^{2n+1} \\
\big\downarrow{\scriptstyle \zeta} & & \big\downarrow{\scriptstyle p^{r-s}} & & \big\downarrow{\scriptstyle =} \\
S^{2n+1}\{p^r\} & \longrightarrow & S^{2n+1} & \xrightarrow{p^r} & S^{2n+1}
\end{array}
\qquad
\begin{array}{ccccc}
S^{2n+1}\{p^r\} & \longrightarrow & S^{2n+1} & \xrightarrow{p^r} & S^{2n+1} \\
\big\downarrow{\scriptstyle \eta} & & \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle p^{r-s}} \\
S^{2n+1}\{p^s\} & \longrightarrow & S^{2n+1} & \xrightarrow{p^s} & S^{2n+1}
\end{array} \tag{2.3}$$

For the remainder of this paper, let $p \geq 5$.

Theriault's universality property implies that the maps in (2.2) extend to H-maps $\zeta \colon \mathrm{BD}(n,s) \to \mathrm{BD}(n,r)$ and $\eta \colon \mathrm{BD}(n,r) \to \mathrm{BD}(n,s)$ where the ζ and η maps are unique and

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there are commutative diagrams

$$
(2.4)\qquad
\begin{array}{ccccc}
S^{2n-1} & \xrightarrow{p^{r-s}} & S^{2n-1} & \xrightarrow{=} & S^{2n-1} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{BD}(n,s) & \xrightarrow{\zeta} & \mathrm{BD}(n,r) & \xrightarrow{\eta} & \mathrm{BD}(n,s) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega S^{2n+1} & \xrightarrow{=} & \Omega S^{2n+1} & \xrightarrow{\Omega p^{r-s}} & \Omega S^{2n+1}
\end{array}
$$

Therefore, for $n > 1$ we have a commutative diagram with rows and columns fibration sequences

$$
(2.5)\qquad
\begin{array}{ccccc}
* & \longrightarrow & \Omega^2 S^{2n+1} & \xrightarrow{=} & \Omega^2 S^{2n+1} \\
\downarrow & & \downarrow & & \downarrow \\
S^{2n-1}\{p^{r-s}\} & \longrightarrow & S^{2n-1} & \xrightarrow{p^{r-s}} & S^{2n-1} \\
\downarrow{=} & & \downarrow & & \downarrow \\
S^{2n-1}\{p^{r-s}\} & \longrightarrow & \mathrm{BD}(n,s) & \xrightarrow{\zeta} & \mathrm{BD}(n,r)
\end{array}
$$

and for all $n \geq 1$ we have a commutative diagram with rows and columns fibration sequences

$$
(2.6)\qquad
\begin{array}{ccccc}
* & \longrightarrow & S^{2n-1} & \xrightarrow{=} & S^{2n-1} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega S^{2n+1}\{p^{r-s}\} & \longrightarrow & \mathrm{BD}(n,r) & \xrightarrow{\eta} & \mathrm{BD}(n,s) \\
\downarrow{=} & & \downarrow & & \downarrow \\
\Omega S^{2n+1}\{p^{r-s}\} & \longrightarrow & \Omega S^{2n+1} & \xrightarrow{\Omega p^{r-s}} & \Omega S^{2n+1}.
\end{array}
$$

The uniqueness of ζ and η clearly implies that $\zeta \circ \zeta = \zeta$ and $\eta \circ \eta = \eta$.

We note that the universality property implies that, if $r \geq 1$, the composition $\eta \circ \zeta\colon \mathrm{BD}(n,r) \to \mathrm{BD}(n,r+1) \to \mathrm{BD}(n,r)$ is the p-th power map and, if $r \geq 2$, so is the composition $\zeta \circ \eta\colon \mathrm{BD}(n,r) \to \mathrm{BD}(n,r-1) \to \mathrm{BD}(n,r)$. Furthermore, the diagram below commutes

$$
(2.7)\qquad
\begin{array}{ccc}
\mathrm{BD}(n,r) & \xrightarrow{p=\eta\circ\zeta=\zeta\circ\eta} & \mathrm{BD}(n,r) \\
\downarrow & & \downarrow \\
\Omega S^{2n+1}\{p^r\} & \xrightarrow{\Omega p} & \Omega S^{2n+1}\{p^r\} \\
\downarrow & & \downarrow \\
\Omega S^{2n+1} & \xrightarrow{\Omega p} & \Omega S^{2n+1}
\end{array}
$$

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These equalities allow us to easily compute any composition $\eta \circ \zeta$ and any composition $\zeta \circ \eta$. For example, $\eta \circ \zeta\colon \mathrm{BD}(n,r) \to \mathrm{BD}(n,r+t) \to \mathrm{BD}(n,r+t-s)$ equals $\zeta \circ p^{t-s} = p^{t-s} \circ \zeta$ if $t, t-s > 0$.

If we compose the modified James-Hopf invariant $\bar{h}_{p^k}$ with the map in (2.1), we get a map

$$H_{p^k}\colon \Omega P^{2n+1}(p^r) \to \mathrm{BD}(p^k n, r)$$

and a commutative diagram with the columns fibration sequences:

$$\begin{array}{ccc} \Omega F^{2n+1}\{p^r\} & \xrightarrow{\bar{H}_{p^k}} & S^{2p^k n-1} \\ \downarrow & & \downarrow \\ \Omega P^{2n+1}(p^r) & \xrightarrow{H_{p^k}} & \mathrm{BD}(p^k n, r) \\ \downarrow & & \downarrow \\ \Omega S^{2n+1} & \xrightarrow{h_{p^k}} & \Omega S^{2p^k n+1}. \end{array} \tag{2.8}$$

We shall call the map H_{p^k} the Anick-James-Hopf invariant.

3 Selick's Lifting Method

In this section, we apply a method due to Selick [12] to construct a lifting of $\Omega(p) \circ \Omega(H_{p^k})$, the loops on the composition of the Anick-James-Hopf invariant with the p-th power map.

Selick's lifting method is based on two facts. First, if $A \to \Omega S^{2n+1}$ is any map with A a space of category $< p^k$, then the composition with the p^k-th James-Hopf invariant followed by the p-th power,

$$A \longrightarrow \Omega S^{2n+1} \xrightarrow{h_{p^k}} \Omega S^{2p^k n+1} \xrightarrow{\Omega(p)} \Omega S^{2p^k n+1},$$

is null homotopic. Second, if G is a topological group with classifying space BG and Milnor filtration $B_j G$, then the composite map $G \xrightarrow{\Sigma} \Omega\Sigma G = \Omega B_1 G \subseteq \Omega BG$ is a homotopy equivalence with $G \to \Omega B_j G$ an H-map for $j > 1$. Selick wrote his proofs only for the case $k = 1$, but, as he knew, they work without change for $k \geq 1$.

Let $1 < j < p^k$. Since $B_j G$ is a space of category j, it follows that the composite map

$$B_j(\Omega^2 S^{2n+1}) \subseteq B(\Omega^2 S^{2n+1}) = \Omega S^{2n+1} \xrightarrow{h_{p^k}} \Omega S^{2p^k n+1} \xrightarrow{\Omega(p)} \Omega S^{2p^k n+1}$$

is null homotopic. The range being simply connected, we can assume that the homotopy is basepoint preserving.

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If we restrict (2.8) to B_jG, we get a diagram

$$
\begin{array}{ccccc}
B_j(\Omega^2 F^{2n+1}\{p^r\}) & \xrightarrow{\bar{H}_{p^k}} & S^{2p^k n-1} & \xrightarrow{p} & S^{2p^k n-1} \\
\downarrow & & \downarrow & & \downarrow \\
B_j\big(\Omega^2 P^{2n+1}(p^r)\big) & \xrightarrow{H_{p^k}} & \mathrm{BD}(p^k n, r) & \xrightarrow{p} & \mathrm{BD}(p^k n, r) \\
\downarrow & & \downarrow & & \downarrow \\
B_j\Omega^2 S^{2n+1} & \xrightarrow{h_{p^k}} & \Omega S^{2p^k n+1} & \xrightarrow{\Omega(p)} & \Omega S^{2p^k n+1}.
\end{array}
\tag{3.1}
$$

The above basepoint preserving null homotopy of $\Omega(p) \circ h_{p^k}$ on $B_j(\Omega^2 S^{2n+1})$ yields, via the covering homotopy property, a homotopy of $p \circ H_{p^k}$ defined on $B_j\big(\Omega^2 P^{2n+1}(p^r)\big)$. This covering homotopy terminates at a map $\bar{H}$ of $B_j\big(\Omega^2 P^{2n+1}(p^r)\big)$ into the fibre $S^{2p^k n-1}$ and since the original homotopy is basepoint preserving it may be constructed to be a stationary homotopy on $B_j(\Omega^2 F^{2n+1}\{p^r\})$. Hence, we get a commutative diagram

$$
\begin{array}{ccc}
B_j(\Omega^2 F^{2n+1}\{p^r\}) & \xrightarrow{p\circ\bar{H}_{p^k}} & S^{2p^k n-1} \\
\downarrow & & \downarrow{=} \\
B_j\big(\Omega^2 P^{2n+1}(p^r)\big) & \xrightarrow{\bar{H}} & S^{2p^k n-1} \\
\downarrow{=} & & \downarrow \\
B_j\big(\Omega^2 P^{2n+1}(p^r)\big) & \xrightarrow{p\circ H_{p^k}} & \mathrm{BD}(p^k n, r).
\end{array}
\tag{3.2}
$$

If we loop (3.2), inject G into ΩB_jG as described in the first paragraph of this section, and include the maps σ and $\bar{\sigma}$ from the end of Section 1, we get a commutative diagram of H-maps as follows

$$
\begin{array}{ccccc}
\Omega W^{2n+1}\{p^r\} & \xrightarrow{\Omega\bar{\sigma}} & \Omega^2 F^{2n+1}\{p^r\} & \xrightarrow{\Omega(p)\circ\Omega(\bar{H}_{p^k})} & \Omega S^{2p^k n-1} \\
\downarrow & & \downarrow & & \downarrow{=} \\
\Omega T^{2n+1}(p^r) & \xrightarrow{\Omega\sigma} & \Omega^2 P^{2n+1}(p^r) & \xrightarrow{H} & \Omega S^{2p^k n-1} \\
\downarrow{=} & & \downarrow{=} & & \downarrow \\
\Omega T^{2n+1}(p^r) & \xrightarrow{\Omega\sigma} & \Omega^2 P^{2n+1}(p^r) & \xrightarrow{\Omega(p)\circ\Omega(H_{p^k})} & \Omega\,\mathrm{BD}(p^k n, r).
\end{array}
\tag{3.3}
$$

4 Lifting the Lift

In this section we construct a lift of the map H in (3.3).

The main technical result of [10], slightly extended as in [11], is a map $T^{2n+1}(p^r) \to \Pi_r$ such that, if we compose this map with the map $S^{2n-1} \times \Pi_{r+1} \to T^{2n+1}(p^r)$, the result fits into horizontal fibration sequences

$$\begin{array}{ccccc} C(n) \times \Pi_1 & \xrightarrow{1\times\Pi\zeta} & C(n) \times \Pi_{r+1} & \longrightarrow & \Pi_r \\ \downarrow & & \downarrow & & \downarrow = \\ S^{2n-1} \times \Pi_1 & \xrightarrow{1\times\Pi\zeta} & S^{2n-1} \times \Pi_{r+1} & \longrightarrow & \Pi_r \end{array} \tag{4.1}$$

where $C(n)$ is the fibre of the double suspension $S^{2n-1} \to \Omega^2 S^{2n+1}$.

Consider the diagram of [10] in which the rows and columns are all fibration sequences, $C(n) \to S^{2n-1} \xrightarrow{\Sigma^2} \Omega^2 S^{2n+1}$ is the fibration sequence of the double suspension, and the left hand column is the evident product:

$$\begin{array}{ccccc} C(n) \times \Pi_{r+1} & \longrightarrow & T^{2n+1}(p^r) & \longrightarrow & \Omega S^{2n+1}\{p^r\} \\ \downarrow & & \downarrow = & & \downarrow \\ S^{2n-1} \times \Pi_{r+1} & \longrightarrow & T^{2n+1}(p^r) & \longrightarrow & \Omega S^{2n+1} \\ \downarrow & & \downarrow & & \downarrow p^r \\ \Omega^2 S^{2n+1} & \longrightarrow & * & \longrightarrow & \Omega S^{2n+1}. \end{array} \tag{4.2}$$

The fact that $S^{2n+1}\{p^r\}$ is an H-space with a null homotopic p^r-th power map [9], together with (4.1) and (4.2) looped to make everything an H-map, easily shows, using standard lifting properties of fibrations as in the proof of Proposition 1.2 of [10], that the p^r-th power map on $\Omega T^{2n+1}(p^r)$ factors as

$$\begin{aligned} \Omega(p^r)\colon \Omega T^{2n+1}(p^r) &\longrightarrow \Omega C(n) \times \Omega\Pi_1 \xrightarrow{1\times\Pi\zeta} \Omega C(n) \\ \times\, \Omega\Pi_{r+1} &\longrightarrow \Omega S^{2n-1} \times \Omega\Pi_{r+1} \longrightarrow \Omega T^{2n+1}(p^r). \end{aligned} \tag{4.3}$$

Notice that $W^{2n+1}\{p^r\} = S^{2n-1} \times \Pi_{r+1}$ and use the above paragraph and the fact that H in (3.3) is an H-map to conclude that $\Omega(p^r) \circ H \circ \Omega\sigma = H \circ \Omega\sigma \circ \Omega(p^r)$ factors as

$$\begin{aligned} \Omega T^{2n+1}(p^r) &\longrightarrow \Omega C(n) \\ \times\, \Omega\Pi_1 &\longrightarrow \Omega W^{2n+1}\{p^r\} \xrightarrow{\Omega\bar\sigma} \Omega^2 F^{2n+1}\{p^r\} \xrightarrow{\Omega(p)\circ\Omega(\bar H_{p^k})} \Omega S^{2p^k n-1}. \end{aligned}$$

Since $C(n)$ and Π_1 are both H-spaces with null homotopic p-th power maps [5, 9], it follows that $\Omega(p^r) \circ H \circ \Omega\sigma$ is null homotopic and thus that we have a lift of $H \circ \Omega\sigma\colon \Omega T^{2n+1}\{p^r\} \to \Omega S^{2p^k n-1}$ to a map $K\colon \Omega T^{2n+1}\{p^r\} \to \Omega S^{2p^k n-1}\{p^r\}$.

5 Proof of the Decomposition Theorem for the Double Loop Space

From Section 2 we get a commutative diagram with the columns fibration sequences

$$\begin{array}{ccccc} S^{2p^kn-1}\{p^{r+1}\} & \xrightarrow{=} & S^{2p^kn-1}\{p^{r+1}\} & \xrightarrow{\eta} & S^{2p^kn-1}\{p^r\} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{BD}(p^kn,r) & \xrightarrow{\eta} & \mathrm{BD}(p^kn,r-1) & \xrightarrow{\zeta} & \mathrm{BD}(p^kn,r) \\ \downarrow\zeta & & \downarrow\zeta & & \downarrow\zeta \\ \mathrm{BD}(p^kn,2r+1) & \xrightarrow{\eta} & \mathrm{BD}(p^kn,2r) & \xrightarrow{=} & \mathrm{BD}(p^kn,2r) \end{array} \tag{5.1}$$

with $\zeta\circ\eta = p$ in the middle row. Since the spaces $D(p^k,0)$ do not exist, we are required to assume that $r \geq 2$. Clearly, the lower left hand square and the upper right hand square are both homotopy pullbacks, a fact that will be preserved if we apply the loop functor to (5.1).

Hence, the maps

$$K\colon \Omega T^{2n+1}(p^r) \to \Omega S^{2p^kn-1}\{p^r\}$$

and

$$\Omega\eta\circ\Omega H_{p^k}\circ\Omega\sigma : \Omega T^{2n+1}(p^r) \to \Omega^2 P^{2n+1}(p^r) \to \Omega\,\mathrm{BD}(p^kn,r) \to \Omega\,\mathrm{BD}(p^kn,r-1)$$

yield a map

$$L\colon \Omega T^{2n+1}(p^r) \to \Omega S^{2p^kn-1}\{p^{r+1}\}$$

and hence a map into the product

$$\bar{L}\colon \Omega T^{2n+1}(p^r) \to \Omega\Pi_{r+1}.$$

We claim that the composition $\Omega\Pi_{r+1} \to \Omega T^{2n+1}(p^r) \to \Omega\Pi_{r+1}$ is a homotopy equivalence, which as mentioned in Section 0 is equivalent to proving the product decomposition in (0.1).

Sections 1 and 2 imply that ΩH_{p^k} maps the transgression τ of τ_k in $H_{2p^kn-2}\big(\Omega T^{2n+1}(p^r)\big)$ to the generator $u(2p^kn-2,r)$ of $H_{2p^kn-2}\big(\Omega\,\mathrm{BD}(p^kn,r)\big)$. Then $\Omega\eta$ sends it to the generator $u(2p^kn-2,r-1)$ of $H_{2p^kn-2}\big(\Omega\,\mathrm{BD}(p^kn,r-1)\big)$. From the homological properties of the fibration in the middle column of (5.1), it follows that L sends τ to the generator $v(2p^kn-2,r+1)$ of $H_{2p^kn-2}(\Omega S^{2p^kn-1}\{p^{r+1}\})$. From the proof of Theorem 6.1 in [10], we see that $v(2p^kn-2,r+1)$ is the only primitive element in $H_{2p^kn-2}(\Omega\Pi_{r+1})$. It follows that $\bar{L}$ sends τ to $v(2p^kn-2,r+1)$ in $H_{2p^kn-2}(\Omega\Pi_{r+1})$. From [4], [10], we see that $\Omega\Pi_{r+1} \to \Omega T^{2n+1}(p^r)$ sends $v(2p^kn-2,r+1)$ to τ. Now Theorem 6.1 in [10], a generalization of an atomicity result in [12], says that any self map of $\Omega\Pi_{r+1}$ which sends $v(2p^kn-2,r+1)$ to itself and hence its $(r+1)$-st Bockstein $u(2p^kn-3,r+1)$ to itself is a homotopy equivalence. Thus, the composition in the preceding paragraph is a homotopy equivalence.

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