Glasgow Math. J. **45** (2003) 217–224. © 2003 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S0017089503001186. Printed in the United Kingdom

THE CONSERVATION OF NUMBER PRINCIPLE IN REAL ALGEBRAIC GEOMETRY

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(Received 7 December, 2001; accepted 8 November, 2002)

Abstract. The classical conservation of number principle is an important result in algebraic geometry. We present a version of this principle suitable for the study of topological properties of real algebraic varieties. Our self-contained topological proof does not depend on the intersection theory of algebraic cycles. Some applications are included.

2000 *Mathematics Subject Classification.* 14P25, 14C25.

1. Introduction and results. The goal of this note is to give self-contained topological proofs of certain results in real algebraic geometry, which heretofore required techniques of intersection theory (Chow rings, algebraic equivalence of cycles, etc.) [**1, 8, 9**]. The main results are a suitable version of the conservation of number principle (Theorem 1.4) and an application of this principle concerning topological properties of fibers of a real algebraic morphism (Theorem 1.7).

Throughout this note the term *real algebraic variety* designates a locally ringed space isomorphic to an algebraic subset of $\mathbb{R}^n$, for some n, endowed with the Zariski topology and the sheaf of $\mathbb{R}$-valued regular functions. Morphisms between real algebraic varieties will be called *regular maps*. Basic facts on real algebraic varieties and regular maps can be found in [**4**]. Every real algebraic variety carries also the Euclidean topology, which is determined by the usual metric topology on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions related to real algebraic varieties will refer to the Euclidean topology.

Given a compact real algebraic variety X, we denote by $H_d^{\mathrm{alg}}(X, \mathbb{Z}/2)$ the subgroup of the homology group $H_d(X, \mathbb{Z}/2)$ generated by the homology classes of d-dimensional Zariski closed subsets of X [**2, 3, 4, 6**]. Assuming that X is nonsingular, we let $H^c_{\mathrm{alg}}(X, \mathbb{Z}/2)$ denote the inverse image of $H_d^{\mathrm{alg}}(X, \mathbb{Z}/2)$ under the Poincaré duality isomorphism

$$D_X : H^c(X, \mathbb{Z}/2) \to H_d(X, \mathbb{Z}),\ D_X(\alpha) = \alpha \cap [X],$$

where $c + d = \dim X$ and $[X]$ is the fundamental class of X.

The groups $H_d^{\mathrm{alg}}(-, \mathbb{Z}/2)$ and $H^c_{\mathrm{alg}}(-, \mathbb{Z}/2)$ have the expected functorial properties: If $f : X \to Y$ is a regular map between compact nonsingular real algebraic varieties, then the induced homomorphisms

$$f_* : H_*(X, \mathbb{Z}/2) \to H_*(Y, \mathbb{Z}/2),\ f^* : H^*(Y, \mathbb{Z}/2) \to H^*(X, \mathbb{Z}/2)$$

https://doi.org/10.1017/S0017089503001186 Published online by Cambridge University Press

satisfy

$$f_*\big(H_d^{\mathrm{alg}}(X,\mathbb{Z}/2)\big) \subseteq H_d^{\mathrm{alg}}(Y,\mathbb{Z}/2),\ f^*\big(H^c_{\mathrm{alg}}(Y,\mathbb{Z}/2)\big) \subseteq H^c_{\mathrm{alg}}(X,\mathbb{Z}/2).$$

Furthermore,

$$H^*_{\mathrm{alg}}(X,\mathbb{Z}/2) = \bigoplus_{c\geq 0} H^c_{\mathrm{alg}}(X,\mathbb{Z}/2)$$

is a subring of the cohomology ring $H^*(X,\mathbb{Z}/2)$. Proofs of these facts are in **[2, 3, 6]** (**[2, 3]** contain topological proofs).

Assume that X is compact and nonsingular. A cohomology class α in $H^k_{\mathrm{alg}}(X,\mathbb{Z}/2)$ is said to be *algebraically equivalent to* 0 if there exist a compact nonsingular irreducible real algebraic variety T, two points t_0 and t_1 in T, and a cohomology class σ in $H^k_{\mathrm{alg}}(X\times T,\mathbb{Z}/2)$ such that $\alpha = \sigma_{t_1} - \sigma_{t_0}$, where given t in T, one defines $i_t : X \to X\times T$ by $i_t(x) = (x,t)$ for all x in X, and sets $\sigma_t = i_t^*(\sigma)$. We denote by $\mathrm{Alg}^k(X)$ the set of all cohomology classes in $H^k_{\mathrm{alg}}(X,\mathbb{Z}/2)$ that are algebraically equivalent to 0.

EXAMPLE 1.1. Let X be a compact nonsingular irreducible real algebraic variety of dimension n. Obviously, $H^n_{\mathrm{alg}}(X,\mathbb{Z}/2) = H^n(X,\mathbb{Z}/2)$. We assert that given any two distinct points t_0 and t_1 in X, the cohomology class α in $H^n_{\mathrm{alg}}(X,\mathbb{Z}/2)$, Poincaré dual to the homology class in $H_0^{\mathrm{alg}}(X,\mathbb{Z}/2)$ represented by $\{t_0, t_1\}$, belongs to $\mathrm{Alg}^n(X)$. Indeed, let σ in $H^n_{\mathrm{alg}}(X\times X,\mathbb{Z}/2)$ be the cohomology class Poincaré dual to the homology class in $H_n^{\mathrm{alg}}(X\times X,\mathbb{Z}/2)$ represented by the diagonal

$$\Delta = \{(x,t) \in X\times X \mid x = t\}.$$

For any point t in X, the map $i_t : X \to X\times X$, defined by $i_t(x) = (x,t)$ for all x in X, is transverse to Δ and hence $D_X(i_t^*(\sigma))$ is the homology class in $H_0(X,\mathbb{Z}/2)$ represented by $i_t^{-1}(\Delta)$. Since $i_t^*(\sigma) = \sigma_t$ and $i_t^{-1}(\Delta) = \{t\}$, we get $\alpha = \sigma_{t_1} - \sigma_{t_0}$. Thus α belongs to $\mathrm{Alg}^n(X)$ as asserted. Note that $\alpha \neq 0$ if t_0 and t_1 belong to distinct connected components of X.

In a straightforward manner one can prove the following result.

PROPOSITION 1.2. *For any compact nonsingular real algebraic variety X, the set $\mathrm{Alg}^k(X)$ is a subgroup of $H^k_{\mathrm{alg}}(X,\mathbb{Z}/2)$. If α is in $\mathrm{Alg}^k(X)$ and γ is in $H^\ell_{\mathrm{alg}}(X,\mathbb{Z}/2)$, then $\alpha\cup\gamma$ is in $\mathrm{Alg}^{k+\ell}(X)$. If moreover, δ is in $\mathrm{Alg}^m(Y)$, where Y is a compact nonsingular real algebraic variety, then $\gamma\times\delta$ is in $\mathrm{Alg}^{\ell+m}(X\times Y)$.*

The group $\mathrm{Alg}^k(-)$ also has nice functorial properties.

PROPOSITION 1.3. *Let $f : X \to Y$ be a regular map between compact nonsingular real algebraic varieties. Then*

(i) $f^*(\mathrm{Alg}^k(Y)) \subseteq \mathrm{Alg}^k(X)$,

(ii) $(D_Y^{-1}\circ f_*\circ D_X)(\mathrm{Alg}^{n-k}(X)) \subseteq \mathrm{Alg}^{p-k}(Y)$, *where* $n = \dim X$ *and* $p = \dim Y$.

Propositions 1.2 and 1.3 will be proved in Section 2.

Given a compact nonsingular real algebraic variety X, two cohomology classes α_1 and α_2 in $H^k_{\mathrm{alg}}(X,\mathbb{Z}/2)$ are said to be *algebraically equivalent* if $\alpha_1 - \alpha_2$ is in $\mathrm{Alg}^k(X)$.

For α in $H^k(X,\mathbb{Z}/2)$ and β in $H^\ell(X,\mathbb{Z}/2)$, where $k+\ell = \dim X$, we denote by $\alpha\bullet\beta$ the intersection number of α and β, that is, $\alpha\bullet\beta := \langle\alpha\cup\beta, [X]\rangle$. Thus $\alpha\bullet\beta$ is an element of $\mathbb{Z}/2$.

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The next result is called the *conservation of number principle*.

THEOREM 1.4. *Let X be a compact nonsingular real algebraic variety. Assume that α_1, α_2 in $H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$ are algebraically equivalent and β_1, β_2 in $H^\ell_{\mathrm{alg}}(X, \mathbb{Z}/2)$ are algebraically equivalent. If $k + \ell = \dim X$, then $\alpha_1 \bullet \beta_1 = \alpha_2 \bullet \beta_2$.*

As a consequence we immediately obtain the following fact.

COROLLARY 1.5. *For any compact nonsingular real algebraic variety X, one has*

$$\dim_{\mathbb{Z}/2}\left(H^k(X, \mathbb{Z}/2)/H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)\right) \geq \dim_{\mathbb{Z}/2} \mathrm{Alg}^\ell(X),$$

where $k + \ell = \dim X$.

Proof. By Theorem 1.4, $\alpha \bullet \beta = 0$ for all α in $H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$ and all β in $\mathrm{Alg}^\ell(X)$. The proof is complete since

$$H^k(X, \mathbb{Z}/2) \times H^\ell(X, \mathbb{Z}/2) \to \mathbb{Z}/2, \quad (\alpha, \beta) \to \alpha \bullet \beta$$

is a dual pairing [**7,** Proposition 8.13].

EXAMPLE 1.6. Note that

$$X = \{(x, y, z) \in \mathbb{R}^3 \mid ((x^2 + y^2) - 1)((x^2 + y^2) - 2) + z^2 = 0\}$$

is a nonsingular Zariski closed surface in $\mathbb{R}^3$, homeomorphic to a torus, and

$$C = \{(u, v) \in \mathbb{R}^2 \mid (u^2 - 1)(u^2 - 2) + v^2 = 0\}$$

is a compact nonsingular Zariski closed curve in $\mathbb{R}^2$, with two connected components C_+ containing $(1, 0)$ and C_- containing $(-1, 0)$. The map $\pi : X \to C$, $\pi(x, y, z) = (x^2 + y^2, z)$, is regular, $\pi(X) = C_+$, and $\pi : X \to C_+$ is a smooth (of class $\mathcal{C}^\infty$) circle bundle over C_+. Let β be the cohomology class in $H^1(C, \mathbb{Z}/2)$ Poincaré dual to the homology class in $H_0(C, \mathbb{Z}/2)$ represented by $\{(1, 0), (-1, 0)\}$. In view of Example 1.1, β is in $\mathrm{Alg}^1(C)$. It follows from Proposition 1.3(i) that $\pi^*(\beta)$ belongs to $\mathrm{Alg}^1(X)$. By construction, $\pi^*(\beta) \neq 0$ and hence $\mathrm{Alg}^1(X) \neq 0$. Applying Corollary 1.5, we get $H^1_{\mathrm{alg}}(X, \mathbb{Z}/2) \neq H^1(X, \mathbb{Z}/2)$. Since $H^1(X, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^2$, we have $H^1_{\mathrm{alg}}(X, \mathbb{Z}/2) = \mathrm{Alg}^1(X) \cong \mathbb{Z}/2$.

If $X^n = X \times \cdots \times X$ is the n-fold product, then, in view of the last statement of Proposition 1.2, $\mathrm{Alg}^k(X^n) \neq 0$ for $1 \leq k \leq n$.

This example was first used by Joost van Hamel (unpublished) to illustrate a somewhat different phenomenon.

Our next result can also be deduced from Theorem 1.4.

THEOREM 1.7. *Let $f : X \to Y$ be a regular map between compact nonsingular real algebraic varieties. If Y is irreducible, then given two regular values y_1 and y_2 of f, the smooth manifolds $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are cobordant.*

This result is of interest if y_1 and y_2 belong to distinct connected components of Y. A different proof of Theorem 1.7 can be found in [**5**].

Proofs of Theorems 1.4 and 1.7 are given in Section 3.

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2. Proof of the propositions. Given real algebraic varieties X and T, a point t in T, and a cohomology class τ in $H^k(X \times T, \mathbb{Z}/2)$, we set $\tau_t = i_t^*(\tau)$, where $i_t : X \to X \times T$ is defined by $i_t(x) = (x, t)$ for all x in X.

It is convenient to give the following characterization of cohomology classes algebraically equivalent to 0.

LEMMA 2.1. *For any compact nonsingular real algebraic variety X, given a cohomology class α in $H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$, the following conditions are equivalent:*

(a) *α is algebraically equivalent to* 0,

(b) *there exist a compact nonsingular irreducible real algebraic variety T, two points t_0 and t_1 in T, and a cohomology class τ in $H^k_{\mathrm{alg}}(X \times T, \mathbb{Z}/2)$ such that $\tau_{t_0} = 0$ and $\tau_{t_1} = \alpha$.*

Proof. Suppose that (a) holds. Then there exist a compact nonsingular irreducible real algebraic variety T, two points t_0 and t_1 in T, and a cohomology class σ in $H^k_{\mathrm{alg}}(X \times T, \mathbb{Z}/2)$ such that $\alpha = \sigma_{t_1} - \sigma_{t_0}$. Let $\pi : X \times T \to X$ be the canonical projection. Since $i_{t_0} \circ \pi \circ i_t = i_{t_0}$ for every point t in T, setting $\tau = \sigma - \pi^*(i_{t_0}^*(\sigma))$, we get

$$\tau_t = i_t^*(\sigma) - i_t^*\big(\pi^*\big(i_{t_0}^*(\sigma)\big)\big) = \sigma_t - \big(i_{t_0} \circ \pi \circ i_t\big)^*(\sigma) = \sigma_t - \sigma_{t_0}.$$

In particular, $\tau_{t_1} = \sigma_{t_1} - \sigma_{t_0} = \alpha$ and $\tau_{t_0} = 0$. Hence (b) is satisfied.

The proof is complete since it is obvious that (b) implies (a).

Proof of Proposition 1.2. In order to prove that $\mathrm{Alg}^k(X)$ is a subgroup of $H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$ it suffices to show that given α and β in $\mathrm{Alg}^k(X)$, the sum $\alpha + \beta$ is in $\mathrm{Alg}^k(X)$. By Lemma 2.1, there exist compact nonsingular irreducible real algebraic varieties T and U, and cohomology classes σ in $H^k_{\mathrm{alg}}(X \times T, \mathbb{Z}/2)$ and τ in $H^k_{\mathrm{alg}}(X \times U, \mathbb{Z}/2)$ such that $\sigma_{t_0} = 0$, $\sigma_{t_1} = \alpha$ for some t_0, t_1 in T and $\tau_{u_0} = 0$, $\tau_{u_1} = \beta$ for some u_0, u_1 in U. Given t in T and u in U, let $i_t : X \to X \times T$, $j_u : X \to X \times U$, $e_{(t,u)} : X \to X \times T \times U$ be the maps defined by $i_t(x) = (x, t)$, $j_u(x) = (x, u)$, $e_{(t,u)}(x) = (x, t, u)$ for all x in X. Denoting by $\pi : X \times T \times U \to X \times T$ and $\rho : X \times T \times U \to X \times U$ the canonical projections, we have $\pi \circ e_{(t,u)} = i_t$ and $\rho \circ e_{(t,u)} = j_u$. Thus, setting $\xi = \pi^*(\sigma) + \rho^*(\tau)$, we get

$$\begin{aligned}
\xi_{(t,u)} &= e_{(t,u)}^*(\pi^*(\sigma) + \rho^*(\tau)) \\
&= \big(\pi \circ e_{(t,u)}\big)^*(\sigma) + \big(\rho \circ e_{(t,u)}\big)^*(\tau) \\
&= i_t^*(\sigma) + j_u^*(\tau) \\
&= \sigma_t + \tau_u.
\end{aligned}$$

In particular, $\xi_{(t_0,u_0)} = \sigma_{t_0} + \tau_{u_0} = 0$ and $\xi_{(t_1,u_1)} = \sigma_{t_1} + \tau_{u_1} = \alpha + \beta$. Hence $\alpha + \beta$ is in $\mathrm{Alg}^k(X)$. We proved that $\mathrm{Alg}^k(X)$ is a subgroup of $H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$.

Let $p : X \times T \to X$ be the canonical projection and set $\eta = \sigma \cup p^*(\gamma)$. Since $p \circ i_t$ is the identity map of X, we get

$$\eta_t = i_t^*(\sigma \cup p^*(\gamma)) = i_t^*(\sigma) \cup i_t^*(p^*(\gamma)) = \sigma_t \cup (p \circ i_t)^*(\gamma) = \sigma_t \cup \gamma.$$

In particular, $\eta_{t_0} = \sigma_{t_0} \cup \gamma = 0 \cup \gamma = 0$ and $\eta_{t_1} = \sigma_{t_1} \cup \gamma = \alpha \cup \gamma$. Thus $\alpha \cup \gamma$ is in $\mathrm{Alg}^{k+\ell}(X)$.

It remains to prove that $\gamma \times \delta$ is in $\mathrm{Alg}^{\ell+m}(X \times Y)$. By Lemma 2.1, there exist a compact nonsingular irreducible real algebraic variety T, two points t_0 and t_1 in T, and a cohomology class θ in $H^m_{\mathrm{alg}}(Y \times T, \mathbb{Z}/2)$ such that $\theta_{t_0} = 0$ and $\theta_{t_1} = \delta$. Since

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$\gamma \times \theta = q^*(\gamma) \cup r^*(\theta)$, where $q : X \times Y \times T \to X$ and $r : X \times Y \times T \to Y \times T$ are the canonical projections, it follows that $\gamma \times \theta$ belong to $H^{\ell+m}_{\text{alg}}(X \times Y \times T, \mathbb{Z}/2)$. For each t in T, we have $(\gamma \times \theta)_t = \gamma \times \theta_t$. In particular, $(\gamma \times \theta)_{t_0} = \gamma \times \theta_{t_0} = \gamma \times 0 = 0$ and $(\gamma \times \theta)_{t_1} = \gamma \times \theta_{t_1} = \gamma \times \delta$. Hence $\gamma \times \delta$ is in $\text{Alg}^{\ell+m}(X \times Y)$.

Proof of Proposition 1.3. (i) Let β be an element of $\text{Alg}^k(Y)$. By Lemma 2.1, there exist a compact nonsingular irreducible real algebraic variety T, two points t_0 and t_1 in T, and a cohomology class τ in $H^k_{\text{alg}}(Y, \mathbb{Z}/2)$ such that $\tau_{t_0} = 0$ and $\tau_{t_1} = \beta$. For t in T, let $i_t : X \to X \times T$ and $j_t : Y \to Y \times T$ be the maps defined by $i_t(x) = (x, t)$ for all x in X and $j_t(y) = (y, t)$ for all y in Y. Denoting by $i : X \to X$ the identity map, we have $(f \times i) \circ i_t = j_t \circ f$. Thus, setting $\sigma = (f \times i)^*(\tau)$, we obtain

$$\sigma_t = i_t^*((f \times i)^*(\tau)) = ((f \times i) \circ i_t)^*(\tau) = (j_t \circ f)^*(\tau) = f^*(j_t(\tau)) = f^*(\tau_t).$$

In particular, $\sigma_{t_0} = f^*(\tau_{t_0}) = f^*(0) = 0$ and $\sigma_{t_1} = f^*(\tau_{t_1}) = f^*(\beta)$, and hence $f^*(\beta)$ is in $\text{Alg}^k(X)$. This completes the proof of (i).

(ii) Let α be an element of $\text{Alg}^{n-k}(X)$. By Lemma 2.1, there exist a compact nonsingular irreducible real algebraic variety T, two points t_0 and t_1 in T, and a cohomology class σ in $H^{n-k}_{\text{alg}}(X \times T, \mathbb{Z}/2)$ such that $\sigma_{t_0} = 0$ and $\sigma_{t_1} = \alpha$.

Given a point t in T, let $e_t : \{t\} \hookrightarrow T$ be the inclusion map. For any cohomology class η in $H^s(T, \mathbb{Z}/2)$, we define the element $\epsilon_t(\eta)$ of $\mathbb{Z}/2$ by setting $\epsilon_t(\eta) = 1$ if $s = 0$ and $e_t^*(\eta) \neq 0$, and $\epsilon_t(\eta) = 0$ in all other cases.

For any λ in $H^r(X, \mathbb{Z}/2)$ and any μ in $H^r(Y, \mathbb{Z}/2)$, we have

$$i_t^*(\lambda \times \eta) = \epsilon_t(\eta)\lambda, \quad j_t^*(\mu \times \eta) = \epsilon_t(\eta)\mu,$$

where the i_t and j_t are the maps defined as in (i). If e is the identity map of T, then

$$\begin{aligned}
\big(D_Y \circ j_t^* \circ D^{-1}_{Y\times T} \circ (f \times e)_* \circ D_{X\times T}\big)(\lambda \times \eta) &= \big(D_Y \circ j_t^* \circ D^{-1}_{Y\times T} \circ (f \times e)_*\big)(D_X(\lambda) \times D_T(\eta)) \\
&= \big(D_Y \circ j_t^* \circ D^{-1}_{Y\times T}\big)(f_*(D_X(\lambda)) \times D_T(\eta)) \\
&= D_Y\big(j_t^*\big(D_Y^{-1}(f_*(D_X(\lambda))\big) \times \eta)\big) \\
&= D_Y\big(\epsilon_t(\eta) D_Y^{-1}(f_*(D_X(\lambda)))\big) \\
&= \epsilon_t(\eta) f_*(D_X(\lambda)) \\
&= f_*(D_X(\epsilon_t(\lambda)\lambda)) \\
&= (f_* \circ D_X \circ i_t^*)(\lambda \times \eta).
\end{aligned}$$

Since r and s are arbitrary, it follows from Künneth's theorem for cohomology that

$$D_Y \circ j_t^* \circ D^{-1}_{Y\times T} \circ (f \times e)_* \circ D_{X\times T} = f_* \circ D_X \circ i_t^*$$

as homomorphisms from $H^*(X \times T, \mathbb{Z}/2)$ into $H_*(Y, \mathbb{Z}/2)$, and hence

$$j_t^* \circ D^{-1}_{Y\times T} \circ (f \times e)_* \circ D_{X\times T} = D_Y^{-1} \circ f_* \circ D_X \circ i_t^*.$$

Setting now $\tau = (D^{-1}_{Y\times T} \circ (f \times e)_* \circ D_{X\times T})(\sigma)$, we obtain

$$\tau_t = j_t^*(\tau) = \big(D_Y^{-1} \circ f_* \circ D_X \circ i_t^*\big)(\sigma) = \big(D_Y^{-1} \circ f_* \circ D_X\big)(\sigma_t).$$

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In particular,

$$\tau_{t_0} = \big(D_Y^{-1} \circ f_* \circ D_X\big)\big(\sigma_{t_0}\big) = \big(D_Y^{-1} \circ f_* \circ D_X\big)(0) = 0$$
$$\tau_{t_1} = \big(D_Y^{-1} \circ f_* \circ D_X\big)\big(\sigma_{t_1}\big) = \big(D_Y^{-1} \circ f_* \circ D_X\big)(\alpha).$$

Hence $(D_Y^{-1} \circ f_* \circ D_X)(\alpha)$ is in $\mathrm{Alg}^{p-k}(Y)$, and the proof of (ii) is complete.

3. Proofs of the theorems. We begin with the following result.

LEMMA 3.1. *Let X be a compact nonsingular real algebraic variety of dimension n. Then for any cohomology class α in $\mathrm{Alg}^n(X)$, one has $\langle \alpha, [X]\rangle = 0$.*

Proof. Choose a finite subset S of X representing the homology class $D_X(\alpha) = \alpha \cap [X]$ in $H_0(X, \mathbb{Z}/2)$. By [**7,** p. 239], $\langle \alpha, [X]\rangle = \epsilon(\alpha \cap [X])$, where $\epsilon : H_0(X, \mathbb{Z}/2) \to \mathbb{Z}/2$ is the augmentation homomorphism. Hence, denoting by $\#S$ the number of elements of S, we get

$$\langle \alpha, [X]\rangle = \#S \,(\mathrm{mod}\ 2).$$

In order to complete the proof it suffices to show that $\#S$ is an even integer.

Suppose that $\#S$ is an odd integer. We obtain a contradiction as follows. Let Y be a real algebraic variety consisting of one point and let $f : X \to Y$ be the unique possible map. Obviously, $(D_Y^{-1} \circ f_* \circ D_X)(\alpha) \neq 0$ in $H^0(Y, \mathbb{Z}/2) \cong \mathbb{Z}/2$. On the other hand, by Proposition 1.3(ii), $(D_Y^{-1} \circ f_* \circ D_X)(\alpha)$ is in $\mathrm{Alg}^0(Y)$. However, since Y consists of one point, it follows from the definition that $\mathrm{Alg}^0(Y) = 0$. Thus we have a contradiction and the proof is complete.

Proof of Theorem 1.4. By assumption, $\alpha_1 - \alpha_2$ is in $\mathrm{Alg}^k(X)$ and $\beta_1 - \beta_2$ is in $\mathrm{Alg}^\ell(X)$. Therefore, in view of Proposition 1.2, $(\alpha_1 - \alpha_2) \cup \beta_1$ and $\alpha_2 \cup (\beta_1 \cup \beta_2)$ are in $\mathrm{Alg}^{k+\ell}(X)$. Hence

$$\langle \alpha_1 \cup \beta_1, [X]\rangle - \langle \alpha_2 \cup \beta_1, [X]\rangle = \langle (\alpha_1 - \alpha_2) \cup \beta_1, [X]\rangle = 0,$$
$$\langle \alpha_2 \cup \beta_1, [X]\rangle - \langle \alpha_2 \cup \beta_2, [X]\rangle = \langle \alpha_2 \cup (\beta_1 - \beta_2), [X]\rangle = 0,$$

where the last equality in either line is a consequence of Lemma 3.1. It follows that $\langle \alpha_1 \cup \beta_1, [X]\rangle = \langle \alpha_2 \cup \beta_2, [X]\rangle$, which is equivalent to $\alpha_1 \bullet \beta_1 = \alpha_2 \bullet \beta_2$. The proof is complete.

The proof of Theorem 1.7 requires some preparation. All manifolds we use will be smooth (of class $\mathcal{C}^\infty$), paracompact and without boundary. Let M be a smooth manifold and let N be a smooth submanifold of M. Assume that N is a closed subset of M. We denote by τ_N^M the Thom class of N in M; thus τ_N^M is in $H^k(M, M \setminus N; \mathbb{Z}/2)$, where $k = \dim M - \dim N$. If $N = \{x\}$, we shall write τ_x^M instead of $\tau_{\{x\}}^M$. Clearly, τ_x^M is just the unique generator of the group $H^m(M, M \setminus \{x\}, \mathbb{Z}/2) \cong \mathbb{Z}/2$, $m = \dim M$. As usual, $w_i(M)$ will denote the ith Stiefel-Whitney class of M.

Given a topological space T, we let $\epsilon_T : H_0(T, \mathbb{Z}/2) \to \mathbb{Z}/2$ denote the augmentation homomorphism.

Proof of Theorem 1.7. Let $n = \dim X$, $p = \dim Y$, and $k = n - p$. For any point y in Y, let β_y denote the cohomology class in $H^p(Y, \mathbb{Z}/2)$ Poincaré dual to the homology class in $H_0(Y, \mathbb{Z}/2)$ represented by y. By Example 1.1, given y_1 and y_2

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in Y, the cohomology class $\beta_{y_1} - \beta_{y_2}$ belongs to $\mathrm{Alg}^p(Y)$. In view of Proposition 1.3(i), $f^*(\beta_{y_1} - \beta_{y_2}) = f^*(\beta_{y_1}) - f^*(\beta_{y_2})$ is in $\mathrm{Alg}^p(X)$ and hence Theorem 1.4 implies that

$$\alpha \bullet f^*\left(\beta_{y_1}\right) = \alpha \bullet f^*\left(\beta_{y_2}\right)$$

for every cohomology class α in $H^k_{\mathrm{alg}}(X, \mathbb{Z}/2)$. It is known that $w_i(X)$ is in $H^i_{\mathrm{alg}}(X, \mathbb{Z}/2)$ for all $i \geq 0$ [**2, 3**]. Thus, given nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = k$, we have

$$\left(w_{i_1}(X) \cup \ldots \cup w_{i_r}(X)\right) \bullet f^*\left(\beta_{y_1}\right) = \left(w_{i_1}(X) \cup \ldots \cup w_{i_r}(X)\right) \bullet f^*\left(\beta_{y_2}\right). \qquad (1)$$

Let us set

$$n_{i_1 \ldots i_r}(f, y) = \left(w_{i_1}(X) \cup \ldots \cup w_{i_r}(X)\right) \bullet f^*(\beta_y).$$

Note that

$$n_{i_1 \ldots i_r}(f, y) = 0 \text{ for } y \text{ in } Y \setminus f(X), \qquad (2)$$

since y in $Y \setminus f(X)$ implies $f^*(\beta_y) = 0$.

If y in $f(X)$ is a regular value of f, then $f^{-1}(y)$ is a smooth submanifold of X of dimension k. We assert

$$n_{i_1 \ldots i_r}(f, y) = \left\langle w_{i_1}(f^{-1}(y)) \cup \ldots \cup w_{i_r}(f^{-1}(y)), [f^{-1}(y)]\right\rangle. \qquad (3)$$

Suppose that (3) holds. If y_1 and y_2 are regular values of f, then (1), (2), and (3) guarantee that $f^{-1}(y_1)$ and $f^{-1}(y_2)$ have the same Stiefel-Whitney numbers. Hence, by Thom's theorem [**11**], the smooth manifolds $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are cobordant. Thus it remains to prove (3).

In order to simplify notation set $F = f^{-1}(y)$. Let $\bar{f} : (X, X \setminus F) \to (Y, Y \setminus \{y\})$ be the map defined by f. Since y is a regular value of f, we have

$$\bar{f}^*\left(\tau^Y_y\right) = \tau^X_F.$$

Moreover the following diagram is commutative:

$$\begin{array}{ccc} H^p(Y, Y \setminus \{y\}; \mathbb{Z}/2) & \xrightarrow{\bar{f}^*} & H^p(X, X \setminus F; \mathbb{Z}/2) \\ \psi\downarrow & & \varphi\downarrow \\ H^p(Y, \mathbb{Z}/2) & \xrightarrow{f^*} & H^p(X, \mathbb{Z}/2), \end{array}$$

where φ and ψ are the canonical homomorphisms. Since $\psi(\tau^Y_y) = \beta_y$, it follows that

$$f^*(\beta_y) = f^*\left(\psi\left(\tau^Y_y\right)\right) = \varphi\left(\bar{f}^*\left(\tau^Y_y\right)\right) = \varphi\left(\tau^X_F\right). \qquad (4)$$

Note that if $e : F \hookrightarrow X$ is the inclusion map, then

$$\left\langle \alpha \cup \varphi\left(\tau^X_F\right), [X]\right\rangle = \langle e^*(\alpha), [F]\rangle \qquad (5)$$

https://doi.org/10.1017/S0017089503001186 Published online by Cambridge University Press

for every cohomology class α in $H^p(X, \mathbb{Z}/2)$. Indeed, (5) can be proved by direct computation:

$$\begin{aligned}\langle \alpha \cup \varphi(\tau_F^X), [X] \rangle &= \epsilon_X\big((\alpha \cup \varphi(\tau_F^X)) \cap [X]\big) \\ &= \epsilon_X\big(\alpha \cap (\varphi(\tau_F^X) \cap [X])\big) \\ &= \epsilon_X(\alpha \cap e_*([F])) \\ &= \epsilon_X(e_*(e^*(\alpha) \cap [F])) \\ &= \epsilon_F(e^*(\alpha) \cap [F]) \\ &= \langle e^*(\alpha), [F] \rangle,\end{aligned}$$

where the third equality holds since $\varphi(\tau_F^X) \cap [X] = e_*([F])$ [**10,** Problem 11.C], the fifth equality is a consequence of naturality of augmentation, and the other equalities are standard properties of the $\cup$, $\cap$, and $\langle , \rangle$ products [**7**].

Furthermore, since the normal vector bundle of F in X is trivial, we have $e^*(w_i(X)) = w_i(F)$ for all $i \geq 0$, and hence

$$e^*\big(w_{i_1}(X) \cup \ldots \cup w_{i_r}(X)\big) = w_{i_1}(F) \cup \ldots \cup w_{i_r}(F). \tag{6}$$

Now, making use of (4), (5), and (6), we get

$$\begin{aligned} n_{i_1 \ldots i_r}(f, y) &= \langle w_{i_1}(X) \cup \ldots \cup w_{i_r}(X) \cup f^*(\beta_y), [X] \rangle \\ &= \langle w_{i_1}(X) \cup \ldots \cup w_{i_r}(X) \cup \varphi(\tau_F^X), [X] \rangle \\ &= \langle e^*\big(w_{i_1}(X) \cup \ldots \cup w_{i_r}(X)\big), [F] \rangle \\ &= \langle w_{i_1}(F) \cup \ldots \cup w_{i_r}(F), [F] \rangle, \end{aligned}$$

which proves (3). Hence the proof is complete. □

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