2
Why QCD?

A possible approach to a theory like QCD is just to state its definition, and then immediately proceed deductively. However, this begs the question of why we should use this theory and not some other. Moreover, the approach is quite abstract, and the initial connection to the real physical world is missing.

Instead, I will take a quasi-historical approach, after first stating the theory. Such an approach is suitable for newcomers since their background in QCD is like that of its inventors/discoverers, i.e., little or none. There were several lines of development, all of which powerfully converged on a unique theory from key aspects of experimental data. Of course, we see this much more readily in retrospect than was apparent at the time of the original work, and my account is selective in focusing on the issues now seen to be the most significant. A historical approach also enables the introduction of ideas and methods that do not specifically depend on QCD: e.g., deeply inelastic scattering and the parton model.

I have tried to make the presentation self-contained, in summarizing the relevant experimental phenomena and their consequences for theory. The reader is only assumed to have a working knowledge of relativistic quantum field theory. Inevitably there are issues, ideas, experiments, and historical developments which will be unfamiliar to many readers, and for which a complete treatment needs much more space. I give references for many of these. In addition, there are several references that are global to the whole chapter and that the reader should refer to for more detail. A detailed historical account from the point of view of a physicist is given in the excellent book by Pais (1986). A good account of the phenomena is given by Perkins (2000). Standard books on quantum field theory also refer to them; see, for example, Sterman (1993); Peskin and Schroeder (1995); Weinberg (1995, 1996); Srednicki (2007). A comprehensive account of experimental results is given by the Particle Data Group in Amsler *et al*. (2008); this includes up-to-date authoritative summaries of measurements and their theoretical interpretation.

Naturally, QCD is not the whole story; there are known electromagnetic, weak and gravitational interactions, and presumably if we examine phenomena at short enough distances, beyond the reach of current experimental probes, we are likely to need new theories. But within the domain of the strong interaction at accessible scales, there is an amazing uniqueness to the structure of QCD.

2.1 QCD: statement of the theory

An expert in quantum field theory could simply define QCD as a standard Yang-Mills theory with a gauge group SU(3) and several multiplets of Dirac fields in the fundamental (triplet) representation of SU(3).

In more detail, QCD is specified by its set of field variables and its Lagrangian density $\mathcal{L}$. The Dirac fields $\psi_{\rho a f}$ are called quark fields, and the gauge fields A^{α}_{μ} are called gluon fields. On the quark fields the indices ρ, a, and f are respectively a Dirac index, a "color" index taking on three values, and a "flavor" index. The gauge group acts on the color index. Currently the flavor index has six known values u, d, s, c, b, t (or "up", "down", "strange", "charm", "bottom", and "top"). On the gluon field, the color index α has eight values, for the generators of SU(3), and μ is a Lorentz vector index. The important role played by the color charge leads to the theory's name, "quantum chromodynamics" or QCD. Of course, the names "color" and "flavor", and the names of the quark flavors, are whimsical inventions unrelated to their everyday meanings.

To deal with the renormalization of the UV divergences of QCD (Sec. 3.2) we distinguish between bare and renormalized quantities (fields, coupling and masses). We define QCD by a Lagrangian written in terms of bare quantities, which are distinguished by a subscript 0 or (0). The gauge-invariant Lagrangian is the standard Yang-Mills one:

$$\mathcal{L}_{\rm GI} = \bar{\psi}_0 (i \not{D} - m_0) \psi_0 - \frac{1}{4} (G^{\alpha}_{(0)\,\mu\nu})^2. \tag{2.1}$$

The full Lagrangian used for perturbation theory will add to this some terms to implement gauge fixing by the Faddeev-Popov method; see Sec. 3.1. The covariant derivative is given by

$$D_\mu \psi_0 \stackrel{\rm def}{=} \left(\partial_\mu + i g_0 t^\alpha A^\alpha_{(0)\,\mu} \right) \psi_0, \tag{2.2}$$

where t^α are the standard generating matrices[1] of the SU(3) group, acting on the color indices of ψ. The gluon field strength tensor is

$$G^{\alpha}_{(0)\,\mu\nu} \stackrel{\rm def}{=} \partial_\mu A^\alpha_{(0)\,\nu} - \partial_\nu A^\alpha_{(0)\,\mu} - g_0 f_{\alpha\beta\gamma} A^\beta_{(0)\,\mu} A^\gamma_{(0)\,\nu}, \tag{2.3}$$

where $f_{\alpha\beta\gamma}$ are the (fully antisymmetric) structure constants of the gauge group, defined so that $[t_\alpha, t_\beta] = i f_{\alpha\beta\gamma} t_\gamma$. The Lagrangian is invariant under local (i.e., space-time-dependent) SU(3) transformations:

$$\psi_{(0)\,\rho a f}(x) \mapsto \left[e^{-i g_0 \omega_\alpha(x) t^\alpha} \right]_{ab} \psi_{(0)\,\rho b f}(x), \tag{2.4a}$$

$$A^\alpha_{(0)\,\mu}(x) t^\alpha \mapsto \frac{-i}{g_0} e^{-i g_0 \omega_\alpha(x) t^\alpha} D_\mu e^{i g_0 \omega_\alpha(x) t^\alpha}. \tag{2.4b}$$

The quark fields have been redefined, as is always possible (Weinberg, 1973a), so that the mass matrix is diagonal:

$$\bar{\psi}_0 m_0 \psi_0 = m_{0\,u} \bar{u}_0 u_0 + m_{0\,d} \bar{d}_0 d_0 + m_{0\,s} \bar{s}_0 s_0 + \ldots \tag{2.5}$$

[1] $t^\alpha = \frac{1}{2}\lambda^\alpha$, where the standard λ^α are given in, e.g., Amsler *et al*. (2008, p. 338).

Here separate symbols are used for the fields for different flavors of quark: $u_{0\,\rho a} = \psi_{0\,\rho a u}$, etc.

The renormalized masses of the quarks are given in Table 2.2 below, along with the masses of the other elementary particles of the Standard Model. Large fractional uncertainties for the light quark masses arise because quarks are in fact confined inside color-singlet hadrons, which gives considerable complications in relating the mass parameters to data.

For their electromagnetic interactions, we need the quark charges:

$$e_d = e_s = e_b = -1/3, \qquad e_u = e_c = e_t = 2/3, \tag{2.6}$$

in units of the positron charge.

The only significant freedom in specifying QCD is in the set of matter fields, the quarks. At the time of discovery of QCD, only the u, d and s quarks were known; the c quark came slightly later. The discovery of the b and t quarks needed high enough collision energies to produce them. There have been many conjectures about possible new heavy quarks, both scalar and fermion, possibly in non-triplet color representations, but searches so far have been unsuccessful (Amsler *et al.*, 2008). The decoupling theorem (Appelquist and Carazzone, 1975) for heavy fields ensures that we can ignore the heavy fields if experimental energies are too low to make the corresponding particles.

A complete theory of strong, electromagnetic, and weak interactions is made by combining QCD with the Weinberg-Salam theory to form the Standard Model of elementary particle physics, summarized in Sec. 2.7.

2.2 Development of QCD

Why we should postulate the QCD Lagrangian and study QCD as the unique field theory for the strong interaction? An answer to this question should be at a high level and broad, since QCD is a high-level theory, intended to cover a broad range of phenomena, i.e., all of the strong hadronic interaction.

Starting in the 1950s, as accelerator energies increased, elementary particle physics gradually became a separate subject, distinct from nuclear physics. Several, not entirely distinct, strands of research led to the discovery of QCD in 1972–1973:

1. The quark model of hadron states.
2. The (successful) search for a theory of the weak interactions of leptons, including the weak interactions of hadrons.
3. Current algebra, i.e., the analysis of the currents for the (approximate) flavor symmetries of the strong interaction, including their relationships to the electroweak interactions of hadrons.
4. The theoretical development of non-abelian gauge theories.
5. Deeply inelastic lepton scattering and the measurement that the strong interaction is quite weak at short distances.

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It is almost paradoxical that many of the key issues involved the weak and electromagnetic interactions; much of the research on pure strong-interaction phenomena was not critical to the discovery of QCD.

2.2.1 Quantum fields

Always present was the notion of quantum field theory. Soon after the discovery of quantum mechanics, it was apparent that quantum fields formed an appropriate candidate framework in a search for an all-encompassing underlying theory of known interactions.

First, the basic dynamical variables are local in a field theory, so that there are separate variables to discuss, for example, an experiment in Illinois yesterday and an experiment in Switzerland tomorrow. This happens even in non-relativistic quantum theory. A theory of interacting quantum Schrödinger fields is readily constructed; this theory can be shown (Fetter and Walecka, 1980; Brown, 1992) to be equivalent to a collection of ordinary quantum mechanical theories in terms of N-body wave functions, but now for any N and with specified inter-particle interactions. In contrast, an ordinary Schrödinger equation for a wave function concerns, for example, only one particular electron and proton. But a quantized field theory can be formulated to describe all possible electrons and nuclei. Thus it encompasses all of atomic and molecular physics, not to mention chemistry, etc. Of course to take account of radiative phenomena, one also needs the electromagnetic field.

Since quantum field theories are intrinsically many-body theories, they are suitable for the construction of quantum theories that obey Einstein's special relativity. Once sufficient energy is available in a collision, particles can be created, so that a framework where particles are conserved is wrong. Fig. 2.3 below serves as an icon of this: it shows the multiparticle outcome of one particular positron-proton collision.

Furthermore, it is natural in relativity that fields obey local field equations, written in terms of fields and their derivatives. A non-local interaction would involve action at a distance, and would require enormous conspiracies to avoid faster-than-light propagation, etc.

To obtain a quantum field theory, it is sensible to start by postulating fields that correspond to observed particles, and then asking what interactions, governed by non-linear terms in the field equations, give observed phenomena. This approach was successful for the electromagnetic interaction and gave us the theory called QED. With a long delay to allow the full formulation of the needed non-abelian gauge theories, this approach was also successful for the weak interaction. Considerable restrictions were applied to the candidate theories, concerning self-consistency and renormalizability.

But for the strong interaction, there was a failure of this obvious approach, where one searches for a theory written in terms of fields for observed hadrons, initially the nucleons and pions. In retrospect, the reason is obvious: hadrons are composite, with the size of the bound states (Hofstadter, Bumiller, and Yearian, 1958), around $1\,\text{fm} = 1 \times 10^{-15}\,\text{m}$, being much less than the range of the strong nucleon-nucleon potential and the inter-nucleon separation in atomic nuclei (Hofstadter, 1956).

During the 1960s it became conventional, instead, to suppose that something other than a quantum field theory was needed for the strong interaction, an ultimately fruitless quest.[2] See the later chapters of Pais (1986) for a historical account.

2.2.2 Quark model

In strong-interaction physics very many unstable particle-like states, or resonances, have been discovered (Amsler *et al.*, 2008). They are generically termed hadrons. No fundamental distinction between the unstable and stable hadrons appeared to exist, stable hadrons simply being those that have no available decay channels. One natural hypothesis is that these states are bound states of more elementary particles, which turned out to be the actual case. The establishment of this view, starting in the 1950s, was quite non-trivial, however. Tightly coupled with these developments was the discovery that the strong interaction is approximately invariant under an internal symmetry, called SU(3) flavor symmetry; see, e.g., Gell-Mann (1962).

Within QCD, SU(3) flavor transformations are applied to the u, d and s quark fields, and would give an exact symmetry if the masses of the u, d and s quarks were equal. We get a useful approximate symmetry because the masses (and hence the mass differences) of these lightest three quarks are substantially less than a "normal hadronic mass scale", characterized by the proton mass. The c, b and t quarks (not known until after the construction of QCD) are singlets under SU(3) flavor transformations.

Flavor SU(3) symmetry is to be carefully distinguished from the later-discovered color symmetry group, which is also mathematically SU(3).

Gell-Mann (1964) and Zweig (1964a, b) constructed the quark model, in which baryons (like the proton and neutron) are bound states of three quarks, and mesons are bound states of a quark and antiquark. For the hadrons known at the time, they used three spin-$\frac{1}{2}$ quarks (u, d and s), with the fractional charge assignments of (2.6).

Now the u, d and s quarks are in the triplet representation of flavor SU(3). It follows (Gell-Mann, 1964) that baryons can be classified into multiplets that are singlet, octet and decuplet under SU(3), while the mesons are singlets and octets. Prior to the discovery of a satisfactory theory of the strong interaction, i.e., QCD, it was useful to investigate the consequences of the flavor symmetry abstractly, independently of any assumptions about a quark substructure or a Hamiltonian; see Sec. 2.2.4. Patterns of mass splitting within hadron multiplets can be understood quantitatively by using perturbation theory applied to symmetry-breaking terms in the strong-interaction Hamiltonian, with the hypothesis that symmetry-breaking terms are in an SU(3) octet. These terms are now identified with quark mass terms in QCD. See Amsler *et al.* (2008, Ch. 14) for a recent review and further references.

Each flavor of quark appeared to need three varieties (called "colors") in order for the spin-statistics theorem to hold. This is seen most easily for the $\Delta^{++}(1232)$

[2] Although the quest for a non-QFT theory of the strong interaction failed, it did lead to the invention of string theory, which now leads a prominent life as a candidate fundamental theory of everything including gravity.

baryon.[3] It is a ground-state baryon of spin $\frac{3}{2}$ consisting of three u quarks, so both the space and spin wave functions are totally symmetric.[4] But a side effect of the color hypothesis is that each meson (e.g., π^+) has an extra eight color states, which are not observed. An extra assumption is needed to prohibit the extra states.

Furthermore, there is a complete failure to detect isolated quarks in high-energy collisions, which requires the hypothesis that quarks are permanently confined in hadrons. Quark confinement obviously makes it harder to deduce from data the correct bound-state structure.

Thus there was a continued introduction of new hypotheses, which led to great scepticism (Zweig, 1980). Nevertheless, the situation was the unusual one of a correct general idea being forced by data into a unique implementation. In favor of the quark model, calculations with phenomenological interquark potentials allowed calculations for the energies of excited hadrons (non-ground-state hadrons), in essential agreement with data.

Around 1972, Fritzsch and Gell-Mann (1972) and Fritzsch, Gell-Mann, and Leutwyler (1973) had the inspiration that a non-abelian gauge theory, with an SU(3) gauge symmetry applied to the color degree of freedom, could not only give all these properties of the quark model, but could also solve other puzzles involving current algebra and the weak interactions of hadrons: Secs. 2.2.3 and 2.2.4.

Somewhat tentatively they proposed exactly the theory now known as QCD, missing only the heavy quarks, which in any event decouple from lower-energy physics and therefore do not affect the arguments. Understanding of the dynamics of the theory was still missing, in particular for the observations in deeply inelastic scattering: Sec. 2.3.

As regards the quark model, the unifying hypothesis suggested by the structure of QCD is that of "color confinement", that all observed states are color singlet. It simultaneously solves the quark confinement problem and the lack of extra meson states, and it is a natural conjecture, since gluons couple to color charge. Already in lowest-order perturbation theory it can be seen that the gluon exchange energy for a quark-antiquark pair is attractive for the color singlet state and repulsive for the color octet state: problem 2.1. Of course, perturbation theory for a generic strong-interaction quantity is at best a rough approximation. Even so, although a real demonstration of color confinement from QCD has still not been found, the hypothesis is consistent with all the evidence, theoretical and experimental.

The terms in the QCD Lagrangian that correspond to differences of quark masses give an operator in the Hamiltonian that transforms as an octet under *flavor* SU(3). This is exactly what had previously been assumed to explain mass splittings in the hadronic flavor multiplets.

[3] In this notation, the number denotes the mass in MeV, i.e., 1232 MeV, while the Δ^{++} denotes the quark and isospin content of the state (Amsler *et al.*, 2008, Ch. 8), which in this case corresponds to a baryon of isospin 3/2 with charge +2.

[4] The possibility that there are other types of particle statistics than Bose or Fermi was considered under the names of "para-statistics" or "quark statistics". But it was shown by Doplicher, Haag, and Roberts (1974) that all these possibilities are equivalent to ordinary Bose or Fermi statistics supplemented by selection rules on the allowed states. See also Drühl, Haag, and Roberts (1970). So the color solution is generic.

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2.2.3 Weak interactions

By the early 1970s there was a leading candidate for electroweak interactions of leptons, the Weinberg-Salam theory (Weinberg, 1967; Salam, 1968). This theory used spontaneous symmetry breaking to give mass to the weak gauge-bosons. It became a genuine candidate theory after it was shown how to successfully quantize and renormalize non-abelian gauge theories, and has since become fully established. This work solved severe consistency problems of theories with massive charged vector fields.

How is the theory to be extended to include hadrons? We treat the situation perturbatively in the electroweak interactions, using a decomposition of the complete Hamiltonian as

$$H = H_{\text{SI}} + H_{0,\,\text{lept}} + H_{\text{I, EW}} + H_{\text{SI-EW}}. \tag{2.7}$$

Here H_{SI} is the full strong-interaction Hamiltonian, not yet known around 1970, $H_{0,\,\text{lept}}$ is the free Hamiltonian for non-hadronic fields, $H_{\text{I, EW}}$ is the interaction Hamiltonian for electroweak interactions, and $H_{\text{SI-EW}}$ gives the coupling between hadronic fields and the electroweak fields.

We now do time-dependent perturbation theory with the unperturbed Hamiltonian including the *full* strong-interaction part, i.e., $H_0 = H_{\text{SI}} + H_{0,\,\text{lept}}$. Useful information can be extracted without either knowing or solving the full strong-interaction theory. The reason is that the couplings between the strong-interaction fields and the electroweak gauge fields were found from phenomenological evidence to involve currents for hadronic flavor symmetries. We write these in the form

$$H_{\text{SI-EW}} = \int \mathrm{d}^3\boldsymbol{x} \sum_A j_A^\mu W_{A,\mu} + \text{Higgs terms}, \tag{2.8}$$

where $W_{A,\mu}$ are the electroweak gauge fields $W^\pm$, Z and γ, while j_A^μ are the hadronic currents to which they couple. For consistency of the electroweak theory, the hadronic currents must be conserved, apart from the effects of their couplings to the electroweak fields. In fact, the currents, within the strong-interaction sector, are not quite conserved, which appears to be somewhat inconsistent. The inconsistency is solved retrospectively by the full Standard Model, where the non-conservation is caused by quark mass terms in QCD. Since quark masses arise from the vacuum expectation value of the Higgs field in the Yukawa couplings for the quarks, the lack of conservation of the flavor currents within QCD is essentially associated with weak interactions.

This form of perturbation theory, where the unperturbed Hamiltonian contains the full strong-interaction Hamiltonian leads to normal Feynman perturbation theory only for the electroweak fields (leptons, photon, etc.). In the strong-interaction part, the electroweak gauge fields are coupled to matrix elements of currents. For example, the decay of the neutron to $p + e + \bar{\nu}_e$ (Fig. 2.1) has an amplitude

$$\big\langle p, \text{out} \big| j_-^\mu(0) \big| n, \text{in} \big\rangle \, \frac{-i g_{\mu\lambda}}{q^2 - m_W^2} \bar{u}_e \gamma^\lambda (1 - \gamma_5) v_\nu \times \text{couplings}, \tag{2.9}$$

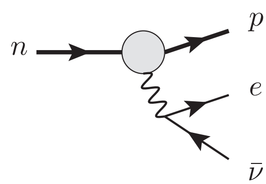

Fig. 2.1. Lowest-order weak interaction for neutron decay.

where j_-^μ is the hadronic current to which the W^+ couples, u_e and v_ν are standard Dirac spinors for the states of the leptons, $q = p_n - p_p$ is the momentum transfer, and p_n and p_p are the 4-momenta of the neutron and proton.

2.2.4 Current algebra

For further references and for a more detailed historical account of the issues treated in this section, see Pais (1986, Ch. 21).

Initially, with no known theory of the strong interaction, and with no complete theory of the weak interaction, it was measured that the weak interactions of hadrons involved current matrix elements as in (2.9). This led to the subject of current algebra, i.e., the study of hadronic current operators. The current coupled to the W boson appears as one of the currents for an approximate symmetry group of the strong interactions. This group, a chiral SU(3) ⊗ SU(3) group, will be discussed further in the context of QCD in Sec. 3.8, together with its more exact SU(2) ⊗ SU(2) subgroup. Explicit breaking of the symmetry is caused by the relatively small mass terms for the u, d and s quarks in QCD.

It was found that the symmetries are spontaneously broken to a "vector" SU(3) or SU(2), with the pions being the Goldstone bosons for the SU(2) ⊗ SU(2) case. The explicit symmetry breaking by quark masses implies that the pion is not massless but simply much lighter than other hadrons. The residual vector SU(3) symmetry is the one that is prominent in the quark model: Sec. 2.2.2.

Many consequences of the Ward identities for these symmetries were derived, in particular soft pion theorems. See, e.g., Treiman, Jackiw, and Gross (1972). One dramatic example is the Goldberger-Treiman relation that gives a relation between the matrix element in (2.9) and the long-distance part of the pion exchange contribution to the nucleon-nucleon potential; it thus relates a measurement of a weak-interaction quantity to an apparently very different quantity in pure strong-interaction physics.

Studies of symmetries require understanding of commutators of currents. This led to the study of matrix elements of two currents, like $\langle P \,|\, j^\mu(x) j^\nu(0) \,|\, P \rangle$, which are investigated experimentally in deeply inelastic scattering: Sec. 2.3.

A natural problem was now to find a theory that supports current algebra, i.e., a theory in which the currents are ordinary Noether currents and have the commutation relations postulated in current algebra. What excited Fritzsch and Gell-Mann (1972) and Fritzsch, Gell-Mann, and Leutwyler (1973) was that their proposed QCD Lagrangian not only could explain the quark model but naturally gave current algebra. The symmetry properties of the quark mass terms are exactly those used for the symmetry-breaking part of the strong-interaction Hamiltonian in current algebra.

Around 1970 it was found that the derivation of certain Ward identities for products of three currents fails in real field theories. It was found, moreover, that the resulting anomalies are correctly calculated within lowest-order perturbation theory; higher-order corrections are exactly zero according a theorem due to Adler and Bardeen. The methods of current algebra then enabled the decay rate for $\pi^0 \to \gamma\gamma$ to be calculated to the extent that the masses of the u and d quarks are small. Agreement with the observed decay rate is obtained if each flavor of quark has three color states. See Peskin and Schroeder (1995, Ch. 19).

Another line of argument related to current algebra was by Weinberg (1973a, b), who considered weak-interaction corrections to strong-interaction phenomena. In a generic candidate theory for the strong interaction, loop graphs have unsuppressed contributions from momenta around the W mass. The resulting violations of strong-interaction symmetries (e.g., parity) would be electromagnetic in strength, contrary to observation. Weinberg showed that this problem is avoided if the strong interaction is mediated by exchange of bosons whose symmetries commute with those for the electroweak bosons. This is the case for QCD, where color SU(3) commutes with the electroweak gauge group. The revolutionary consequence is that flavor symmetries were demoted from fundamental properties of the strong interaction to apparently accidental and approximate symmetries that occur because of the small size of the Yukawa couplings of the Higgs field to the light quarks.

2.2.5 Non-abelian gauge theories

The discovery of QCD needed a parallel track of purely theoretical work to formulate non-abelian gauge theories and establish their consistency. The initial formulation was by Yang and Mills (1954), who beautifully generalized the concept of local gauge invariance from the abelian symmetry of QED to a non-abelian group. Their attempt to apply their theory to the actual strong interaction foundered on the prejudice that the fields in the Lagrangian should correspond to observed particles, contrary to the now-known reality.

But the theoretical idea remained. With the discovery of the concept of spontaneous symmetry breaking, Weinberg (1967) and Salam (1968) found what is in fact the correct theory of electroweak interactions. At about the same time, Faddeev and Popov (1967) showed how to quantize such theories consistently. After this, it was quickly found how to derive Ward identities and thence to show that Yang-Mills theories, possibly including spontaneous symmetry breaking, are renormalizable.

With this, non-abelian gauge theories became fully fledged consistent field theories, setting the stage for the developments outlined in the preceding sections.

2.3 Deeply inelastic scattering

In parallel with work just described, the remaining developments that led to the establishment of QCD as the theory of strong interactions concerned deeply inelastic scattering of leptons (DIS). Since this process remains an important subject of study in QCD, we now examine those aspects that do not depend on knowing the strong-interaction Lagrangian.

We consider scattering of a lepton of momentum l^μ on a hadron N of momentum P^μ to an outgoing lepton of momentum l'^μ plus anything:

$$l + N(P) \longrightarrow l' + X. \tag{2.10}$$

The symbol X has a standard connotation, that we work with an *inclusive* cross section, i.e., a cross section differential in lepton momentum l', with a sum and integral over all possible states for the X part of the final state. Effectively only the lepton is treated as being detected.

There are a number of cases with different types of lepton for which there is experimental data: $e + N \longrightarrow e + X$, $e + N \longrightarrow \nu + X$, $\mu + N \longrightarrow \mu + X$, $\nu + N \longrightarrow \nu + X$, $\nu + N \longrightarrow (e \text{ or } \mu) + X$. When the momentum transfer at the lepton side is large, as we will see, we effectively have a powerful microscope into the initial-state hadron N. In actual data, N is either a proton or a heavier nucleus. Scattering on a nucleus is often approximated as scattering on an incoherent mixture of protons and neutrons. For more accurate work, "nuclear corrections" are applied to obtain cross sections relative to independent protons and neutrons.

In this section we will only treat the electron-to-electron case, for which the current state of the art for high energy is at the recently shut-down HERA accelerator at the DESY laboratory. There an electron (or positron) beam of energy 27.5 GeV was collided against a proton beam of energy 920 GeV, with a center-of-mass energy of $\sqrt{s} = 318\,\text{GeV}$.

2.3.1 General considerations

Consider a wide-angle scattering of the electron in the center-of-mass frame, Fig. 2.2. The large space-like momentum transfer, $q^\mu = l^\mu - l'^\mu$, for the (essentially point-like) electron suggests that a short-distance scattering is necessary, which would naturally occur off a small constituent of the hadron. If we let the invariant momentum transfer be $Q = \sqrt{-q^2}$, then a natural distance scale is $1/Q$ (in units with $\hbar = c = 1$). At HERA there is data to above $Q = 100\,\text{GeV}$, with a corresponding distance of less than 10^{-2} fm.

An enormous simplification occurs because, at high energy, the hadron is Lorentz-contracted and time-dilated[5] by a large factor, which is about 150 in the center-of-mass at HERA. A hadron like a proton has a size (Hofstadter, Bumiller, and Yearian, 1958) of around 1 fm, so it is reasonable to say that *in the hadron's rest frame* the constituents interact with each other on a time scale of order 1 fm/c. In the boosted hadron, as seen in the center-of-mass frame of the scattering, time dilation implies that the last interaction of the constituents typically occurred a long distance upstream. In the HERA center-of-mass frame, this is of order 100 fm, which is much larger than the scale of the electron scattering.

[5] These concepts are non-trivial (Gribov, 1973, p. 12) for microscopic particles in a quantum field theory, but that does not affect the motivational issues for this section.

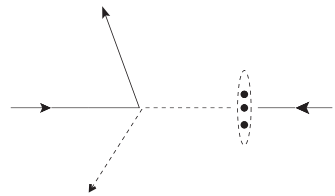

Fig. 2.2. Deeply inelastic scattering of an electron on a proton. The electron comes from the left and the proton from the right. In the diagram, the electron (solid line) is depicted as point-like and the hadron as a Lorentz-contracted extended object. The three dots inside the proton symbolize the three quarks that are its constituents in the quark model. The struck parton is indicated by a dashed line. Drawing a realistic Lorentz contraction would result in a much thinner proton than shown here.

This suggests (Feynman, 1972) that in the short-distance electron-constituent collisions it is a useful approximation to neglect the interactions that bind the constituents into a hadron. Quantitative development of this idea leads first to the "parton model", to be explained in Sec. 2.4, and then to the factorization theorems of QCD, which give a precise and correct mathematical formulation of the intuitive ideas.

In the original DIS experiments at SLAC, in the early 1970s, only the outgoing electron was detected; there was no sensitivity to the rest of the final state. Moreover the electron beam energy was at most 21 GeV on a fixed target. Modern experiments, like the ZEUS and H1 detectors at HERA, can see the hadronic final state. An example event with $Q = 158\,\text{GeV}$ is shown in Fig. 2.3. It supports the intuitive picture: an isolated wide-angle electron recoils against a narrow group of particles, called a jet, which is reminiscent of the scattered constituent. The scattered constituent (the "parton" in Feynman's terminology) does not retain its identity as a single particle except at sufficiently microscopic distances; this is of course compatible with the idea that quarks are permanently confined in hadrons and never appear as isolated single particles. The standard view (Andersson, 1998) is that many quark-antiquark pairs are created by the intense gluon field between an outgoing struck quark and the proton remnant. These form into color-singlet hadrons, mostly pions, that go in roughly the direction of the outgoing quark. The remnants of the proton continue in motion, with excitation and only a small deflection: these cause hits in the detector segments around the beam pipe, at the left of Fig. 2.3(a). Much of the remnant energy is too close to the beam direction to be detected.

2.3.2 Kinematics; structure functions

We work to lowest order in electromagnetism and in this section we will ignore weak interactions.[6] Then the amplitude for a contributing process is represented diagrammatically

[6] Unless Q is at least of order the masses of the W and Z bosons, weak-interaction effects are suppressed, by a factor of Q^2/m_W^2. Higher-order electromagnetic corrections are smaller by a factor of roughly α/π, except for infra-red dominated terms associated with the masslessness of the photon. It is conventional to present data "with the effects of radiative corrections removed", so that higher-order electromagnetic corrections are effectively absent in published data. The formalism is readily extended, with only notational complications, to deal with exchange of weak-interaction bosons. See Sec. 7.1.

(a)

(b)

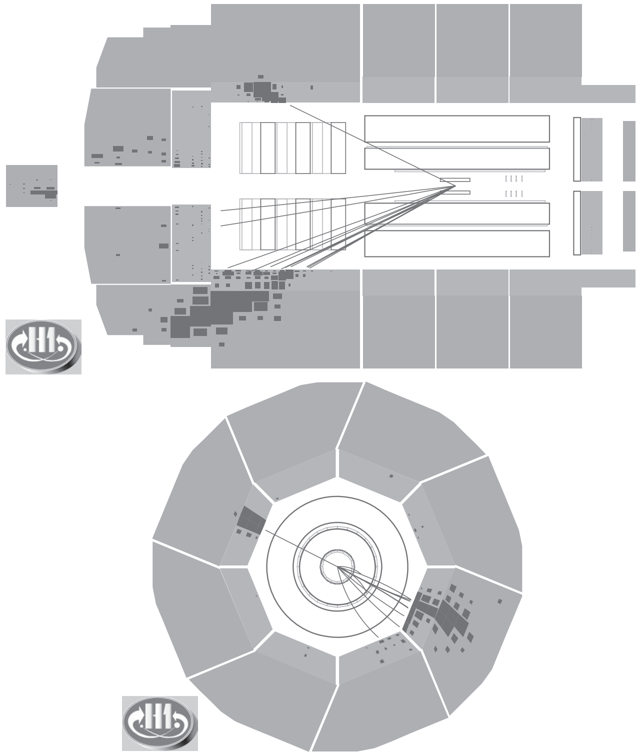

Fig. 2.3. Scattering event in an positron-proton collision in the H1 detector (H1 website, 2010) at a center-of-mass energy of about 320 GeV. The detector is approximately cylindrically symmetric about the center line which contains the beam pipes. Both a side view (a) and an end view (b) are shown. In (a), electrons come from the left, and protons from the right. One isolated track was identified as an electron, and there is a recoiling jet, approximately back-to-back in azimuth. The kinematic variables are $Q^2 = 25\,030\,\mathrm{GeV}^2$ and $y = 0.56$ (see Sec. 2.3.2).

in Fig. 2.4(a), and is a product of a lowest-order leptonic vertex, a photon propagator, and a hadronic matrix element of the electromagnetic current, $\langle X, \text{out} \,|\, j^\mu \,|\, P \rangle$. The two independent Lorentz invariants for the hadron system are $Q^2 \stackrel{\text{def}}{=} -q^2 \geq 0$ and $P \cdot q$, both of which can be computed from the measured momenta l, l' and P, with the momentum of the exchanged photon being $q^\mu = l^\mu - l'^\mu$. The mass of the hadronic final state is then

$$m_X^2 = (P + q)^2 = M^2 + 2P \cdot q - Q^2, \tag{2.11}$$

where M is the mass of the initial-state hadron. A convenient combination of variables is Q and the Bjorken variable

$$x \stackrel{\text{def}}{=} \frac{Q^2}{2P \cdot q}. \tag{2.12}$$

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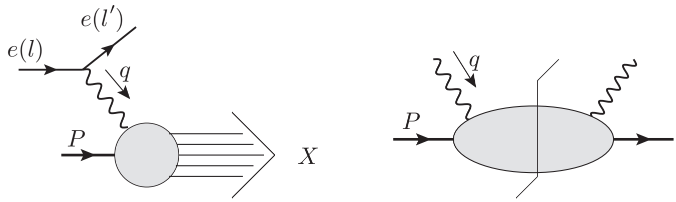

Fig. 2.4. (a) DIS amplitude to lowest order in electromagnetism. (b) Hadronic part squared and summed over final states. For the meaning of the vertical "final-state cut", see the discussion below (2.19).

Kinematically x is restricted to the range $Q^2/(s+Q^2) \leq x \leq 1$ (with fractional corrections of order M^2/Q^2 being neglected). In the parton model we will find that x gives an estimate of the fraction of the initial hadron's momentum that is carried by the struck parton. That the term "momentum fraction" has a useful meaning depends on the relativistic kinematics of the process: Sec. 2.4.

The term "deeply inelastic scattering" (DIS) applies to the region where both Q and m_X are large, so that there is a large momentum transfer and the hadron target is very much excited, inelastically.

Another commonly used variable is

$$y \stackrel{\text{def}}{=} \frac{q \cdot P}{l \cdot P}. \tag{2.13}$$

It lies between 0 and 1. In the rest frame of the hadron, this is the fractional energy loss of the lepton: $(E - E')/E$, so that it is simple to measure in a fixed target experiment. But it is not an independent variable, since

$$Q^2 = xy(s - M^2 - m_e^2). \tag{2.14}$$

The Lorentz-invariant inclusive cross section is then

$$\begin{aligned} E' \frac{\mathrm{d}\sigma}{\mathrm{d}^3 \mathbf{l}'} &\simeq \frac{\pi e^4}{2s} \sum_X \delta^{(4)}(p_X - P - q) \left| \langle l' | j_\lambda^{\text{lept}} | l \rangle \, \frac{1}{q^2} \, \langle X, \text{out} | j^\lambda | P \rangle \right|^2 \\ &= \frac{2\alpha^2}{s Q^4} L_{\mu\nu} W^{\mu\nu}. \end{aligned} \tag{2.15}$$

In the prefactor, we have neglected the electron mass m_e and the hadron mass M compared with $\sqrt{s}$, while the fine-structure constant is $\alpha = e^2/(4\pi)$. The sum over X denotes the usual Lorentz-invariant sum and integral over all hadronic final states. The currents j_λ^{lept} and j^λ are respectively the electromagnetic currents for the leptons and for the hadronic fields. In QCD the electromagnetic current involves a sum over quark flavors:

$$j^\lambda = \sum_f e_f \bar{\psi}_f \gamma^\lambda \psi_f = \tfrac{2}{3}(\bar{u}\gamma^\lambda u + \ldots) - \tfrac{1}{3}(\bar{d}\gamma^\lambda d + \ldots). \tag{2.16}$$

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In the second line of (2.15), we have separated out factors for the leptonic and hadronic parts. The leptonic tensor is obtained from lowest-order Feynman graphs, and in the unpolarized case is

$$L_{\mu\nu} = \tfrac{1}{2}\,\mathrm{Tr}\,\gamma_\nu \not{l}\, \gamma_\mu \not{l}' = 2(l_\mu l'_\nu + l'_\mu l_\nu - g_{\mu\nu} l\cdot l'). \tag{2.17}$$

The hadronic tensor is defined as a complete matrix element,

$$\begin{aligned} W^{\mu\nu}(q, P) &\stackrel{\text{def}}{=} 4\pi^3 \sum_X \delta^{(4)}(p_X - P - q)\, \langle P, S \,|\, j^\mu(0) \,|\, X\rangle\, \langle X \,|\, j^\nu(0) \,|\, P, S\rangle \\ &= \frac{1}{4\pi} \int \mathrm{d}^4 z\;\; e^{iq\cdot z}\, \langle P, S \,|\; j^\mu(z)\, j^\nu(0) \,|\, P, S\rangle \end{aligned} \tag{2.18}$$

in the full strong-interaction theory. The normalization is a standard convention, and the variable S labels the spin state of the target. In general, this may be a mixed state, and the notation $\langle P, S \,|\, \ldots \,|\, P, S\rangle$ is a shorthand for a trace with a spin density matrix: see App. A.7, and especially (A.8) and (A.13), for details. For the usual case of a spin-$\frac{1}{2}$ target, the spin state is determined by its (space-like) spin vector S^μ, which obeys $S \cdot P = 0$. We normalize S^μ as in Amsler *et al.* (2008), so that $S^2 = -M^2$ for a pure state.

To obtain the last line of (2.18), we used a standard result for the transformation of fields under space-time translations:

$$\langle P, S \,|\, j^\mu(z) \,|\, X, \text{out}\rangle = \langle P, S \,|\, j^\mu(0) \,|\, X, \text{out}\rangle\, e^{i(P-p_X)\cdot z}. \tag{2.19}$$

This allows the conversion of the momentum-conservation delta function to an integral over position. Then we used the completeness relation: $\sum_X |X, \text{out}\rangle\, \langle X, \text{out}| = I$.

Diagrammatically, we use the cut-diagram notation of Fig. 2.4(b) to represent $W^{\mu\nu}$. There the vertical line is called a "final-state cut". It represents the final state $|X, \text{out}\rangle$, and implies a sum and integral over all possible out-states $|X, \text{out}\rangle$. The part of the diagram to the left of the final-state cut is an ordinary amplitude $\langle X, \text{out}| j^\nu(0)|P, S\rangle$; in perturbation theory it is a sum over ordinary Feynman graphs with the appropriate on-shell conditions. The part to the right of the cut is a *complex-conjugated* amplitude, in this case $\langle P, S \,|\, j^\mu(0) \,|\, X, \text{out}\rangle = \langle X, \text{out}| j^\mu(0)|P, S\rangle^*$.

The Particle Data Group's definition (Amsler *et al.*, 2008) of $W^{\mu\nu}$ differs in replacing $j^\mu(z)\, j^\nu(0)$ by the commutator $[j^\mu(z), j^\nu(0)]$, but I find it better to use the more obvious definition with the simple product. The other definition is a relic from the period when the commutator was a dominant topic of research. The second term in their commutator $-j^\nu(0)\, j^\mu(z)$ gives a contribution equal to $-W^{\nu\mu}(-q, P)$, so the commutator version can be reconstructed from knowledge of $W^{\mu\nu}$. In fact, for a given value of q, only one of the two terms in the commutator contributes, since, when P is the momentum of a stable single-particle state, only one of $P + q$ and $P - q$ is the momentum of a physical state that can be used for $|X, \text{out}\rangle$.

We now decompose $W^{\mu\nu}$ into fixed tensors times scalar functions. For this we observe that:

- The electromagnetic current is conserved, $\partial \cdot j = 0$, so that $q_\mu W^{\mu\nu} = W^{\nu\mu} q_\mu = 0$.
- $W^{\mu\nu}$ is linear in the spin vector, which is an axial vector.

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- The strong interactions are parity invariant.
- $W^{\mu\nu}$ is a hermitian matrix, i.e., $(W^{\mu\nu})^* = W^{\nu\mu}$.

Then the most general form of the tensor is

$$W^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1(x, Q^2) + \frac{(P^\mu - q^\mu P\cdot q/q^2)(P^\nu - q^\nu P\cdot q/q^2)}{P\cdot q} F_2(x, Q^2)$$
$$+ i\epsilon^{\mu\nu\alpha\beta}\frac{q_\alpha S_\beta}{P\cdot q} g_1(x, Q^2) + i\epsilon^{\mu\nu\alpha\beta}\frac{q_\alpha\left(S_\beta - P_\beta\frac{S\cdot q}{P\cdot q}\right)}{P\cdot q} g_2(x, Q^2). \tag{2.20}$$

The scalar coefficients F_1, F_2, g_1, and g_2 are called structure functions. The invariant antisymmetric tensor $\epsilon_{\kappa\lambda\mu\nu}$ obeys $\epsilon_{0123} = +1$, i.e., $\epsilon^{0123} = -1$, a convention that is not universal.

2.3.3 Breit/brick-wall frame; helicity analysis

For much of our work, it will be convenient to use the so-called Breit frame, where the incoming proton is in the $+z$ direction, and the photon's momentum is all in the $-z$ direction: $q = (0, 0, 0, -Q)$. In the parton-model approximation, we will see that the struck quark gets its 3-momentum exactly reversed in this frame, which is therefore also called the brick-wall frame.

In the Breit frame we define structure functions with simple transformation properties under rotations about the z axis. These are the longitudinal and transverse structure functions:

$$F_{\mathrm{L}} \stackrel{\mathrm{def}}{=} F_2 - 2xF_1; \qquad F_{\mathrm{T}} \stackrel{\mathrm{def}}{=} F_1. \tag{2.21}$$

Then F_{L} corresponds to the components of $W^{\mu\nu}$ in the energy direction, while F_{T} corresponds to the components transverse to q and P.

2.3.4 Cross sections and measurements of structure functions

In the case of unpolarized scattering, which is the most usual situation, we set $S^\mu = 0$. Then (2.15) and (2.20) give

$$\frac{\mathrm{d}^2\sigma^{\mathrm{unpol}}}{\mathrm{d}x\,\mathrm{d}y} \simeq \frac{4\pi\alpha^2}{xyQ^2}\left[\left(1 - y - \frac{x^2y^2M^2}{Q^2}\right) F_2(x, Q^2) + y^2 x F_1(x, Q^2)\right]$$
$$= \frac{4\pi\alpha^2 s}{Q^4}\left[\left(1 - \frac{Q^2}{xs} - \frac{Q^2M^2}{s^2}\right) F_2(x, Q^2) + \frac{Q^4}{xs^2} F_1(x, Q^2)\right]. \tag{2.22}$$

The errors in this formula are due only to the neglect of the electron and hadron masses with respect to $\sqrt{s}$, of the electron mass with respect to Q, and to the use of lowest-order perturbation theory for the electromagnetic interaction. The form of the kinematic dependence multiplying the structure functions is due to the established form of the electromagnetic interaction. Thus measurements of the structure functions are equivalent to measurements of the cross section. Without further knowledge of the strong interaction, a measurement at a single energy $\sqrt{s}$ only determines the x and Q dependence of a combination of structure functions, as is made clear on the second line. Measurements at a minimum of two different energies are needed to separate the structure functions. After that the cross section for all other energies is predicted for values of x and Q that are within the kinematic limits of the first measurements.

The remaining structure functions g_1 and g_2 can be measured with polarized electron beams on a polarized target; see Leader and Predazzi (1982, p. 256).

This finishes the summary of those results and definitions that apply independently of the theory of the strong interactions.

2.4 Parton model

The parton model was formulated by Feynman (1972) and formalized by Bjorken and Paschos (1969) as an idea for understanding DIS in the absence of knowledge of an underlying microscopic theory of the strong interaction. It relies on an intuition stated in Sec. 2.3.1 and symbolized in Fig. 2.2.

Feynman proposed that the photon vertex couples to a single constituent of the target hadron, and that it is useful to neglect the strong interactions of the constituents during the collision with the lepton. The word "parton" is a generic term for one of the constituents under the conditions in which it participates in the short-distance part of a collision. In QCD it is therefore often treated as a collective name for quarks and gluons (and antiquarks).

A quantitative formulation is greatly helped by the relativistic kinematics of the process. Consider a parton of momentum k inside its parent hadron of momentum P. To get from the rest frame of the hadron to the frame of Fig. 2.2, we apply a large boost. We use light-front coordinates (App. B) with the positive z axis in the direction of the hadron; we therefore write $k^\mu = (k^+, k^-, \boldsymbol{k}_\mathrm{T})$, $P^\mu = (P^+, M^2/(2P^+), \boldsymbol{0}_\mathrm{T})$, where $k^\pm = (k^0 \pm k^z)/\sqrt{2}$. We assume that in the rest frame of the hadron, the components of k are appropriate for a constituent of a bound state whose typical scale is M, i.e., that all components of k are of order M (or smaller) in the hadron rest frame. Then after the large boost, k^+ is by far the biggest: it is of order Q, while k^- and $\boldsymbol{k}_\mathrm{T}$ are of order M^2/Q and M. The ratio of the plus momenta is boost invariant, so we define the fractional momentum of the parton by $\xi = k^+/P^+$.

Based on the space-time structure of the reaction, the parton model asserts that we should approximate the inclusive DIS cross section as incoherent scattering of electrons on quasi-free partons. The partons have a probability distribution in fractional momentum ξ and in parton flavor, and the shape of the distribution is determined by the proton's bound state wave function. For the electron-parton interaction, the momentum transfer Q is large, so we approximate the incoming and outgoing partons as massless free particles, and neglect the transverse momentum of the incoming parton. The outgoing parton also has high energy, so the interactions converting it to a hadron final state are also time-dilated, thereby justifying its approximation as a free particle. Most importantly, the *strong* interaction is neglected, and only the lowest-order electromagnetic scattering interaction is used.

Contrary to the impression that might be gained from the literature, the parton model does *not* require that partons are genuinely free massless particles. They are only approximately free, and only for the purposes of estimating a short-distance cross section.

It is by no means obvious, a priori, that the parton model is actually valid. In Ch. 6 and later chapters, we will formulate the parton model in real quantum field theories, and show that modifications are generally needed, because of singularities in the short-distance interactions. Moreover, the concept of a wave function and how to apply Lorentz boosts to it are quite unobvious in relativistic quantum field theories. Nevertheless the parton model has intuitive appeal, so it provides an excellent framework for motivating and organizing a proper treatment. In fact, we will even justify the parton model, in a certain sense, because QCD is asymptotically free; a dimensionless measure of its interactions decreases with distance. The true results are a distorted parton model.

2.4.1 Elementary formulation of parton model; parton densities

We now make a quantitative formulation of the parton model. The logic, as presented here following the original work, involves certain intuitively motivated jumps, the quality of which we can best assess after the more strictly deductive treatment in later chapters.

The hard scattering, i.e., the short-distance scattering of the electron and parton, occurs at a particular time. The proton is in a state consisting of some number of partons, whose fractional plus momenta are $\xi_1, \xi_2, \dots$, which sum to unity: $\sum \xi_i = 1$. There is a probability distribution over states and the hard scattering samples any one particular parton. So we postulate that there is a number distribution of partons $f_j(\xi)$. Thus $f_j(\xi)\,\mathrm{d}\xi$ is the expectation of the number of partons of flavor j with fractional momentum ξ to $\xi + \mathrm{d}\xi$. Standard terminology is to call $f_j(\xi)$ a "parton density" or a "parton distribution function". In QCD, the flavor index takes on values for up-quark, anti-up-quark, gluon, etc. If the two u quarks and the d quark in a proton shared its energy roughly equally, we would expect the quark densities to be peaked at around $\xi \sim 1/3$ and the u quark density to be approximately twice the d quark density. We would expect the other quark and antiquark densities to be smaller. In a real QFT, these other densities are in fact non-zero, because of the presence of quark-antiquark pairs from the interaction terms of the Hamiltonian, as can be seen later from the formal operator definitions of parton densities.

We can interpret the initial insight for the parton model in Feynman-graph notation with the aid of Fig. 2.5(a). A parton of momentum k scatters off the virtual photon; it then goes into the final state, undergoing "hadronization" interactions that convert it to observable hadrons. Topologically this diagram is in fact the most general one possible. The parton model consists of an assertion of the typical momenta involved and that the final-state hadronization interactions cancel. In the parton model, the struck quark momentum k has a large plus component, and relatively small minus and transverse components (in the Breit frame), while the outgoing parton $k + q$ has low invariant mass. The final-state interactions rearrange the content of the final state, but time dilation of the outgoing parton suggests that

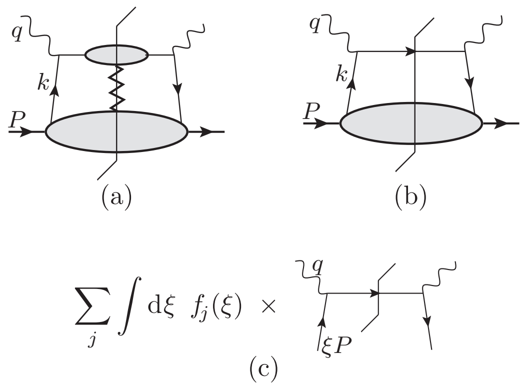

Fig. 2.5. Parton scattering in DIS. (a) Including hadronization and final-state interactions of struck parton. (b) Handbag diagram obtained after cancellation of hadronization and final-state interactions in graph (a). (c) Parton model with parton density and lowest-order DIS on partonic target.

these happen on a long time-scale, and therefore do not greatly affect the probability that a scattering has occurred. That is, the final-state interactions cancel to a first approximation in the *inclusive* cross section. Thus we can approximate graph (a) by the "handbag" graph (b), where the final-state interactions of the quark are ignored.

An analysis can be made from the handbag diagram itself, but that is postponed to Ch. 6. Here we just work with the parton-model assertion of incoherent lowest-order electromagnetic scattering on partons governed by parton densities, as embodied in Fig. 2.5(c).

2.4.2 *Quark-parton model calculation*

It is convenient to use the Breit frame, and to write the light-front coordinates of q and P as

$$q^\mu = \left(-x_N P^+, \frac{Q^2}{2x_N P^+}, \mathbf{0}_T \right), \qquad P^\mu = \left(P^+, \frac{M^2}{2P^+}, \mathbf{0}_T \right). \tag{2.23}$$

In this equation, x_N is the Nachtmann variable (Nachtmann, 1973)

$$x_N = \frac{2x_{Bj}}{1 + \sqrt{1 + 4M^2 x_{Bj}^2 / Q^2}}, \tag{2.24}$$

which differs from the Bjorken variable $x_{Bj} = Q^2/2P \cdot q$ by a power-suppressed correction.

In the partonic scattering we replace k by its plus component: $k \mapsto (\xi P^+, 0, \mathbf{0}_T)$, in accordance with our discussion of the sizes of the components of k. We also approximate the outgoing parton as massless and on-shell. We let $d\sigma_{lj}^{\text{partonic}}$ be the differential cross section for lepton-parton scattering with the following kinematics:

$$l + (\xi P^+, 0, \mathbf{0}_T) \to l' + k', \tag{2.25}$$

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where the outgoing parton momentum k' is massless and on-shell. Within the parton model, the partonic cross section is computed at lowest order. Then the parton model asserts that the inclusive DIS cross section is

$$d\sigma = \sum_j \int d\xi \; f_j(\xi) \; d\sigma_{lj}^{\text{partonic}}, \tag{2.26}$$

where the sum is over parton flavors. This formula relates a cross section with a hadron target to a cross section with a calculable partonic cross section. Naturally, these two kinds of cross section should be chosen to be differential in the same variables.

There now follow corresponding formulae for the structure tensor and for the structure functions. Now, in (2.15), we see a factor $1/s$ in converting the hadronic structure tensor to a cross section. But at the partonic level, there is instead a factor $1/\xi s$, because the lepton-parton scattering has a squared center-of-mass energy $(\xi P + q)^2 \simeq 2\xi P \cdot q$, up to power-law corrections. Then the parton model approximation for $W^{\mu\nu}$ is

$$W_{\text{PM}}^{\mu\nu} = \sum_j \int \frac{d\xi}{\xi} f_j(\xi) \, C_{j,\,\text{partonic}}^{\mu\nu}, \tag{2.27}$$

with a factor $1/\xi$ compared with (2.26). Here $C_{j,\,\text{partonic}}^{\mu\nu}$ is like $W^{\mu\nu}$ but computed on a free massless parton of type j and momentum $\hat{k}^\mu = (\xi P^+, 0, \mathbf{0}_{\text{T}})$, and with neglect of all interactions.

When the partons are quarks of spin $\frac{1}{2}$, we have

$$\begin{aligned} C_{j,\text{partonic}}^{\mu\nu} &= e_j^2 \frac{1}{4\pi} \frac{1}{2} \operatorname{Tr} \hat{\not{k}} \gamma^\mu (\not{q} + \hat{\not{k}}) \gamma^\nu \, 2\pi \delta((q + \hat{k})^2) \\ &= e_j^2 \left(2\hat{k}^\mu \hat{k}^\nu + q^\mu \hat{k}^\nu + \hat{k}^\mu q^\nu - g^{\mu\nu} q \cdot \hat{k} \right) \frac{x}{Q^2} \delta(\xi - x), \end{aligned} \tag{2.28}$$

where e_j is the electric charge of quark j (in units of the positron charge). It immediately follows that

$$F_2^{\text{QPM}} = \sum_j e_j^2 x \, f_j(x), \qquad F_1^{\text{QPM}} = \frac{1}{2x} F_2^{\text{QPM}}, \tag{2.29}$$

where "QPM" means "quark-parton model" (to distinguish these formulae from the correct factorization formulae in QCD). In this calculation, the incoming and outgoing quarks are approximated as massless and on-shell. The on-shell condition for the outgoing parton results in the parton momentum fraction ξ being set to the measurable Bjorken variable x (up to ignored power-law corrections). The measured variable y, defined in (2.13), equals $(1 - \cos\theta)/2$ in the parton model, where θ is the scattering angle of the lepton-parton collision. Thus, a measurement of x and Q in an event immediately gives an estimate of the parton kinematics, Fig. 2.3 providing an illustration of a typical event.

2.4.3 Bjorken scaling

A prediction of the parton model embodied in (2.29) is that at fixed x the structure functions are independent of Q (at large Q of course). This is called "Bjorken scaling", and, as we

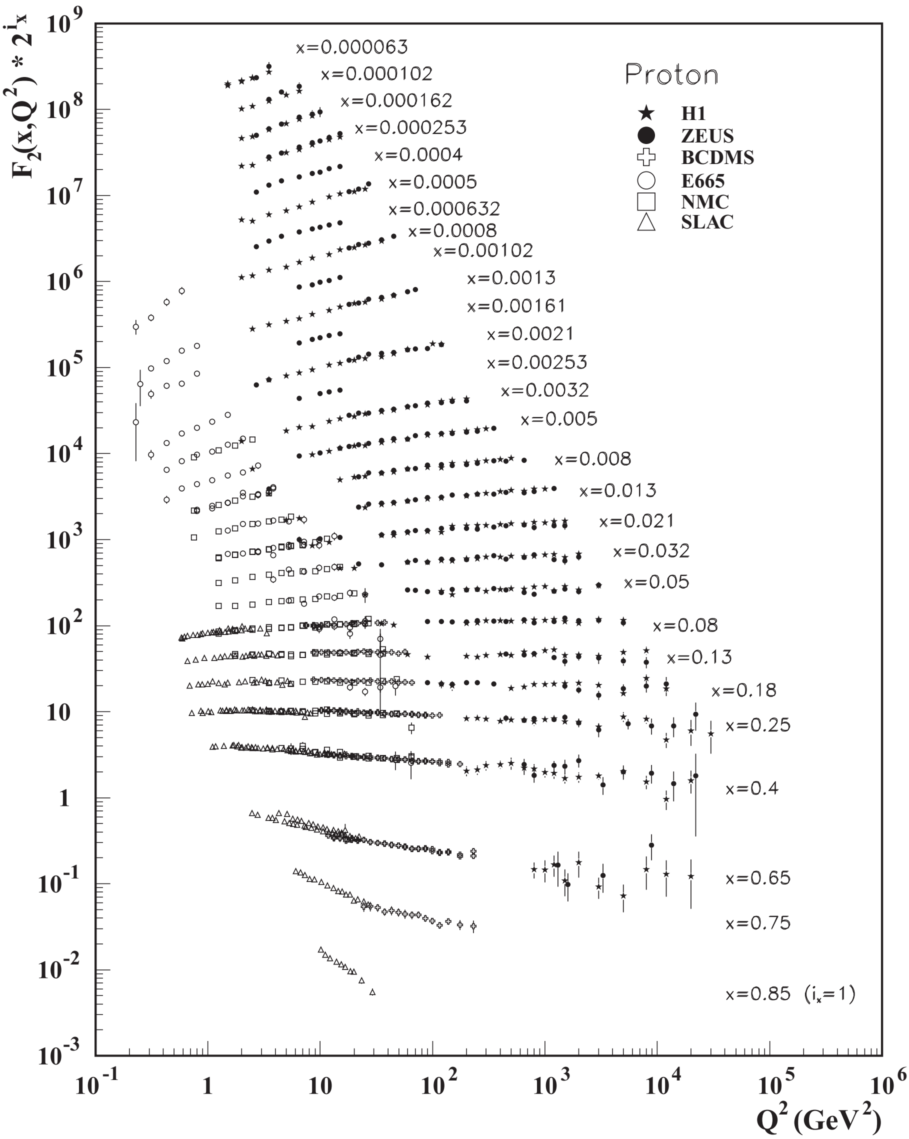

Fig. 2.6. Compilation by the Particle Data Group of data on F_2 on a proton target. For the purpose of separating the different sets of data the values of F_2 have been multiplied by 2^{i_x}, where i_x is the number of the x bin, ranging from $i_x = 1$ for $x = 0.85$ to $i_x = 28$ for $x = 0.000\,063$. Reprinted from Amsler *et al.* (2008), with permission from Elsevier.

will see in Chs. 8 and 11, it is violated after allowing for QCD interactions. We will see that measurements of scaling violation allow a deduction of the strength of the strong interaction. Current data are shown in Fig. 2.6. It can be seen that Bjorken scaling is approximately true at moderate x, for example between 0.1 and 0.5. This region is relevant

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Fig. 2.7. Initial state and final state for QPM, with conserved right-handed helicity for the quark. The small arrows indicate the spin.

to a model where a substantial amount of the momentum of the proton is carried by three similar quarks, with a typical x of around $1/3$. One might expect the intuitive picture to be less reliable at extreme values of x, so the greater scaling violations as x gets close to 1 or 0 are not in violation of the spirit of the parton model.

2.4.4 Callan-Gross relation and parton spin

Observe that the longitudinal structure function is $F_{\mathrm{L}}^{\mathrm{QPM}} = 0$ in the QPM, a result first obtained by Callan and Gross (1969). It is a simple consequence of conservation of angular momentum about the z axis in the brick-wall frame, as in Fig. 2.7. The electromagnetic interaction preserves the helicity of the massless quark (Sterman, 1993, p. 215), i.e., its spin *relative* to its *direction of motion*. The quark's 3-momentum is reversed in the collision, so relative to a fixed axis its spin is reversed. There are no transverse momenta in this calculation, so there is no orbital angular momentum about the z axis. So one unit of spin is transferred from the virtual photon, which must therefore be transverse, not longitudinal.

2.4.5 Field theory implementation of parton model

In Ch. 6 we will show how to convert the parton-model idea into formal statements in QFT, with definitions of parton densities as expectation values of certain operators. We will find that the parton model is exactly correct only in certain simple field theories. In more general cases, notably QCD, modifications are needed: Chs. 8 and 11.

Particularly in retrospect the parton model was a natural conjecture, but when first formulated, in the absence of an underlying microscopic theory, it was controversial. The need for modifying it in real QCD underscores the basis for the initial scepticism.

Some of the first parts of this development were obtained before the discovery of QCD, and provided important hints that pointed uniquely to the structure of QCD.

2.5 Asymptotic freedom

A powerful argument by Callan and Gross (1973) used the operator product expansion and the renormalization group to show that exact Bjorken scaling in DIS requires there to be an ultra-violet fixed point of the strong-interaction theory at zero coupling. Hence the observed approximate Bjorken scaling implies that the strong interaction is relatively weak at short distances.

Since the strong interaction is strong at large distances, this led to a search for theories that are asymptotically free, i.e., for which the effective coupling goes to zero at zero distance. One result was the demonstration by Coleman and Gross (1973) that no field theory constructed using only scalar, Dirac, and abelian gauge fields can be asymptotically free. This left only non-abelian gauge theories, which just slightly earlier had been quantized and proved renormalizable. If these theories also failed to give asymptotic freedom, then it would be strong evidence that no quantum field theory could describe strong interactions, a view that was quite popular at the time: there were indeed absolute arguments by Landau and Pomeranchuk based on apparently universally fundamental principles that the effective coupling always had to increase at short distances; see 't Hooft (1999).

Then Gross and Wilczek (1973a, b) and Politzer (1973) calculated the lowest-order renormalization-group β function for the Yang-Mills theory, and demonstrated its asymptotic freedom, even with quark fields present.[7] The previously formulated QCD Lagrangian is therefore able to explain (approximate) Bjorken scaling. The rising coupling in the infrared, even if it does not by itself imply color confinement, is compatible with it and is a precondition that the standard connection between fields and particles can be completely destroyed for quarks and gluons.

The result then is that for the first time there was a unique viable and complete theory of the strong interaction, QCD. Previously mysterious phenomena were direct consequences of the Lagrangian (2.1). From now on we can proceed deductively.

2.6 Justification of QCD

I now summarize the powerful arguments that pick out QCD as the unique field theory of the strong interaction. The following list involves a rearrangement and even a reversal of the historical logic.

1. We can treat any theory of currently known physics as a low-energy effective theory (Weinberg, 1995, p. 499) obtained from some more exact theory. In the normal quantum field theory framework it is a theorem that the low-energy theory is renormalizable. This applies to leading power in the ratio of a large mass scale for the exact theory to currently available energies. To agree with observations, the theory is Poincaré invariant to a very good approximation (Liberati and Maccione, 2009).
2. Bjorken scaling implies either actual asymptotic freedom, or at least a decreasing coupling at currently accessible energies. Hence the theory must be a non-abelian gauge theory with not too many matter fields. See Fig. 3.6 below for a recent plot of measured values of the strong coupling.
3. It must be possible to combine the theory with the known Weinberg-Salam theory of electroweak interactions. Since the couplings are very different, we cannot have anything except a direct product of the SI gauge group and the EW gauge group. Let us call the SI gauge fields "gluon" fields and the SI matter fields "quark" fields (which could be Dirac and/or scalar).

[7] In fact, the calculation of this coefficient had already been made slightly earlier by 't Hooft (see 't Hooft, 1999) and in 1969 by Khriplovich (1970). Even earlier, Vanyashin and Terentyev (1965) computed a negative beta function in Yang-Mills theory, but their calculation did not include the not-yet-known ghost contribution. But these authors did not immediately recognize the significance of their results for a theory of strong-interaction physics.

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4. Because the electroweak and strong-interaction gauge groups commute, there are no direct gluon couplings to W, Z, and Higgs fields.
5. Thus the strong-interaction theory is the QCD Lagrangian, possibly supplemented only by extra quark fields. It is the original Yang-Mills Lagrangian, but with a different gauge group and with extra fermion fields. No further terms are permitted in the gauge-invariant Lagrangian without violating renormalizability.
6. We now identify the gauge group and the matter fields:
   (a) Asymptotic freedom together with the masslessness of the gluons implies that the effective coupling increases out of the perturbative range for low mass scales or large distances. This allows the connection between fields and directly observable particles to be lost.
   (b) It also indicates that under suitable conditions, quarks and gluons have approximately free-particle behavior for short distances.
   (c) Colored states tend to be unbound or of higher energy.
   (d) The approximation of the quark model indicates that an SU(3) color group together with three light flavors of Dirac quark is needed to explain the observed spectrum of hadrons.
   (e) Extra quarks, in whatever representation of the color group, are a matter for discovery at higher energy, and of obtaining a suitably consistent structure for the electroweak theory. Consistency requirements concern the lack of anomalies in the electroweak theory.
   (f) Certain measurements are key ones in confirming the determination of the color group, and in the measurement of the number of flavors, during and after the discovery of QCD:
      i. the $\pi^0 \to \gamma\gamma$ decay rate, which is obtained from an anomaly in the vacuum matrix element of three currents;
      ii. the total cross section for e^+e^- annihilation to hadrons at high energy gives a measure of the sum of the charges squared of the accessible quarks – see Ch. 4;
      iii. More detailed jet cross sections in e^+e^- give quite direct measurements of the color-group theory coefficients C_A and C_F, etc. – again, see Ch. 4.

These arguments are primarily structural. They do not depend, for the most part, on detailed numerical predictions of the theory. Such predictions are used mainly in determining which gauge group is needed.

Once we have confidence that the theory is a good approximation to reality, we (i.e., people working on the strong interaction) change our attitude. The mathematics is hard, and when useful, we appeal to the real world as a realization of QCD to help us to determine what results are true. A failure of agreement between theory and experiment is expected to indicate that there is an error either in the theoretical methods or their application, or in the experiments, but it does not normally indicate an error in the theory itself. (An extension of the theory, to add another quark, for example, is not regarded as a breakdown in the theory.)

2.7 QCD in the full Standard Model

Many applications of perturbative QCD concern the interaction of hadrons with non-QCD particles, e.g., DIS, and all kinds of production processes for leptons, the Higgs boson, and many hypothesized particles. To put these in context, I now review the definition of the Standard Model (SM). For details, see any standard textbook, such as Halzen and Martin (1984); Peskin and Schroeder (1995); Quigg (1997).

The SM Lagrangian is

$$\mathcal{L}_{\text{SM}} = -\frac{1}{4}\sum_{\alpha}\left(G_{\alpha\,\mu\nu}\right)^2 + i\sum_{f}\bar{\psi}_f \not{D}\psi_f + D\phi^\dagger \cdot D\phi + M^2\phi^\dagger\phi$$
$$-\frac{\lambda}{4}(\phi^\dagger\phi)^2 - \sum_{ij} h_{ij}\bar{\psi}_{i\,\text{R}}\phi\psi_{j,\text{L}} + \text{gauge-fixing terms, etc.,} \tag{2.30}$$

with the usual modifications for renormalization. Structurally this is like QCD, except for the addition of a scalar "Higgs" field ϕ, with its self-interaction and its Yukawa couplings to the fermion fields. The main features are as follows.

- In the first line, the sum over α is over the 12 generators of the gauge group SU(3) $\otimes$ SU(2) $\otimes$ U(1). We let the gauge fields for the three commuting components of the gauge group be $A^\alpha_\mu(x)$, W^j_μ and B_μ. The renormalized couplings of the three commuting component groups are g_s, g and g' respectively, and the SU(3) subgroup is the QCD group.
- When we are working with pure QCD, without any mention of electroweak interactions, we will often replace the notation g_s by g.
- The fermion fields $\psi_{\rho a f}$ carry different representations of the gauge group, unlike the case of simple QCD.
- The covariant derivative is

$$D_\mu = \partial_\mu + ig_s\sum_{\alpha=1}^{8} T^\alpha_{\text{col}} A^\alpha_\mu + ig\sum_{j=1}^{3} W^j_\mu T^j_{\text{W}} + ig' B_\mu \frac{Y}{2}, \tag{2.31}$$

  where for any given multiplet of fields T_{col} and T_{W} are the generating matrices for the color SU(3) and the SU(2) groups, while Y is the weak hypercharge of the multiplet.
- The fermion fields are arranged in multiplets of left-handed fields and right-handed fields. "Left-handed" fields are $\frac{1}{2}(1-\gamma_5)$ times the Dirac field, and "right-handed" fields have a $\frac{1}{2}(1+\gamma_5)$ factor.
- All known left-handed fields are doublets under SU(2), and all known right-handed fields are singlets under SU(2).
- There are three generations of fermion, and the assignments of quantum numbers to fields are specified in Table 2.1. Here we have extended the Standard Model slightly beyond its original definition to include right-handed neutrino fields, as needed to accommodate the measured neutrino mixing.
- The vacuum expectation value of the Higgs field is given by $\langle 0|\phi|0\rangle = (0, v/\sqrt{2})^{\text{T}}$, with $v = 246\,\text{GeV}$. This breaks three of the electroweak symmetries, with the Z and

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Table 2.1 *Quantum numbers of field multiplets in the Standard Model. The symbols for the fields correspond to the particle names.*

	Color singlet	Y	Color triplet	Y
First generation:	$\begin{pmatrix} \nu_{e\,\mathrm{L}} \\ e_\mathrm{L} \end{pmatrix}$	-1	$\begin{pmatrix} u_\mathrm{L} \\ d_\mathrm{L} \end{pmatrix}$	$\frac{1}{3}$
	(e_R)	-2	(u_R)	$\frac{4}{3}$
	$(\nu_{e\,\mathrm{R}})$	0	(d_R)	$-\frac{2}{3}$

The next two generations (ν_μ, μ, s, c) and (ν_τ, τ, b, t) are exactly similar.

	Color singlet	Y
Higgs field:	$\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	1

photon fields being

$$Z_\mu = \cos\theta_\mathrm{W}\, W^3_\mu - \sin\theta_\mathrm{W}\, B_\mu, \qquad A_\mu = \sin\theta_\mathrm{W}\, W^3_\mu + \cos\theta_\mathrm{W}\, B_\mu, \tag{2.32}$$

where the measured Weinberg angle obeys $\sin^2\theta_\mathrm{W} = 0.22 \pm 0.02$.

- The electroweak couplings are given in terms of the QED coupling and θ_W by

$$g = \frac{e}{\sin\theta_\mathrm{W}}, \qquad g' = \frac{e}{\cos\theta_\mathrm{W}}.$$

- The fermion masses are then obtained from the Yukawa couplings. From global fits to data (Amsler *et al.*, 2008) estimates of the masses of the elementary fields are found (Table 2.2).
- All the formulae for masses, etc., are subject to higher-order electroweak corrections.
- The flavor and mass eigenstates of the two components of the fermion doublets are not aligned, but have mixing given by the CKM and MNS matrices; see, e.g., Amsler *et al.* (2008).

2.8 Beyond the Standard Model

All theories of physics are ultimately approximate, and many possibilities for theories that are better than the SM are under active discussion. To keep agreement with known results, the QFTs considered are generally extensions of the SM, except for the Higgs sector on which there is as yet little direct data. Extensions include both the simple addition of field multiplets and the embedding of the symmetry groups in bigger symmetries, as in Grand Unified Theories and in supersymmetry.

Once gravity enters the picture, space-time becomes dynamical, and so any QFT, including QCD, becomes only an effective low-energy approximation to a radically different kind

Table 2.2 *Standard Model masses for elementary fields, from Amsler* et al. *(2008).*

Leptons and quarks (spin 1/2):

ν	~ 0	d	3.5 to 6 MeV
e	0.511 MeV	u	1.5 to 3.3 MeV
		s	$\sim 104^{+26}_{-34}$ MeV
μ	106 MeV	c	$\sim 1.27^{+0.07}_{-0.11}$ GeV
		b	$\sim 4.20^{+0.17}_{-0.07}$ GeV
τ	1.78 GeV	t	171.2 ± 2.1 GeV

Gauge bosons:

$W^{\pm}$	80.398 ± 0.025 GeV	Z	91.1876 ± 0.0021 GeV

Higgs:

100 to 300 GeV (indirect)

of theory (e.g., string theory), with a very different understanding of space-time. Factorization in QCD remains a vital tool in phenomenological discussions of such theories, because it separates treatment of the ultra-microscopic physics of the new theories from the longer-distance physics which is an integral part of a full scattering process.

For current work in this area, see the proceedings of recent conferences and workshops, e.g., Allanach *et al.* (2006).

2.9 Relation between fields and particles

In a free QFT, there is a direct correspondence between the types of single particle and the fields, and in fact with the normal modes of the corresponding classical field theory. In simple interacting QFTs, this correspondence continues to hold, but it is clear from both QCD and the full Standard Model, that the particle-field correspondence is not general:

- With interactions some of these particles can be become unstable, as exemplified by the muon, with its decay to $e\nu_\mu\bar{\nu}_e$.
- There may be bound states, e.g., atoms. These are not related in a simple way to normal modes of the elementary fields.
- It is also possible that there is no particle, stable or unstable, that corresponds to a particular elementary field of a theory. QCD is an excellent example with its quark and gluon fields. Any corresponding particles are permanently confined, and only behave approximately like particles on short enough distance scales inside collisions. Before the advent of QCD, this possibility was hardly recognized, if at all.
- Moreover, low-energy effective theories approximating a more exact microscopic theory may use fields corresponding to bound states. This is the case for a Schrödinger QFT for atomic physics, which might have fields for atomic nuclei.

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Moreover, one must be careful about what is meant by a particle. One standard definition is from the single-particle states that are used to build up the asymptotic in- and out-states of scattering theory. For this purpose completely stable bound states, like the ground state of a hydrogen atom or even of a large macroscopic object like a planet, are particles. But unstable particles, even relatively long-lived ones like the muon and the neutron, are not particles under this definition.

It is clear that the connection between particles and interacting fields is somewhat impressionistic. Even the usage of the word “particle” is quite fuzzy in the real world. Which objects are called particles, which bound states, and which resonances is essentially a linguistic matter: a matter of convention, and usage, and even of context.

Some confusions in the recent literature should be noted. For example, Weinberg in his excellent textbooks on quantum field theory (Weinberg, 1995, p. 110) bases his logic on the concept of a particle in the strict sense of scattering theory. Then his derivation of perturbation theory requires that the set of one-particle states be unchanged after turning on the interaction in a theory. If this were really necessary, it would immediately rule out conventional Feynman perturbation theory for all known interactions.

Weinberg’s derivation of perturbation theory is for the S-matrix. Instead, if one bases the logic on perturbation theory for (off-shell) Green functions, one no longer has to assume that the particle spectrum is unchanged under perturbations. The particle spectrum and the S-matrix are derived objects involving examination of poles in the Green functions. Thus, for example, the stability or instability of a particular particle can be an accidental consequence of the particular values of parameters of the theory.

It is evidently important to dispose of this issue at the outset, for otherwise most of our work in *perturbative* QCD would be without a foundation. An account of the logic for perturbation theory that is suitable from our perspective is given in Sterman’s textbook (Sterman, 1993).

Exercises

2.1 (a) Show how to compute a particle-particle potential from the non-relativistic limit of a first-order $2 \longrightarrow 2$ scattering amplitude. You might do this by comparing the Born approximation in QED with the Born approximation in non-relativistic potential scattering. Consider both the case of spin-$\frac{1}{2}$ and spin-0 particles.

(b) Apply this method to QCD to find the lowest-order approximation to the quark-antiquark potential with massive quarks. Separately consider the case that the system is a color singlet and a color octet. You should find that the potential is only attractive for a color-singlet bound state.

2.2 *Review problem:* Define the concept of a structure function. Why is it a useful concept?

2.3 In the parton model approximation, compute the electromagnetic structure functions for a scalar quark (i.e., for a spin-0 quark).

2.4 *Formulation of the structure function method for scalar field exchange instead of vector field:* Suppose you wanted to investigate the consequences of a hypothetical theory with an extra neutral scalar field ϕ that has Yukawa couplings to quarks and to leptons:

$$\mathcal{L}_{\text{int}} = \phi \times \left(h_e \bar{e} e + \sum_i h_i \bar{q}_i q_i \right), \tag{2.33}$$

where h_e and h_i are the couplings to electrons and to quarks of flavor i. (a) What would be an appropriate definition of structure function(s) in this problem? (b) What would be the parton model formula?

Review and revise your answer to problem 2.2 in the light of your answer this problem.

2.5 How do you extend the analysis of problem 2.4 in the presence of interference between scalar and vector exchange?

2.6 Examine the state of the knowledge about current algebra just before the discovery of QCD, e.g., in Treiman, Jackiw, and Gross (1972). How does this compare with the description in this chapter?

The rest of this problem is best done after finishing learning about QCD. During your studies of QCD, determine the extent to which the work in Treiman *et al*. (1972) is (a) true in QCD, (b) needs modification, or (c) still needs proof. How much remains relevant to current research and/or to understanding QCD and the strong interaction?

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