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ONE-POINT EXTENSIONS OF LOCALLY PARA-H-CLOSED SPACES

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Abstract

A space X is para-H-closed if every open cover of X has a locally-finite open refinement (not necessarily covering the space) whose union is dense in X. In this paper, we study one-point para-H-closed extensions of locally para-H-closed spaces.

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Introduction

Only Hausdorff topological spaces are considered.

DEFINITION 1. Let γ be an open cover of a topological space X. Then λ is *para-H-closed refinement* of γ if λ is a locally-finite collection of open subsets of X refining γ and such that $\bigcup\lambda$ is dense in X.

DEFINITION 2. A space X is *para-H-closed* if every open cover of X has a para-H-closed refinement.

DEFINITION 3. A space X is *locally para-H-closed* if every point has a neighbourhood whose closure is para-H-closed.

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DEFINITION 4. A space (Y, σ) is said to be a *one-point extension* of a space (X, τ) if $X \subset Y$, $\tau = \{O \cap X: O \in \sigma\}$, $|Y \setminus X| = 1$ and $\text{cl}_\sigma(X) = Y$.

DEFINITION 5. Let $E(X)$ be the set of all one-point para-H-closed extensions of a locally para-H-closed but not para-H-closed space X. A space Y in $E(X)$ is said to be a *projective maximum* in $E(X)$ if, for any space Z in $E(X)$, there exists a continuous function f from Y onto Z such that $f(x) = x$ for all x in X. A space M in $E(X)$ is said to be a *projective minimum* in $E(X)$ if, for any Z in $E(X)$, there exists a continuous function f from Z onto M such that $f(x) = x$ for all x in X.

Para-H-closed spaces and locally para-H-closed spaces were defined and studied in [3]. We mention here some basic results about para-H-closed spaces.

THEOREM 1. *A regular space is para-H-closed if and only if it is paracompact.*

THEOREM 2. (i) *Every domain of a para-H-closed space is para-H-closed.*
(ii) *Every Lindelöf Hausdorff space is para-H-closed.*

THEOREM 3. *A space X is para-H-closed if and only if every open cover of X has a σ-locally-finite open refinement* $\lambda = \bigcup_{n\in\omega} \lambda_n$ *such that*

$$\bigcup\{\text{int}[\text{cl}(\bigcup\lambda_n)] : n \in \omega\} = X.$$

For locally-compact spaces, we know that there is only one one-point compactification. For locally-H-closed spaces, F. Obreano [1] and J. Porter [2] have shown that there may not be a unique one-point H-closed extension. Locally-H-closed spaces, however, do possess a projective maximum and also a projective minimum one-point H-closed extensions. For locally para-H-closed spaces, we show that while there is a projective maximum in the set of all one-point para-H-closed extensions, there is no projective minimum in general.

The following notation will be fixed throughout the rest of the paper. Let (X, τ) be a locally para-H-closed space which is not para-H-closed. Let $\Phi = \{\gamma:$ γ is an open cover of X without a para-H-closed open refinement$\}$.

For each $\gamma \in \Phi$, let $\Omega_\gamma = \{\lambda: \lambda$ is a locally-finite collection of open sets in X refining $\gamma\}$. For each $\gamma \in \Phi$, let $\Psi_\gamma = \{X \setminus \text{cl}(\bigcup\lambda): \lambda \in \Omega_\gamma\}$. Note that for each $\gamma \in \Phi$, Ψ_γ has the finite intersection property.

Let ∇_γ be the open filter generated by Ψ_γ, and let ζ_γ be an open ultrafilter generated by Ψ_γ.

We shall let $\Lambda = \bigcap\{\zeta_\gamma: \gamma \in \Phi\}$, and let $\Pi = \bigcap\{\nabla_\gamma: \gamma \in \Phi\}$.
It is easy to see that for each $\gamma \in \Phi$, ∇_γ and ζ_γ are free filters.

LEMMA 1. *Let* Λ *be as defined above. Then*

$$\Lambda = \{U \in \tau\colon X \setminus \operatorname{int}(\operatorname{cl}(U)) \text{ is para-}H\text{-closed}\}.$$

PROOF. Let $U \in \tau$ be such that $X \setminus \operatorname{int}(\operatorname{cl}(U))$ is para-H-closed. Let $\gamma \in \Phi$. Consider ζ_γ. Suppose $U \notin \zeta_\gamma$. Then $X \setminus \operatorname{cl}(U) \in \zeta_\gamma$. Consider $\xi = \{O \cap (X \setminus \operatorname{int}(\operatorname{cl}(U)))\colon O \in \gamma\}$, which is an open cover of $X \setminus \operatorname{int}(\operatorname{cl}(U))$. There is a para-$H$-closed refinement κ of ξ in $X \setminus \operatorname{int}(\operatorname{cl}(U))$ consisting of open subsets of $X \setminus \operatorname{int}(\operatorname{cl}(U))$.

Let $\lambda = \{K \cap (X \setminus \operatorname{cl}(U))\colon K \in \kappa\}$. Then λ is an open collection in X refining γ. Also λ is locally-finite in X and its union is dense in $X \setminus \operatorname{int}(\operatorname{cl}(U))$. Thus $\lambda \in \Omega_\gamma$, which implies that $X \setminus \operatorname{cl}(\bigcup\lambda) \in \zeta_\gamma$. But $X \setminus \operatorname{cl}(\bigcup\lambda) = \operatorname{int}(\operatorname{cl}(U))$. Therefore $\operatorname{int}(\operatorname{cl}(U)) \in \zeta_\gamma$ and $X \setminus \operatorname{cl}(U) \in \zeta_\gamma$, which is a contradiction. Therefore $U \in \zeta_\gamma$. Now let us suppose that $U \in \Lambda$ and show that $X \setminus \operatorname{int}(\operatorname{cl}(U))$ is para-H-closed. Let κ be an open cover of $X \setminus \operatorname{int}(\operatorname{cl}(U))$ without a para-H-closed open refinement. For each $K \in \kappa$, let K' be open in X such that $K' \cap X \setminus \operatorname{int}(\operatorname{cl}(U)) = K$, and let $K'' = K \cap (x \setminus \operatorname{cl}(U))$. Let $\kappa_1 = \{K'\colon K \in \kappa\} \cup \{\operatorname{int}(\operatorname{cl}(U))\}$ and $\kappa_2 = \{K''\colon K \in \kappa\} \cup \{\operatorname{int}(\operatorname{cl}(U))\}$. Then κ_1 is an open cover of X and $\operatorname{cl}(\bigcup\kappa_1) = \operatorname{cl}(\bigcup\kappa_2) = X$. Suppose κ_1 has a para-H-closed open refinement in X, say η_1. Then $\eta_2 = \{H \cap (X \setminus \operatorname{cl}(U))\colon H \in \eta_1\}$ is a para-H-closed open refinement of κ in $X \setminus \operatorname{int}(\operatorname{cl}(U))$, which is a contradiction. Thus $\kappa_1 \in \Phi$. Now $\operatorname{int}(\operatorname{cl}(U)) \in \kappa_1$, which implies that $\{\operatorname{int}(\operatorname{cl}(U))\} \in \Omega_{\kappa_1}$. This further implies that $X \setminus \operatorname{cl}(U) \in \zeta_{\kappa_1}$. But $U \in \Lambda$ means $U \in \zeta_{\kappa_1}$, which leads us to the desired contradiction.

THEOREM 4. *Suppose* (X, τ) *is a locally para-H-closed space which is not para-H-closed. Let* Λ *be as defined above. Let* $X^* = X \cup \{p\}$ *be such that* $p \notin X$. *Then*

(a) $\tau^* = \tau \cup \{\{p\} \cup G\colon G \in \Lambda\}$ *is a Hausdorff topology on* X^*,

(b) *The space* (X^*, τ^*) *is a one-point para-H-closed extension of* (X, τ),

(c) *The space* (X^*, τ^*) *is a projective maximum in the set of all one-point para-H-closed extensions of* (X, τ).

PROOF. (a) Since Λ is a free open filter on X, it is easy to see that τ^* is a topology on X^*. Let us show that it is Hausdorff. Let $x \neq p$. Then there exists an open set U_x in X such that $x \in U_x$ and $\overline{U}_x$ is para-H-closed. By Lemma 1, $X \setminus \overline{U}_x \in \Lambda$. This is true because $X \setminus \overline{U}_x$ is an open domain. Thus $x \in U_x \varepsilon \tau^*$ and $\{p\} \cup (X \setminus \overline{U}_x) \in \tau^*$, so there are disjoint τ^*-open neighbourhoods of x and p.

(b) Let μ be a τ^*-open cover of X^* by basic open sets. Then there exists G in Λ such that $\{p\} \cup G \in \mu$. Let $\mu' = \mu \setminus \{\{p\} \cup G\}$. Let $\xi = \{E \cap X\colon E \in \mu'\} \cup \{G\}$. Then ξ is a τ-open cover of X. Also since $G \in \Lambda$, $X \setminus \overline{G}^\circ$ is para-H-closed in

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X. Now $\xi_1 = \{E \cap X\colon E \in \mu'\}$ is an open cover of $X \setminus \overline{G}^{\circ}$. So there is a para-$H$-closed open refinement λ of ξ_1 in $X \setminus \overline{G}^{\circ}$. Let $\lambda_1 = \{V \cap (X \setminus \overline{G})\colon V \in \lambda\}$, and let $\beta = \lambda_1 \cup \{\{p\} \cup G\}$. Then β is the required para-H-closed open refinement of μ in X^*. Therefore X^* is para-H-closed.

(c) Let (Y, σ) be a one-point para-H-closed extension of (X, τ). We must show that there exists a continuous function f from (X^*, τ^*) onto (Y, σ) which leaves X pointwise fixed. Let $Y = X \cup \{r\}$. Define $f(x) = x$ for each x in X, and define $f(p) = r$. Let U be a σ-open neighbourhood of r. For each x in X, there exists a σ-open set U_x in Y such that x belongs to U_x and $r \notin \mathrm{cl}_\sigma(U_x)$. Now $\gamma = \{U_x\colon x \in X\} \cup \{U\}$ is a σ-open cover of Y. So there is a para-H-closed open refinement $\lambda = \{V_x\colon x \in X\} \cup \{V\}$ of γ in (Y, σ) such that $V_x \subset U_x$ for each x in X and $V \subset U$. Let $W = \bigcup\{V_x\colon x \in X\}$. Then

$$r \notin \bigcup\{\mathrm{cl}_\sigma V_x\colon x \in X\} = \mathrm{cl}_\sigma\left[\bigcup\{V_x\colon x \in X\}\right] = \mathrm{cl}_\sigma(W).$$

Observe that $r \in \mathrm{cl}_\sigma(V)$. In fact $r \in \mathrm{int}_\sigma[\mathrm{cl}_\sigma(V)]$. Since Y is para-H-closed, $\mathrm{cl}_\sigma(W)$ is also para-H-closed. But $W \subset X$, and $\mathrm{cl}_\sigma(W) = \mathrm{cl}_\tau(W)$. Thus $\overline{W}$ is para-H-closed in X. Let $G = V \cap X$. Then $\mathrm{cl}_\sigma(G) = \mathrm{cl}_\sigma(V)$, and $\mathrm{int}_\sigma[\mathrm{cl}_\sigma(V)] = \mathrm{int}_\tau[\mathrm{cl}_\tau(G)] \cup \{r\}$. Also $Y \setminus \mathrm{int}_\sigma[\mathrm{cl}_\sigma(V)] = X \setminus \mathrm{int}_\tau[\mathrm{cl}_\tau(G)] \subset \mathrm{cl}_\tau(W)$, which is para-$H$-closed. This implies that $X \setminus \mathrm{int}_\tau[\mathrm{cl}_\tau(G)]$ is para-H-closed. Thus by Lemma 1, $G \in \Lambda$. So $G \cup \{p\} = (V \cap X) \cup \{p\}$ is a τ^*-open neighborhood of p, and $f(G \cup \{p\}) = V \subset U$. Therefore f is continuous at p. But f is continuous at each x in X, too. This completes the proof of (c).

Theorem 5. *Let (X, τ) be a locally paracompact, non-locally-H-closed, non-para-H-closed space. Then (X, τ) does not have a projective minimum in the set of all of its one-point para-H-closed extensions.*

Proof. Let p be a point not in X and $Y = X \cup \{p\}$. Let (Y, σ) be any para-H-closed extension of (X, τ). Let $q \in X$ be such that there exists $U_q \in \tau$ with the following properties:

(i) $q \in U_q$.

(ii) $\mathrm{cl}_\tau(U_q)$ is not H-closed.

(iii) $p \notin \mathrm{cl}_\sigma(U_q)$.

Let Γ be a free filter-base of open subsets of $\mathrm{cl}_\sigma(U_q)$ such that, for every $F \in \Gamma$, there exists $F' \in \Gamma$ with $\mathrm{cl}(F') \subset F$.

Define a coarser topology σ' on Y by enlarging the neighbourhoods at p. Let the new neighbourhoods at p be of the type $O \cup \mathrm{int}_\tau(F)$, where O is any open neighbourhood of p in (Y, σ) and $F \in \Gamma$. Then (Y, σ') is a Hausdorff extension of (X, τ) and is strictly coarser than (Y, σ). We claim that (Y, σ') is para-H-closed. Let γ be an open cover of (y, σ'). There exist an open neighbourhood O_0 of p in (Y, σ) and $F_0 \in \Gamma$ such that $O_0 \cup F_0 \subset U_0$ for some $U_0 \in \gamma$. There exists $F_1 \in \Gamma$

such that $\mathrm{cl}(F_1) \subset F_0$. Let λ be a para-H-closed refinement of γ in (Y, σ). Let $\xi = \{V \setminus (\mathrm{cl}_\tau(F_1) \cup \{p\}) \colon V \in \lambda\} \cup \{O_0 \cup F_0\}$. Then ξ is the required refinement of γ in (Y, σ'). Hence (Y, σ') is para-H-closed.

THEOREM 6. *Let (X, τ) be a locally H-closed space which is not para-H-closed. Then (X, τ) has a projective minimum in the set of all of its one-point para-H-closed extensions.*

PROOF. Let (X^*, τ^*) be the projective minimum of (X, τ) in the set of all one-point H-closed extensions of (X, τ). (See [2]). Then (X^*, τ^*) is also a projective minimum in the set of all one-point para-H-closed extensions of X.

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