REVERSIBLE TOPOLOGICAL SPACES

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1. Reversibility

We propose to study a topological property which is not new, but seems not to have been systematically investigated.

DEFINITION. *A topological space* (X, T) *is called reversible if it has no strictly stronger topology* T' *such that* (X, T) *and* (X, T') *are homeomorphic, equivalently, if it has no strictly weaker topology* T' *such that* (X, T) *and* (X, T') *are homeomorphic.*

The choice of the adjective "reversible" is explained by the following Lemma.

LEMMA 1. *A space is reversible if and only if each continuous one-to-one map of the space onto itself is a homeomorphism.*

Let (X, T) be a reversible topological space and $f : X \to X$ continuous, one to one, and onto. Let $T' = \{G \subset X : f[G] \in T\}$. Then T' is stronger than T since if $G \in T'$, $G = f^{-1}f[G] \in T$. Moreover $f : (X, T') \to (X, T)$ is a homeomorphism. Since X is reversible it follows that $T' = T$, hence f is a homeomorphism.

The converse is proved similarly.

Thus reversible spaces occupy a place in the category of topological spaces and maps, similar to that of spaces obeying the closed graph theorem in the category of linear topological spaces and linear maps.

Reversibility is a topological property. We shall give examples, and point out, as well, that reversibility is not implied by such properties as connectedness, local compactness, or second countability, even for topological groups. Nor is reversibility hereditary or productive.

Many examples of non-reversible spaces are known and have been published. The examples given below are chosen to answer some natural questions. Some of the examples and theorems given are folklore.

EXAMPLE 1. Consider the map $x \to \{x_n/n\}$ from the normed space of all sequences $\{x_n\}$ of complex numbers with only finitely many non-zero terms, $||x|| = \sup |x_n|$, to itself. It is continuous, one to one, onto, but not a homeomorphism.

EXAMPLE 2. Let R be either the set of all real numbers or the set of all rational numbers. We give R its customary open sets and, besides, all singletons $\{x\}$ with $x < 0$, $x \in R$. (Thus the negative half of R has the discrete topology.) The map $x \to x+1$ is continuous, one-to-one, onto, but not a homeomorphism.

Example 2 shows that a countable metric space need not be reversible; the discrete topology shows that it may be reversible.

2. Extremal properties

A topological space is reversible if its topology is maximal or minimal with respect to some property. The discrete and indiscrete topologies are, respectively, maximal, minimal. The cofinite topology, [20], § 9.1, problem 29, is minimal T_1, hence reversible. A compact Hausdorff space is minimal Hausdorff and maximal compact — thus reversible on two grounds. (See [20], § 9.5, theorem 3.) As shown by Urysohn, there are non compact minimal Hausdorff topologies, see [6], p. 421, [2], [14].

Such a space could not be locally compact by the following result of Katetov, [8], Lemma 6, Theorem 3, and Hewitt, [6], Remark (9).

LEMMA 2. *Every locally compact Hausdorff space X can be given a weaker compact Hausdorff topology. A countable T_3 space has a weaker compact Hausdorff topology if and only if it contains no subset which is dense-in-itself. The second "compact" may be replaced by "locally compact".*

The proof of the first part is trivial. One removes an arbitrary point from the space and takes the one point compactification of the remainder. The second part follows from the Baire category theorem.

There are also non-Hausdorff maximal compact topologies. The first examples seem to have been given by V. K. Balachandran, A. Ramanathan, and Hing Tong (independently) in 1948. The simplest is probably the one-point compactification of the rationals. See [12], [15]. Such a topology is reversible of course.

3. Reversible spaces

It is well known that the Euclidean spaces R^n are reversible. For $n = 1$ this follows from the inverse function theorem; it can also be seen from the fact that R^1 is maximal among connected, locally connected topologies. It is not maximal connected since a simple extension, [11], of its topology by means of the set of rationals is connected; nor is it minimal Hausdorff, by Lemma 2. For arbitrary n, the reversibility of R^n follows from the Brouwer invariance of dimension theorem.

THEOREM 1. *A locally Euclidean space is reversible.*

Let f be a one-to-one continuous map of such a space X onto itself; let $x \in X$, $y = f(x)$, and let V be a neighborhood of y with $\phi : V \to W$ a homeomorphism onto an open subset W of Euclidean n-space. Then $f^{-1}[V]$ contains a neighborhood U of x with $\psi : U \to Z$ a homeomorphism onto an open subset Z of Euclidean n-space. Let $g = \phi \circ f \circ \psi^{-1} : Z \to W$. Then g is one-to-one and continuous. Hence, by the Brouwer theorem, g is a homeomorphism of Z into an open neighborhood of $\phi(y)$ and $f^{-1} = \psi \circ g^{-1} \circ \phi^{-1}$ is continuous in a neighborhood of y.

Remark: A proof of theorem 1 may be based on ideas borrowed from Banach algebra. See § 8. part (e).

COROLLARY. *Every Lie group, and every space of the form $D \times R^n$ where D is discrete, is reversible.*

We shall see that in this respect R^n is better behaved than a compact Hausdorff space. (Theorem 7.)

4. Non-reversible spaces

We are indebted to Victor Klee for the following two theorems, the first of which replaces an earlier and less satisfactory version.

THEOREM 2. *No infinite dimensional normed space is reversible.*

Let p be a strictly weaker norm for the infinite dimensional normed space X. (For example let u be a discontinuous automorphism of X^*, and for $f \in X^*$, $x \in X$ define $q(f) = ||f|| + ||uf||$, and $p(x) = \sup \{|f(x)| : q(f) \leqq 1\}$. See [20], § 7.5, Problem 10 and Example 8; also [10], chapter 11, problem I.) Define $T : X \to X$ by $Tx = (p(x)/||x||)x$. Since $||Tx|| \leqq p(x)$ and p is continuous it follows that T is continuous. But $T^{-1}(x) = (||x||/p(x))x$, thus T is one-to-one and onto, and T^{-1} is not continuous at O (although it is continuous everywhere else) since there exists a sequence $\{x_n\}$ in X with $||x_n|| > n, p(x_n) < 1$, ([20], § 4.2, Fact vi.) so that $T^{-1}(x_n/n||x_n||) \not\to 0$.

THEOREM 3. *No infinite dimensional F space is reversible.*

Let s be the space of all sequences of complex numbers with $!x! = \sum 2^{-n}|x_n|/(1+|x_n|), d(x, y) = !x-y!$. It is proved in [4] that every F space which is not normable is a product of spaces, one of which is s. It is sufficient, in view of this fact and Theorem 2, to show that s is non-reversible. Let E be the complex plane, and let $U \subset E$ be $\{(x, y) : y > 0$, or $y = 0$ and $x \geqq 0\}$ (upper half plane and non-negative X-axis). There is a continuous one-to-one map f of $U^{\aleph_0}$ onto s whose inverse is not continuous; for example, if $x = \{x_n\}$, let $f(x) = \{x_1^2, x_2, x_3, x_4, \cdots\}$. There also exists, ([1]), a homeomorphism h from $U^{\aleph_0}$ onto s. Then $h \circ f$ is a continuous one-to-one map of s onto itself which is not a homeomorphism.

EXAMPLE 3. *The space Q of rational numbers is not reversible.*

The proof of Example 4 could be used; the following is briefer and more elegant. It is sufficient to show that Q can be given a strictly weaker metric topology T, for (Q, T) will be a countable metric space which is dense-in-itself, hence homeomorphic with Q, ([17], p. 11, and [16], p. 141, corollary 1.)

Let R be the reals and consider the one point compactification of $R \sim \{0\}$. Giving the point at infinity the name 0 we arrive at R again but with a weaker (compact) topology. Call this space R_0. Let T be the topology induced on Q by R_0. Since R_0 is a plane figure 8, T is metrizable and the proof is concluded. (Note, T is strictly weaker than the Euclidean topology for Q since, in T, $n \to 0$ while $\{n\}$ is not convergent in the Euclidean topology.)

EXAMPLE 4. *The space of irrational numbers is not reversible.*

In the exposition of this example, notations such as (a, b) for intervals will refer to intervals of irrational numbers, even if a, b are rational. It is sufficient to show that $I = (-1, 1)$ is not reversible. Choose an arbitrary irrational u in I. Let I' be the same as I but with the topology strengthened by adjoining the set $[u, 1)$ to the class of open sets, a simple extension in the sense of [11]. Let $J = (0, 1) \cup (2, 4)$. Then J is homeomorphic with I. We shall show that I' is homeomorphic with J, hence with I. Now $I' = (0, u) \cup [u, 1)$, the union of two disjoint open and closed subsets, and so the proof will be complete when we show that $[u, 1)$ is homeomorphic with $(2, 4)$, each having its natural (Euclidean) topology. To this end, we construct three sequences. Let $\{a_n\}$ be a strictly increasing sequence of rational numbers with $a_0 = 2$, $a_n \to \sqrt{8}$; $\{b_n\}$ a strictly decreasing sequence of rationals with $b_0 = 4$, $b_n \to \sqrt{8}$; $\{c_n\}$ a strictly decreasing sequence of rationals with $c_0 = 1$, $c_n \to u$. For $n = 1, 2, \cdots$, let f_n be a homeomorphism from (c_n, c_{n-1}) onto $(a_{\frac{1}{2}(n-1)}, a_{\frac{1}{2}(n+1)})$ if n is odd, onto $(b_{\frac{1}{2}n}, b_{\frac{1}{2}n-1})$ if n is even. The required homeomorphism of $[u, 1)$ onto $(2, 4)$ is given by $f(u) = \sqrt{8}$ and $f|(c_n, c_{n-1}) = f_n$. (Notice that each a, b, and c interval is open and closed.)

THEOREM 4. *There exists a reversible space which is the union of two disjoint non-reversible subspaces. There exists a non-reversible space which is the union of two disjoint reversible subspaces, each of which is open and closed.*

The first part follows from Examples 3 and 4. The second part is given in the following example.

EXAMPLE 5. Let X be the space consisting of the positive integers, 0, and the reciprocals of the positive integers. A map carrying $2n$ onto $1/2n$

for all n cannot have a continuous inverse. Such a map which is one-to-one, continuous, and onto is easy to construct. Thus X is not reversible.

Remarks 1. The spaces given for Theorem 4 are very special, the first is R, the second a closed countable subspace of R.

2. In the second part of Theorem 4, the subspaces are closed and open; this condition cannot also be placed on the first part of the theorem.

THEOREM 5. *An open and closed subset of a reversible space is reversible. Neither open nor closed can be omitted.*

A continuous map of the subspace onto itself can be extended to the whole space. Example 5 shows that open cannot be omitted. Example 8, below, shows that closed cannot be omitted.

5. Components and discrete products

THEOREM 6. *Let X be a topological space with finitely many components. Then X is reversible if and only if each component is reversible. The result is false if "finitely" is replaced by "countably".*

Necessity follows from Theorem 5, and the last remark follows from Example 3. Now suppose that $X = \cup \{C_i : i = 1, 2, \cdots n\}$, each C_i a component of X. Let $f : X \to X$ be one-to-one, continuous, onto. Then $f[C_i]$ is connected for each i hence it is C_j for some j. Thus f permutes the components and it follows that for each i, there is an integer k, $1 \leqq k \leqq n$ such that $f^k[C_i] = C_i$ and so by hypothesis, f^k is a homeomorphism of C_i onto itself. Then $f^{-1} = f^{k-1} \circ (f^k)^{-1}$ is continuous on C_i. Since each C_i is open and closed, f is a homeomorphism.

COROLLARY. *Let D be a finite discrete space and Y a reversible connected space or a compact Hausdorff space, then $D \times Y$ is reversible. Both results are false if D is allowed to be countable.*

The first is immediate from Theorem 6. The second because $D \times Y$ is compact Hausdorff. For the counterexamples, let us call a topological space Y *fissionable* if it is the disjoint union of two copies of itself. For example a half-open interval (of reals) is fissionable, while no Euclidean space is. Let D be the positive integers with the discrete topology and Y a connected fissionable space, $Y = A \cup B$ with A, B disjoint and each homeomorphic with Y. Then $D \times Y$ is not reversible for we may map $1 \times Y$ onto $1 \times A$, $2 \times Y$ onto $1 \times B$, and $n \times Y$ onto $(n-1) \times Y$ for $n = 3, 4, \cdots$. This yields a continuous one-to-one map of $D \times Y$ onto itself, but the inverse map is not continuous since it maps $1 \times Y$ onto the non-connected set $(1 \times Y) \cup (2 \times Y)$.

To give an example in which Y is compact Hausdorff (suggested by

S. L. Gulden; see also Example 6) let Y be a perfect, nowhere dense subset of the line. It follows immediately from [16], Theorem 7, that Y is fissionable, say $Y = A \cup B$ as before. Then with D the positive integers, $D \times Y$ is not reversible, for Y includes infinitely many disjoint copies of itself and we may map each $2n \times Y$, $n = 1, 2, \cdots$, onto one of these so that $\cup\, 2n \times Y$ is mapped onto $1 \times Y$. Mapping $n \times Y$ onto $(n-1) \times Y$ for $n = 3, 5, 7, \cdots$ yields a map from $D \times Y$ onto itself whose inverse is not continuous since it carries $1 \times Y$ onto the non-compact set $\cup\, 2n \times Y$.

Remark. It is interesting that in the first construction just given we could take $Y = [0, 1)$, but not $(0, 1)$ by the Corollary to Theorem 1. (Of course $(0, 1)$ is not fissionable.)

EXAMPLE 6. *An extremally disconnected non-reversible space.* The example is $N \times \beta N$, N the positive integers with the discrete topology, and βN its Stone-Cech compactification, ([20], § 14.3). Since N is fissionable, βN is also; hence the construction given in the Corollary of Theorem 6 applies.

THEOREM 7. *The product of two reversible spaces need not be reversible. This is true even if one is discrete and the other a connected or compact Hausdorff space.*

This follows from the Corollary to Theorem 6.

The following example extends Theorem 7 by increasing the allowable assumptions.

EXAMPLE 7. *There exists a non-reversible locally compact abelian group. The group is the product of a discrete space and a compact connected metric group. The map with discontinuous inverse is also a homomorphism.*

(We have been informed that L. C. Robertson has also constructed continuous automorphisms, without continuous inverse, of a locally compact group, in a 1965 U.C.L.A. thesis.)

Let G be the full direct product of a countable number of copies of the circle group C with Euclidean topology. (This is the product with coordinatewise multiplication.) Let F be the full direct product of a countable number of copies of D, where D is the circle group. Let F be given the discrete topology. Then $F \times G$ is the required example. Define $T : F \times G \to F \times G$ by $T(f, g) = (u, v)$, $u_n = f_{n+1}$, $v_{n+1} = g_n$ for $n = 1, 2, \cdots$, $v_1 = x_1$. Then T is a continuous automorphism of $F \times G$ whose inverse is not continuous.

EXAMPLE 8. *A non-reversible open subspace of a reversible space.* Let D be the positive integers with the discrete topology. Then $D \times [0, 1]$ is easily seen to be rigid, but its open subset $D \times (0, 1]$ is not, as pointed out in the Remark before Example 6.

6. Heredity

Theorem 5 shows that reversibility is not hereditary. Identification of hereditarily reversible spaces appears difficult; we know only one non-trivial example, given in Example 9 below.

Trivial examples of hereditarily reversible spaces are

(a) Any discrete space,

(b) Any space with only finitely many open sets, in particular any indiscrete space and any finite space,

(c) The cofinite topology. (See § 2.)

EXAMPLE 9. *A hereditarily reversible non-discrete Hausdorff space.* The space is also countable and extremally disconnected (compare Example 6); it is not first countable, however, since, being extremally disconnected, it would be discrete, [5], Theorem 1.3. Let N, βN be as in Example 6, and let $x \in \beta N \sim N$. (All the definitions and results needed for this example may be found in [20] § 14.3, Application 2.) Let $X = N \cup \{x\}$, with the relative topology of βN. Let S be a subspace of X; we shall show that S is reversible. This is trivial if $x \notin S$ since then S is discrete, so we may assume $x \in S$. Let T be the topology of S; let T' be any strictly stronger topology. There exists a subset F of S which is T' closed, but not T closed. Then $x \in \bar{F} \sim F$, taking the T-closure, since x is the only possible limit point of F. It follows that $x \in \overline{S_0 \sim F}$, where $S_0 = S \sim \{x\}$, since the characteristic function of F can be extended from N to a continuous real function defined on X. Now if we take T' closures, we have $x \notin \bar{F}$ since $\bar{F} = F$, $x \notin \overline{S_0 \sim F}$ since T' is stronger than T, and so $x \notin \bar{F} \cup \overline{S_0 \sim F} = \bar{S}_0$. Thus, in the T' topology, x is an isolated point of S. Since all other points of S are isolated in the T topology which is weaker than T', it follows that T' is discrete, thus not homeomorphic with T.

The same argument proves:

THEOREM 8. *$N \cup \{x\}$, where $x \in \beta N \sim N$, has a maximal non-discrete topology.*

THEOREM 9. *There exists a non-discrete topology such that any simple extension,* [11], *is discrete.*

7. Partial orders

Let t be the set of all topologies on a set X, and for $T, T' \in t$, define $T \succ T'$ if T is stronger than and homeomorphic with T'. The reversible topologies are precisely the isolated points in the partially ordered set $(t, \succ)$. J. D. Weston [19] has introduced what appears to be a very important partial ordering; he writes $T \geqq T'$ if T is stronger than T' and for each $x \in X$, each neighborhood of x in either of the two topologies, as a subset of X with the other topology, is dense in a neighborhood of x.

We have $T \succ T'$ but not $T \geqq T'$ for the topologies obtained from Example 2, and $T \geqq T'$ but not $T \succ T'$ if T, T' are the norm and weak topologies on a Banach space. For T, T' as in Example 1 we have both $T \succ T'$ $T \geqq T'$. Any topology which is minimal or maximal with respect to some property is isolated in $(t, \succ)$, but in $(t, \geqq)$ any complete metric topology is minimal T_2 but not necessarily isolated; the discrete topology is isolated, the indiscrete topology is a minimum.

8. Miscellany

(a) A reversible space may have a strictly stronger topology with the same homeomorphism class, [21].

(b) Suppose that X is a non-reversible locally compact Hausdorff space, that $X = \cup \{K_n : n = 1, 2, \cdots\}$ where each K_n is compact, included in the interior of K_{n+1}, and has connected complement. Let $D = \{x : x$ has a neighborhood which is not a neighborhood of x in the strictly weaker homeomorphic topology.$\}$ Then D is closed, not compact; moreover D is connected in the weaker topology, [18].

(c) Let $T(X)$ be the number of $T_{3\frac{1}{2}}$ topologies on a $T_{3\frac{1}{2}}$ space (X, T) which are weaker than T. Lower bounds for $T(X)$ are given in [7]. Example 2 shows that the cardinal of the class of those members of $T(X)$ which are homeomorphic with T may be as large as that of X.

(d) Let us say that a space has property KP (compact preserving) if every stronger homeomorphic topology has the same compact sets, equivalently if every one-to-one continuous map of the space onto itself preserves non-compactness. For example any pseudofinite space has the KP property. *Then any reversible space has the KP property and any k space with the KP property is reversible.* (For k spaces see [9], p. 230, [13], p. 76.)

Proof: The identity map from the weaker topology is continuous on each compact set, hence continuous.

(e) If $C_1[X]$ denotes the Banach algebra of all bounded continuous complex functions on a non-reversible $T_{3\frac{1}{2}}$ space X, let A denote the subalgebra consisting of those functions which are continuous when X is given a strictly weaker homeomorphic topology; A is a proper subalgebra since it determines the topology of X. (See for example, [20], § 9.4, Lemma 3, and, for the following remarks, §§ 14.4, 14.5; B^* and C^* are synonymous.) Moreover A is a C^* subalgebra which separates points of X and contains constants; by the Stone-Weierstrass theorem, A must not separate points of βX, thus there are distinct scalar homomorphisms of $C_1[X]$ which agree on A. Finally, A is isometric, and star isomorphic with $C_1[X]$.

It is clear from this type of argument that if X' is X with the weaker topology, $\beta X'$ is a continuous, not one-to-one, image of βX; indeed it is a quotient of βX obtained by identifying points which $C_1[X']$ fails to separate. A discussion of these ideas is given in [6], §§ 2, 3, especially Theorem A.

(f) A non-reversible connected subset of R^2 may be obtained by joining all the rationals on the X-axis to the point (0, 1).

9. Questions

I. Is the product of a two-point discrete space and a reversible space reversible? In other words, can "connected" be omitted in the Corollary to Theorem 6? (Compare Theorem 4.)

II. Which topologies are homeomorphic with a simple extension, [11]? (The stronger topology in Example 4 is a simple extension of the original one.)

III. A non-reversible topology gives rise to a linearly ordered sequence of topologies. What is the cardinality of a maximal chain of homeomorphic topologies? In particular when is it countable? (Note that it is possible to have $T \subset T' \subset T''$, T homeomorphic with T'' but not with T'. In Example 2, let T be the given topology, T'' the (stronger) topology induced by the given map, and T' a simple extension of T by means of some subset of [0, 1).)

IV. We conjecture that every weaker Hausdorff topology for Q is homeomorphic with the Euclidean topology. (See Example 3 where this is noted for every such metrizable topology.) An affirmative answer to this conjecture would yield the known result that Q has no weaker minimal Hausdorff topology. [3].

V. Is every connected, locally compact, Hausdorff group reversible? (See Example 7.)

Added in proof. Maximal non-discrete topologies (Theorem 8) are called Ultraräumen by O. Frolich, *Mathematische Annalen* 156 (1964), p. 80.

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