Can. J. Math., Vol. XXXVII, No. 4, 1985, pp. 700-709

GENERALIZED $L(f)$ SPACES

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1. Introduction. Given any set Γ, let $\mathscr{P}$ be the family of all finite subsets of Γ. Let $f:[0, \infty) \to \mathbf{R}$ satisfying: (1) $f(x) = 0$ if and only if $x = 0$, (2) f is increasing, (3) $f(x + y) \leqq f(x) + f(y)$ for all $x, y \geqq 0$, and (4) f is continuous at zero from the right. Such an f is called a modules. Let C be the set of all moduli, and $F = \{f_v \in C:v \in \Gamma\}$. $Q(\Gamma)$ will denote the set of all such F, s. For each $F \in Q(\Gamma)$ let

$$L_\Gamma(F) = \{x \in R^\Gamma: \sum f_v(|x(v)|) < \infty\},$$

the summation is taken over Γ, and set

$$|x|_F = \sum f_v(|x(v)|) \text{ for all } x \in L_\Gamma(F).$$

If Γ is countable $Q(\Gamma)$ will be denoted by Q and $L_\Gamma(F)$ by $L(F)$. Let

$$L_\Gamma^1 = \{x \in R^\Gamma: \sum |x(v)| < \infty\} \quad \text{and}$$

$$L_\Gamma^\infty = \{x \in R^\Gamma: \sup|x(v)|\} < \infty.$$

Note that

$$L_\Gamma^1 = l^1(\Gamma) = l_d^1 \quad \text{and} \quad L_\Gamma^\infty = L^\infty(\Gamma),$$

see [**4**, **5** and **6**].

Definition 1.1. Let $F \in Q(\Gamma)$. F is called a *suborder* if there exists $r > 0$, with

$$\inf\{f_v(r):v \in \Gamma\} > 0,$$

such that for any $A \in \mathscr{P}$, there is $v_0 \in A$ such that for all $v \in A$ and every $x \in [0, r]$,

$$f_{v_0}(x) \leqq f_v(x).$$

The set of all suborders in $Q(\Gamma)$ is denoted by $Q(\Gamma)^*$.

Definition 1.2. Two elements F and G in $Q(\Gamma)$ are said to be *in order* if and only if there is an $r > 0$ which puts F and G in order, i.e., there is $B \leqq \Gamma$ such that for all $x \in [0, r]$, if $v \in B$, then $f_v(x) \leqq g_v(x)$. If $B = \Gamma$, then we write $F \leqq G$.

Received March 8, 1983 and in revised form May 3, 1984.

https://doi.org/10.4153/CJM-1985-037-1 Published online by Cambridge University Press

Several authors studied special cases of $L_\Gamma(F)$ spaces. When Γ is countable and $f_v = f \in C$ for all $v \in \Gamma$, then $L_\Gamma(F)$ is just the $L(f)$ space which was introduced by Ruckle [7] and was investigated by Deeb and Hussein [**1**, **2** and **3**]. When Γ is countable and $f_v(x) = x^{p_v}$, $0 < p_v \leqq 1$, then $L_\Gamma(F)$ is the space $l(p_v)$ investigated by Simons [**8**]. If the cardinality of Γ is d and $f_v(x) = x^p$, $0 < p \leqq 1$, for all $v \in \Gamma$, then $L_\Gamma(F)$ is the space $l^p(\Gamma) = l^p_d$, see [**4**, **5**] and [**6**].

In Section 2 of this paper we show that $L_\Gamma(F)$ is a complete metrizable topological vector space. We investigate some of the topological properties of $L_\Gamma(F)$ space. In Section 3, we characterize those elements of $Q(\Gamma)$ of which the dual of $L_\Gamma(F)$ is L^∞_Γ. In Section 4 we investigate local completeness of $L_\Gamma(F)$. We give a sufficient condition for $L(F)$ to contain l^p, $0 < p \leqq 1$. Separability of $L_\Gamma(F)$ is also investigated and we prove that $L_\Gamma(F)$ is separable for a countable Γ and nonseparable when Γ is uncountable.

For the terminology of this paper see [**4**]. The authors would like to thank the referee for his many useful comments and productive suggestions.

2. Topological properties of $L_\Gamma(F)$ spaces. In this section we show that $L_\Gamma(F)$ with the topology induced by $|\cdot|_F$ is an F-space and investigate some of its properties. The proofs of the following, Lemma 2.1, Theorem 2.2 and Lemma 2.3, 2.4 are standard.

LEMMA 2.1. *If $x \in L_\Gamma(F)$, then for every $r > 0$ there is a natural number N such that* $\left|\dfrac{x}{N}\right|_F < r$.

THEOREM 2.2. (1) *$L_\Gamma(F)$ is a vector space.*

(2) *$d(x, y) = |x - y|_F$ is a metric on $L_\Gamma(F)$.*

(3) *If $u(F)$ is the topology induced on $L_\Gamma(F)$ by the above metric then $(L_\Gamma(F), u(F))$ is a topological vector space* (TVS).

(4) *For every $v \in \Gamma$, the evaluation map $E_v : L_\Gamma(F) \to \mathbf{R}$ defined by*

$$E_v(x) = x(v)$$

is continuous.

(5) *$(L_\Gamma(F), u(F))$ is a complete space.*

LEMMA 2.3. *If $L_\Gamma(G) \subseteq L_\Gamma(F)$, then $L_\Gamma(G)$ is dense in $L_\Gamma(F)$.*

LEMMA 2.4. *Assume $F, G \in Q(\Gamma)$ such that for every $v \in \Gamma$ and all $x \geqq 0$,*

$$f_v(x) \leqq g_v(x),$$

then $L_\Gamma(G) \subseteq L_\Gamma(F)$.

https://doi.org/10.4153/CJM-1985-037-1 Published online by Cambridge University Press

LEMMA 2.5. *Assume $F \leqq G$ and r is the real number which puts F and G in order. Moreover let*

$$\inf\{g_v(r): v \in \Gamma\} = r' > 0.$$

Then $L_\Gamma(G) \subseteq L_\Gamma(F)$ and the inclusion map is continuous.

Proof. Let $x \in L_\Gamma(G)$, then there exists $A \in P$ such that

$$\sum_{v \in \Gamma - A} g_v(|x(v)|) < r',$$

hence

$$g_v(|x(v)|) < r' \leqq g_v(r) \quad \text{for all } v \in \Gamma - A.$$

Therefore $|x(v)| < r$ for all $v \in \Gamma - A$. Now

$$\sum f_v(|x(v)|) \leqq \sum_{v \in A} f_v(|x(v)|) + \sum_{v \in \Gamma - A} g_v(|x(v)|) < \infty,$$

hence $x \in L_\Gamma(F)$.

To show that inclusion is continuous: Let $\epsilon > 0$ be given, and take

$$\partial = \min\{r', \epsilon\}.$$

Hence $|x|_G < \partial$ implies that

$$\sum g_v(|x(v)|) < r' \quad \text{and}$$

$$g_v(|x(v)|) < g_v(r) \quad \text{for all } v \in \Gamma.$$

Therefore

$$|x(v)| < r \quad \text{and}$$

$$\sum f_v(|x(v)|) \leqq \sum g_v(|x(v)|) < \partial \leqq \epsilon.$$

COROLLARY. *If $F \in Q(\Gamma)$ and there exists $r > 0$ such that for every $v \in \Gamma$ and all $x \in [0, r]$, $f_v(x) \geqq x$, then $L_\Gamma(F) \subseteq L^1_\Gamma$.*

LEMMA 2.6. *If $F \in Q(\Gamma)^*$, then $L_\Gamma(F) \subseteq L^1_\Gamma$.*

Proof. Let $F \in Q(\Gamma)^*$, then there exists $r > 0$ with

$$\inf\{f_v(r): v \in \Gamma\} > 0$$

and for any $A \in \mathscr{P}$, there exists $v_0 \in A$ such that for all $v \in A$ and every $x \in [0, r]$,

$$f_{v_0}(x) \leqq f_v(x).$$

Assume $x \in L_\Gamma(F)$ and $x \notin L^1_\Gamma$. Then for any $\epsilon > 0$, there is $A \in P$ such that

$$\sum_{v \in \Gamma - A} f_v(|x(v)|) < \epsilon.$$

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Let A' be a finite subset of $\Gamma - A$ such that

$$\sum_{v \in A} |x(v)| > r \quad \text{and} \quad |x(v)| < r, \quad \text{for all } v \in A'.$$

This is possible because if there is an infinite subset $D \leqq \Gamma$ such that $|x(v)| \geqq r$ for all $v \in D$, then $x \notin L_\Gamma(F)$. Now there is $v_0 \in A'$ such that $f_{v_0}(x) \leqq f_v(x)$ for all $v \in A'$ and every $x \in [0, r]$. Hence

$$f_{v_0}(r) \leqq f_{v_0}\left(\sum_{v \in A'} |x(v)|\right) \leqq \sum_{v \in A'} f_{v_0}(|x(v)|)$$

$$\leqq \sum_{v \in A'} f_v(|x(v)|) \leqq \sum_{v \in \Gamma - A} f_v(|x(v)|) < \epsilon.$$

Since ϵ was arbitrary $f_{v_0}(r) = 0$ which is a contradiction. Consequently

$$L_\Gamma(F) \subseteq L_\Gamma^1.$$

Let $F, G \in Q(\Gamma)$ be in order. For each $v \in \Gamma$ set

$$h_v(x) = \max\{f_v(x), g_v(x)\} \quad \text{and}$$

$$k_v(x) = \min\{f_v(x), g_v(x)\}.$$

Let

$$H = \{h_v : v \in \Gamma\} \text{ and } K = \{k_v : v \in \Gamma\},$$

then $H, K \in Q(\Gamma)$.

LEMMA 2.7. *Let $F, G \in Q(\Gamma)$ be in order, and H, K be as defined above. If*

$$\inf\{f_v(r) : v \in \Gamma\} > 0 \text{ and } \inf\{g_v(r) : v \in \Gamma\} > 0,$$

where r is the real number which puts F and G in order. Then

(1) $L_\Gamma(H) = L_\Gamma(F) \cap L_\Gamma(G)$, *and*

(2) $L_\Gamma(K) = M$,

where M is the subspace of R^Γ generated by $\Gamma_\Gamma(F)$ and $L_\Gamma(G)$.

Proof. (1) is obvious by Lemma 2.5. To prove (2), by Lemma 2.5 we have

$$L_\Gamma(F) \cup L_\Gamma(G) \subseteq L_\Gamma(K).$$

Let $x \in L_\Gamma(K)$. Define

$$A_x = \{v \in \Gamma | f_v(x) \geqq g_v(x)\} \quad \text{and}$$

$$B_x = \{v \in \Gamma | f_v(x) \leqq g_v(x)\}.$$

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Let $y, z \in R^\Gamma$ be defined by $y(v) = x(v)$ if $v \in B_x$ and zero otherwise, and $z(v) = x(v)$ if $v \in A_x$ and is zero otherwise. Then

$$\sum f_v(|y(v)|) = \sum_{v \in B_x} k_v(|x(v)|) < \infty$$

and $y \in L_\Gamma(F)$. Similarly $z \in L_\Gamma(G)$. Hence $x = y + z \in M$ and $L_\Gamma(K) \leqq M$. Hence the result.

If $L_\Gamma(G) \subseteq L_\Gamma(F)$, then we can consider two topologies on $L_\Gamma(G)$, $u(G)$ and the subspace topology of $u(F)$. In the following theorem we investigate when these two topologies coincide.

THEOREM 2.8. *Assume that $F, G \in Q(\Gamma)$, $F \leqq G$, and r is the positive real number which puts F and G in order. Moreover let*

$$\inf\{g_v(r):v \in \Gamma\} = r' > 0.$$

Then the following are equivalent

(1) *$u(G)$ is the subspace topology of $u(F)$.*

(2) *$L_\Gamma(G)$ is closed in $L_\Gamma(F)$.*

(3) *$L_\Gamma(G) = L_\Gamma(F)$.*

Proof. To show (1) implies (2). Let $u(G)$ be the topology induced on $L_\Gamma(G)$ as a subspace of $L_\Gamma(F)$, hence $u(G)$ and $u(F)$ give the same definition of Cauchy sequences in $L_\Gamma(G)$. But $(L_\Gamma(G), u(G))$ is complete, hence $L_\Gamma(G)$ is closed in $(L_\Gamma(F), u(F))$.

(2) implies (3) trivially, (by Lemma 2.3), and (3) implies (1) trivially.

3. On the dual of $L_\Gamma(F)$.

It is well known that the dual of $l^p (0 < p \leqq 1)$ is l^∞, and the dual of $L(f)$ spaces with some conditions on f is l^∞ [**3**]. The dual of $l(p_n)$ spaces, $0 < p_n \leqq 1$ and $\inf\{p_n:n = 1, 2, \dots\} \neq 0$ is l^∞ [**8**]. In this section we characterize those elements of $Q(\Gamma)$ for which the dual of $L_\Gamma(F)$ is L_Γ^∞.

THEOREM 3.1. *Let $F \in Q(\Gamma)$ and suppose that there is $r > 0$ with*

$$\inf\{f_v(r):v \in \Gamma\} = r' > 0,$$

such that for all $v \in \Gamma$ and every $x \in [0, r]$, $f_v(x) \geqq x$. Moreover assume that for every $f > 0$, there is a real number $t > 0$ such that $f_v(f) < \epsilon$ for all $v \in \Gamma$. Then there is a bijection between $L_\Gamma(F)'$ and L_Γ^∞.

Proof. For $y \in L_\Gamma^\infty$, define $y^*:L_\Gamma(F) \to R$ as follows. For any $x \in L_\Gamma(F)$, let

$$D = \{v_1, v_2, \dots\} \subseteq \Gamma$$

such that $x(v) = 0$ for all $v \in \Gamma - D$. Then

$$y^*(x) = \sum_{i=1}^\infty x(v_i)y(v_i) \quad \text{and}$$

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$$|y^*(x)| \leqq |y|_\infty. |x|_1 < \infty.$$

To show that $y^* \in L_\Gamma(F)'$, let $x, z \in L_\Gamma(F)$, $t, s \in R$, then there is $D = \{v_1, v_1, \dots\} \subseteq \Gamma$ such that

$$x(v) = z(v) = 0 \quad \text{for all } v \in \Gamma - D.$$

$$y^*(tx + sz) = \sum_{i=1}^{\infty} (tx(v_i) + sz(v_i))y(v_i) = ty^*(x) + sy^*(z).$$

For continuity of y^*. The inclusion $i:L_\Gamma(F) \subseteq L_\Gamma^1$ is continuous (Lemma 2.5). Hence for any $\epsilon > 0$, there is $z > 0$ such that $|x|_F < z$ implies $|x|_1 < F$. Hence

$$|y^*(x)| \leqq |y|_\infty |x|_1 < \epsilon_1$$

and y^* is continuous at zero, being linear it is continuous.

Now define $h:L_\Gamma^\infty \to L_\Gamma(F)'$ by $h(y) = y^*$. For each $v \in \Gamma$ let $B_v \in L_\Gamma(F)$ be defined by $B_v(u) = 1$ if $u = v$ and zero otherwise. If $y_1, y_2 \in L_\Gamma^\infty$ such that $h(y_1) = h(y_2)$, then

$$y_1^*(B_v) = y_2^*(B_v) \quad \text{for all } v \in \Gamma$$

hence $y_1(v) = y_2(v)$ and $y_1 = y_2$. Therefore h is injective.

To show h is onto let $T \in L_\Gamma(F)'$ and let $T(B_v) = b_v$ for all $v \in \Gamma$. Since T is continuous, then it is bounded, in the sense that it maps bounded sets into bounded sets. Let

$$H = \{B_v : v \in \Gamma\}$$

and let

$$U = \{x \in L_\Gamma(F) : |x|_F < \epsilon\}.$$

Now there is $t > 0$ such that $f_v(t) < \epsilon$ for all $v \in \Gamma$. Hence for every $v \in \Gamma$, $tB_v \in U$ and hence

$$H \subseteq \frac{1}{t} U.$$

Therefore $T(H)$ is bounded and hence norm bounded, i.e., there is an $M > 0$ such that $|T(B_v)| < M$ for all $v \in \Gamma$, so

$$\sup_{v \in \Gamma} |T(B_v)| = \sup_{v \in \Gamma} |b_v| \leqq M.$$

Hence if we define $y(v) = b_v$ for all $v \in \Gamma$, then $y \in L_\Gamma^\infty$. Now since for any $x \in L_\Gamma(F)$ and $D = \{v_1, v_2, \dots\} \subseteq \Gamma$ such that $x(v) = 0$ for all $v \in \Gamma - D$, by the continuity of T,

$$T(x) = T\left(\sum_{i=1}^{\infty} x(v_i)Bv_i\right) = \sum_{i=1}^{\infty} x(v_i)b_i$$

and $T = y^*$.

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THEOREM 3.2. *Let F have the same properties as in the last theorem. Moreover assume $f_v(1) = 1$ for all $v \in \Gamma$. Then $L_\Gamma(F)'$ is isometrically isomorphic to L_Γ^∞.*

To show $|y|_\infty = \|y^*\|$ for any $y \in L_\Gamma^\infty$. Let $x \in L_\Gamma(F)$. If $|X|_F \leqq 1$, then

$$|y^*(x)| \leqq |y|_\infty . |x|_F \leqq |y|_\infty$$

hence $\|y^*\| \leqq |y|_\infty$.

On the other hand since $y^*(B_v) = y(v)$ for all $v \in \Gamma$, then

$$|y(v)| = |y^*(B_v)| \leqq \|y^*\|. |B_v|_F = \|y^*\|.$$

Therefore $|y|_\infty \leqq \|y^*\|$, and the result follows.

4. Local boundedness and separability of $L_\Gamma(F)$. In this section we give a sufficient condition for the space $L(F)$ to be locally bounded, and a sufficient condition for $L(F)$ to contain l^p, $0 < p \leqq 1$. We also study the separability of $L_\Gamma(F)$.

THEOREM 4.1. *Suppose $F \in Q$ satisfies the following properties:*

(1) *$f_n(xy) \leqq f_n(x)f_n(y)$ for all $n = 1, 2, \ldots$ and for all $x, y \geqq 0$,*

(2) *there is a natural number k and a real number $r > 0$ such that $f_n(x) \leqq f_k(x)$ for all n and for all $x \in [0, r]$.*

Then a subset $B \leqq L(F)$ is norm bounded if and only if it is topologically bounded.

Proof. Let B be a norm bounded subset of $L(F)$. Then there exists M such that $|b|_F \leqq M$ for all $b \in B$. Let

$$U = \{x \in L(F):|x|_F < \epsilon\}$$

be a neighborhood of zero. By continuity of f_k, there is $s > 0$ such that $|x| < s$ implies

$$f_k(|x|) < \frac{\epsilon}{M}.$$

Let N be a natural number such that $1/N < \min\{s, r\}$, then

$$f_k\left(\frac{1}{N}\right) < \frac{\epsilon}{M}.$$

Now if $x \in B$ then

$$\sum_{n=1}^\infty f_n\left(\frac{|x(n)|}{N}\right) \leqq \sum_{n=1}^\infty f_n\left(\frac{1}{N}\right) f_n(|x(n)|)$$

$$\leqq \sum_{n=1}^\infty f_k\left(\frac{1}{N}\right) f_n(|x(n)|) < \epsilon$$

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i.e., $x/N \in U$ or $x \in NU$ and $B \leqq NU$. The other way is well known.

COROLLARY 1. *If $F \in Q$ satisfies the hypothesis of Theorem* 4.1, *then $L(F)$ is locally bounded. In particular, if $f \in C$ satisfies*

$$f(xy) \leqq f(x)f(y) \quad \text{for all } x, y \geqq 0,$$

then $L(f)$ is locally bounded.

COROLLARY 2. *If $F = \{f_n : f_n(x) = x^{p_n}\}$ where $0 < p_n \leqq 1$ and $\inf p_n > 0$, then $L(F) = l(p_n)$ is locally bounded. This result was proved in* [**8**].

THEOREM 4.2. *Let $F \in Q$ satisfy the following conditions*:
(1) *$f_n(xy) \geqq f_n(x)f_n(y)$ for all n and every $x, y \geqq 0$.*
(2) *There is an $r > 0$ such that for all n and every $x \in [0, r]$ $f_n(x) \geqq x$.*
Then if $L(F)$ is absolutely p-convex then $l^p \subseteq L(F)$.

Proof. There is a basic neighborhood U of zero, which is absolutely p-convex and contained in the unit ball. For this U there is an $s > 0$ such that

$$\{x: |x|_F < s\} \leqq U \leqq \{x: |x|_F < 1\}.$$

Let $r' = \min\{r, s\}$. Now since $f_n(r) \geqq r$ for all n, then there is a real number $t_n > 0$ such that $f_n(t_n) = r'$. Let $x_k = t_k e_k$. Now

$$|x_k|_F = f_k(t_k) = r' \leqq r$$

which implies that $x_k \in U$. If $a_1, a_2, \ldots, a_n$ are real numbers such that

$$\sum_{k=1}^{n} |a_k|^p \leqq 1,$$

then

$$x = \sum_{k=1}^{n} a_k x_k \in U.$$

Now

$$\sum_{k=1}^{n} f_k(|a_k|) = \frac{1}{r'} \sum_{k=1}^{n} f_k(t_k).f_k(|a_k|)$$

$$\leqq \frac{1}{r'} \sum_{k=1}^{n} f_k(|t_k a_k|) \leqq \frac{1}{r'} |x|_F \leqq \frac{1}{r'}.$$

Let $y \in l^p$ and

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$$S = \sum_{n=1}^{\infty} |y(n)|^p.$$

Choose $M \geqq \max\{S, \epsilon\}$. Then

$$\sum_{n=1}^{\infty} |y(n)|^p \leqq M$$

implies

$$\sum_{n=1}^{\infty} \left(\frac{g(n)}{M^{1/p}}\right)^p \leqq 1$$

and

$$\sum_{k=1}^{n} \left(\frac{y(k)}{M^{1/p}}\right)^p \leqq 1 \quad \text{for all } n = 1, 2, \dots .$$

Hence

$$\sum_{k=1}^{n} f_k\left(\left|\frac{y(k)}{M^{1/p}}\right|\right) \leqq \frac{1}{r'} \quad \text{for } n = 1, 2, \dots,$$

and

$$\sum_{k=1}^{n} f_k(|y(k)|).f_k(M^{-1/p}) \leqq \frac{1}{r'},$$

so

$$\sum_{k=1}^{n} M^{-1/p} f_k(|y(k)|) \leqq \frac{1}{r'},$$

or

$$\sum_{k=1}^{n} f_k(|y(k)|) \leqq \frac{M^{1/p}}{r'}$$

and hence $l \leqq L(F)$.

COROLLARY. *Let F satisfy the hypothesis of Theorem 4.2. If $L(F)$ is locally convex, then $L(F) = l^1$.*

THEOREM 4.3. *The space $L_\Gamma(F)$ is separable for any countable Γ and nonseparable for a noncountable Γ.*

Proof. The first part is obvious. To prove the second part let Γ be uncountable. Suppose $E = \{x_1, x_2, \dots\}$ is a countable dense subset of

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$L_\Gamma(F)$. For every i, let D_i be a countable subset of Γ such that $x_i(v) = 0$ for all $v \in \Gamma - D_i$. Let

$$D = \bigcup_{i=1}^{\infty} D_i.$$

Let $u \in \Gamma - D$, and define $x \in L_\Gamma(F)$ as follows. $x(u) = 1$ and $x(v) = 0$ for all $v \neq u$. Then

$$|x - x_i| \geqq 1 \quad \text{for all } i.$$

Hence if $U = \{y:|a - y|_F < 1\}$ then $U \cap E = \emptyset$ and this contradicts the density of E.

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