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ON GRAPH C^*-ALGEBRAS

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Abstract

Certain C^*-algebras on generators and relations are associated to directed graphs. For a finite graph Γ, C^*-algebra $\mathcal{O}_\Gamma$ is canonically isomorphic to Cuntz-Krieger algebra corresponding to the adjacency matrix of Γ. It is shown that if a countably infinite graph Γ is strongly connected, $\mathcal{O}_\Gamma$ is simple and purely infinite.

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1. Introduction and notation

Let Γ be a countable directed graph. Denote vertices of Γ by $U, V, W \in \mathcal{V}(\Gamma)$ and edges by $u, v, w \in \mathcal{E}(\Gamma)$. If $v \in \mathcal{E}(\Gamma)$ is connecting U and V, call U *the source* of v and V *the range* of v, and write

$$s(v) = U, \quad r(v) = V.$$

Let H be an infinite-dimensional Hilbert space. To every edge $v \in \mathcal{E}(\Gamma)$ we associate a non-zero partial isometry s_v, acting on H, with the following properties:

(i) $s_v s_v^* s_w s_w^* = \delta_{v,w} s_v s_v^*$, for all $v, w \in \mathcal{E}(\Gamma)$;
(ii) $s_v^* s_v s_w^* s_w = \delta_{r(v),r(w)} s_v^* s_v$, for all $v, w \in \mathcal{E}(\Gamma)$;
(iii) $s_v^* s_v s_u s_u^* = \delta_{r(v),s(u)} s_u s_u^*$, for all $u, v \in \mathcal{E}(\Gamma)$;
(iv) $s_v^* s_v = \sum_{r(v)=s(w)} s_w s_w^*$, if the set $\{w \in \mathcal{E}(\Gamma); s(w) = r(v)\}$ is finite.

DEFINITION 1. With the notation as above, we set

$$\mathcal{O}_{\Gamma,\{s_v\}} = C^*(s_v;\ v \in \mathcal{E}(\Gamma))$$

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and call $\mathcal{O}_{\Gamma,\{s_v\}}$ *a Cuntz-Krieger algebra associated to* Γ *and the family* $\{s_v\}$. The corresponding universal C^*-algebra will be denoted $\mathcal{O}_\Gamma$.

REMARK 1. In general, $\mathcal{O}_{\Gamma,\{s_v\}}$ defined above depends on the choice of generating partial isometries. Note also that arguments from [5] show that $\mathcal{O}_\Gamma$ exists, for any Γ.

DEFINITION 2. Let Γ be a directed graph. We call Γ *infinite* if the set $\mathcal{E}(\Gamma)$ is infinite, and *row finite* if, for each vertex, the number of outgoing edges is finite. The *adjacency matrix* of Γ is defined as

$$A_\Gamma(u, v) = \begin{cases} 1, & r(u) = s(v) \\ 0, & r(u) \neq s(v), \end{cases}$$

for all pairs of edges (u, v) in $\mathcal{E}(\Gamma)$.

DEFINITION 3. Let A be a non-negative, $n \times n$-matrix. Call A *irreducible* if for each pair of indices (i, j) from $\{1, \dots, n\}$, there is $k \in \mathbb{N}$ such that $A^k(i, j) \neq 0$. A directed graph Γ is called *strongly connected* (or *transitive*) if for all pairs of vertices (U, V), there exists a path $v_1 \cdots v_k$ such that $s(v_1) = U$ and $r(v_k) = V$.

If Γ is finite, strongly connected and every loop in Γ has an exit—in other words, if A_Γ is an irreducible, non-permutation matrix, $\mathcal{O}_\Gamma$ is canonically isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_{A_\Gamma}$ (see [4]). In particular, $\mathcal{O}_\Gamma$ is simple and purely infinite, and $\mathcal{O}_{\Gamma,\{s_v\}}$ does not depend on the choice of generators.

In this note, we give a simple proof of an analogous theorem for infinite graphs (the theorem is proved at the end of the paper—see Theorem 2.6):

THEOREM 1.1. *Let* Γ *be a countably infinite, strongly connected graph. Then* $\mathcal{O}_\Gamma$ *is simple and purely infinite.*

We should point out that the above theorem has been proved by Laca and Exel (see [6, 16.2 and 14.1]). Using the presentation of $\mathcal{O}_\Gamma$ as the crossed product algebra for a partial dynamical system, they extended to the infinite case some of the main results known to hold in the finite case—including the above criterion for $\mathcal{O}_\Gamma$ to be simple and purely infinite.

An important special case of Theorem 2.6 is when Γ is assumed to be row finite (in which case it suffices to use only relations (i) and (ii)—the usual Cuntz-Krieger relations). This situation has been studied by Kumjian, Pask and Raeburn (see [14, Corollary 3.10.]). Using a groupoid approach, they carry out a detailed analysis of how the distribution of loops affects the structure of $\mathcal{O}_\Gamma$, for any row-finite graph Γ.

In contrast to that, the method used here is just an adaptation of the original proof of Cuntz. Namely, in analogy with $\mathcal{O}_\infty$, we use the fact that

$$\mathcal{O}_\Gamma = \varinjlim \mathcal{E}_{A_n},$$

where we show that each $\mathcal{E}_{A_n}$ is a universal algebra on generators and relations, canonically isomorphic to an extension of some Cuntz-Krieger algebra by a direct sum of a finite number of copies of compact operators. We then modify the proof of [1, Theorem 3.4] to this slightly more general setup. Finally, it is clear that algebras $\mathcal{O}_\Gamma$ satisfy the UCT, so are within the range of Kirchberg's classification (see [12, 13]).

Let us also mention that results similar to Theorem 2.6 appear in [10, 11], albeit in a different setting, and that $\mathcal{O}_\Gamma$ can be realized as a Pimsner algebra $\mathcal{O}_X$, for a suitable choice of bimodule X (see [16]).

2. Preliminaries and results

Let Γ be a countably infinite, directed graph. Unless stated otherwise, it is always assumed that Γ is strongly connected. Relabel the edges of Γ as $v_1, v_2, \ldots$, write $A_\Gamma(i, j)$ for $A_\Gamma(v_i, v_j)$ and denote the partial isometry s_{v_i} by s_i. Also, let A_n stand for the upper-left hand corner of the matrix A_Γ, and

$$\mathcal{M}_{A_n} = \{\mu = s_{i_1} \cdots s_{i_k};\ i_j \in \{1, \ldots, n\} \text{ and } A_n(i_j, i_{j+1}) = 1\}.$$

REMARK 2. It is easy to see that, with Γ as above, there exists an increasing filtration $(\Gamma_i)_{i\in\mathbb{N}}$ of Γ, where each Γ_i is a finite, strongly connected graph. Furthermore, we will assume that the edges of Γ (hence, the generators of $\mathcal{O}_\Gamma$) are labelled in a way compatible with this filtration.

LEMMA 2.1. *Let Γ be a countably infinite directed graph. Then Γ is strongly connected if and only if there exists a strictly increasing sequence of integers $(n_k)_{k\in\mathbb{N}}$ such that each A_{n_k} is an irreducible, non-permutation matrix.*

PROOF. If Γ is infinite and strongly connected, there exists a vertex with at least two outgoing edges. Together with the above ordering, that gives the sequence (A_{n_k}). The other direction is obvious. □

REMARK 3. Assume that Γ is as above and denote $\mathcal{E}_{A_n,\{s_i\}} = C^*\{s_1, \ldots, s_n\}$, $p_i = s_i s_i^*$, $q_i = s_i^* s_i$ and $r_i = s_i^* s_i - \sum_{j=1}^n A_n(i, j) s_j s_j^*$. Since the projections q_i and q_j are either equal or orthogonal, the same holds for r_i and r_j, $i, j = 1, \ldots, n$,

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so denote by $m_1, \dots, m_k$ distinct projections among $r_1, \dots, r_n$, for some $k \le n$. Set $I^{(j)} = C^*\{s_\mu m_j s_\nu^*; \mu, \nu \in \mathscr{M}_{A_n}\}$, for every j, and

$$I_n = I^{(1)} \oplus \cdots \oplus I^{(k)}.$$

We then have $I^{(j)} \cong \mathscr{K}$, for all j, where $\mathscr{K}$ stands for the compact operators on a separable Hilbert space (see [1, Proposition 3.1]). Furthermore, if A_n is assumed to be irreducible and non-permutation,

$$(1) \qquad \mathscr{E}_{A_n,\{s_i\}}/I_n \cong \mathscr{O}_{A_n}.$$

The following result is analogous to [3, Lemma 3.1]:

PROPOSITION 2.2. *Let Γ be a countably infinite, strongly connected directed graph, and let s_i, $i \in \mathbb{N}$, be a set of generators of $\mathscr{O}_{\Gamma,\{s_i\}}$. Then, for any $m \in \mathbb{N}$, the algebra $\mathscr{E}_{A_m,\{s_i\}} = C^*\{s_1, \dots, s_m\}$ does not depend on the choice of generators.*

PROOF. Suppose that $s_i \in B(H)$. As above, we set $r_i = s_i^* s_i - \sum_{j=1}^m A(i,j) s_j s_j^*$. Let $m_0 > m$ be such that for each non-zero r_i, $i = 1, \dots, m$, there is $j(i) \in \{m+1, \dots, m_0\}$ such that

$$r_i s_{j(i)} s_{j(i)}^* = s_{j(i)} s_{j(i)}^*,$$

and let $n > m_0$ be such that A_n is irreducible and non-permutation. Note that Lemma 2.1 (and Remark 2) imply that such m_0 and n exist. We want to construct partial isometries $t_{m+1}, \dots, t_n \in B(H)$ such that $C^*(s_1, \dots, s_m, t_{m+1}, \dots, t_n)$ is canonically isomorphic to $\mathscr{O}_{A_n}$.

Let I be a subset of $\{m+1, \dots, n\}$ defined by: $j \in I$ if and only if there is $1 \le i \le m$ such that $A(i,j) = 1$ and $A(i,k) = 0$, for $k = j+1, \dots, n$. For $j \in I$, let $\tilde{p}_j = r_i - \sum_{k=m+1}^{j} A(i,k) s_k s_k^*$. Note that $\tilde{p}_j = s_j s_j^* + s_i^* s_i - \sum_{k=1}^n A(i,k) s_k s_k^*$, and that $\tilde{p}_j$ does not depend on the choice of i in the above formula. For j not in I, set $\tilde{p}_j = s_j s_j^*$. Define projections $\tilde{q}_j$, for $j = m+1, \dots, n$, by

$$\tilde{q}_j = \sum_{k=1}^m A(j,k) s_k s_k^* + \sum_{k=m+1}^m A(j,k) \tilde{p}_k.$$

Since Γ is strongly connected, every $s_i s_i^*$ is an infinite-dimensional projection, so the same holds for $\tilde{p}_j$ and $\tilde{q}_j$. For $j = m+1, \dots, n$, let t_j be any partial isometry such that $t_j t_j^* = \tilde{p}_j$ and $t_j^* t_j = \tilde{q}_j$. Then

$$s_i^* s_i = \sum_{j=1}^m A(i,j) s_j s_j^* + \sum_{j=m+1}^n A(i,j) t_j t_j^*, \quad i = 1, \dots, m,$$

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and

$$t_i^* t_i = \sum_{j=1}^{m} A(i,j) s_j s_j^* + \sum_{j=m+1}^{n} A(i,j) t_j t_j^*, \quad i = m+1, \dots, n,$$

hence, $\mathscr{A} = C^*(s_1, \dots, s_m, t_{m+1}, \dots, t_n) \cong \mathcal{O}_{A_n}$ canonically.

If $s_i' \in B(H)$, $i = 1, 2, 3, \dots$, is another set of generators for $\mathcal{O}_\Gamma$, the above procedure gives $t_{m+1}', \dots, t_n'$ such that $\mathscr{A}' = C^*\{s_1', \dots, s_m', t_{m+1}', \dots, t_n'\} \cong \mathcal{O}_{A_n}$, and the map

$$s_i \mapsto s_i', \quad i = 1, \dots, m, \qquad t_j \mapsto t_j', \quad j = m+1, \dots, n$$

extends to an isomorphism from $\mathscr{A}$ into $\mathscr{A}'$, mapping $\mathscr{E}_{A_m, \{s_i\}}$ onto $\mathscr{E}_{A_m, \{s_i'\}}$. □

COROLLARY 2.3. *Let* Γ *be as in Proposition* 2.2. *Then* $\mathcal{O}_\Gamma$ *does not depend on the choice of generators.*

REMARK 4. It is clear that Proposition 2.2 and Corollary 2.3 will remain true as long as one can construct a canonical embedding of $\mathscr{E}_{A_{n_k}}$ into some Cuntz-Krieger algebra that does not depend on the choice of generators. This has already been argued by Cuntz and Krieger in [4, Remark 2.15]. Note also that $\mathscr{E}_{A_{n_k}}$ can be described as a C^*-algebra associated to some inverse semigroup (see, for example, [9]).

The following result, due to Cuntz (see [3, Proposition 1.6]), describes simple purely infinite C^*-algebras. We use this in the proof of Theorem 2.6:

PROPOSITION 2.4. *Let the* C^**-algebra* $\mathscr{A}$ *satisfy*:

(i) $\mathscr{A} \neq 0, \mathbb{C}$.

(ii) *For every* $\varepsilon > 0$ *and every positive* $a, b \in \mathscr{A}$, *there is* $c \in \mathscr{A}$ *such that* $\|b - cac^*\| < \varepsilon$.

Then $\mathscr{A}$ *is simple and purely infinite.*

LEMMA 2.5. *Let* A_n *be irreducible. Then there exists a non-unitary isometry* $v \in \mathscr{E}_{A_n}$, *such that*

$$\lim_{k\to\infty} (v^*)^k x v^k = 0, \quad \text{for all } x \in I_n.$$

PROOF. In the case of $\mathcal{O}_\infty$, this has been proved in [1, Proposition 3.1, Remark 2]. Since we do not necessarily have an isometry among the generators of $\mathscr{E}_{A_n}$, we have to construct one. Let p_i and m_i be as in Remark 3. Note that

$$\text{for all } j \text{ there is } i \text{ such that } s_i m_j = s_i s_i^* s_i m_j \neq 0,$$

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and denote that s_i by $\tilde{t}_j$. Also,

$$\text{for all } j \text{ there is } i \text{ such that } s_i p_j = s_i s_i^* s_i p_j \neq 0.$$

Denote that s_i by t_j, and set

$$v = \sum_{i=1}^{n} t_i p_i + \sum_{j=1}^{k} \tilde{t}_j m_j.$$

We immediately get $v^* v = 1$ and $vv^* < 1$, so v is a proper isometry. Let $s_\mu = s_{i_1} s_{i_2} \cdots s_{i_p}$, and note that $v^* s_\mu \neq 0$ implies

$$\begin{aligned} v^* s_\mu &= \left(\sum_{i=1}^{n} p_i t_i^* s_{i_1} + \sum_{j=1}^{k} m_j \tilde{t}_j^* s_{i_1} \right) s_{i_2} \cdots s_{i_p} \\ &= (p_{j_1} + \cdots + p_{j_m} + m_{k_1} + \cdots + m_{k_l}) p_{i_2} (s_{i_2} \cdots s_{i_p}) = s_{i_2} \cdots s_{i_p}. \end{aligned}$$

It remains to be shown that $v^* m_j v = 0, j = 1, \ldots, k$. Since $p_i m_j = 0$, we get

$$v^* m_l = \sum_{i=1}^{n} p_i t_i^* (t_i t_i^*) m_l + \sum_{j=1}^{k} m_j \tilde{t}_j^* (\tilde{t}_j \tilde{t}_j^*) m_l = 0, \quad l = 1, \ldots, k. \qquad \square$$

DEFINITION 4. Let α be the action of $\mathbb{T} = \{z \in \mathbb{C};\ |z| = 1\}$ on $\mathcal{O}_\Gamma$, given on generators by $\alpha_t(s_v) = t s_v$, $v \in \mathcal{E}_\Gamma$, and set

$$P(x) = \int_{\mathbb{T}} \alpha_t(x)\, dt, \quad x \in \mathcal{O}_\Gamma \tag{2}$$

(see [1]).

Now we are ready to prove the result announced in the Introduction. The proof closely follows the proof of [1, Theorem 3.4]:

THEOREM 2.6. *Let Γ be a countably infinite, directed, strongly connected graph. Then $\mathcal{O}_\Gamma$ is simple and purely infinite.*

PROOF. Let positive elements $a, b \in \mathcal{O}_\Gamma$ and $0 < \varepsilon < 1/4$ be given. Let $a = zz^*$, for some $z \in \mathcal{O}_\Gamma$, and let $y \in \mathcal{E}_{A_m}$, for some $m \in \mathbb{N}$, be a finite linear combination of words in s_i, s_i^* such that $\|b - y\| < \varepsilon$. We can assume that $\|P(y)\| = 1$ and $\|z\| = 1$.

From Lemma 2.1 there is $n > m$ such that A_n is irreducible and non-permutation. With $t_1, \ldots, t_n$ as in Proposition 2.2, we consider C^*-algebras $\mathcal{A}_1 = C^*\{s_1, \ldots, s_m, t_{m+1}, \ldots, t_n\}$, $\mathcal{E}_{A_n}$, and $\mathcal{A}_2 = \mathcal{E}_{A_n}/I_n$. Denote by π the quotient map $\mathcal{E}_{A_n} \to \mathcal{E}_{A_n}/I_n$. It follows from (1) and Proposition 2.2 that the map

$$s_i \mapsto \pi(s_i), \quad i = 1, \ldots, m, \qquad t_j \mapsto \pi(s_j), \quad j = m+1, \ldots, n$$

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extends to an isomorphism from $\mathscr{A}_1$ into $\mathscr{A}_2$. Let $P_1(x)$ and $P_2(x)$ stand for $P(x)$ (see (2) above), computed in $\mathscr{A}_1$ and $\mathscr{A}_2$, respectively. Since y is a word in s_i, s_i^*, with i only in $\{1, \dots, m\}$, $P_1(y) = P(y)$. Together with the above isomorphism, that gives

$$\|P_2(\pi(y))\| = \|P_1(y)\| = \|P(y)\| = 1.$$

From [1, Remark 1.13], there is $\hat{w} \in \mathscr{A}_2$ such that $\|\hat{w}\| \leq 1+\varepsilon$, and $\hat{w}\pi(y)\hat{w}^* = 1$. Lifting from the quotient gives

$$wyw^* = 1 + I_n$$

in $\mathscr{E}_{A_n}$, with $\|w\| \leq 1 + 2\varepsilon$. Then, from Lemma 2.5, there is $v \in \mathscr{E}_{A_n}$ and $k \in \mathbb{N}$, such that

$$\|(v^*)^k wyw^*v^k - 1\| < \varepsilon.$$

Hence, we get

$$\|z(v^*)^k wbw^*v^k z^* - a\| < 4\varepsilon,$$

which completes the proof. □

REMARK 5. If a directed graph Γ is row finite and strongly connected, [15, Theorem 4.2.4] gives the K-theory of $\mathscr{O}_\Gamma$:

$$K_0(\mathscr{O}_\Gamma) \cong \tilde{\mathbb{Z}}^\infty / \operatorname{Im}(1 - A_\Gamma^t)\tilde{\mathbb{Z}}^\infty \quad \text{and} \quad K_1(\mathscr{O}_\Gamma) \cong \operatorname{Ker}(1 - A_\Gamma^t)\tilde{\mathbb{Z}}^\infty$$

(see [2, 15]). In case of general Γ, see [7]. Finally, note that the K-theory of $\mathscr{O}_\Gamma$ can be computed in the same way as that of $\mathscr{O}_\infty$ (see [3]). That has been done in [8].

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