ENUMERATION OF GRAPHS WITH GIVEN PARTITION

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1. Introduction. In this paper we use a generalized form of Polya's theorem **(1)** to obtain generating functions for the number of ordinary graphs with given partition and for the number of bicoloured graphs with given bipartition. Both the points and lines of the graphs are taken as unlabelled. These graph enumeration problems were proposed by Harary in his review article **(4)**. Read **(7, 8)** solved the problem for unlabelled general graphs and labelled ordinary graphs.

The generating function obtained here for the number of bicoloured graphs with given bipartition furnishes a formal solution of a problem on (0, 1)-matrices mentioned by Ryser **(9**, p. 444**)** and Mirsky **(6**, p. 113**)**.

2. Definitions and notations. An ordinary finite *graph* consists of a finite set of points X together with a prescribed subset of the set of all unordered pairs of points of X. The point pairs are called the *lines* of the graph. Two lines are said to be *adjacent* if they have a common point; two points x_i, x_j are adjacent if (x_i, x_j) is a line of the graph. The line (x_i, x_j) is said to be *incident* to the points x_i and x_j. The *degree* of a point is the number of lines incident to the point. The *partition* (π_n) *of a graph* with n points and l lines is the n-part partition of $2l$ (zero parts being permitted) formed by the degrees of the points of the graph. Two graphs are said to be *isomorphic* if there is a one-to-one correspondence between their points which preserves adjacency.

A *bicoloured graph* consists of a set X of points coloured blue (say) and a set Y of points coloured red (say) together with a prescribed set of unordered pairs of points (x_i, y_j), $x_i \in X$, $y_j \in Y$. If a bicoloured graph with m blue points and n red points has l lines, then the degrees of the blue points form an m-part partition (π_m) of l and the degrees of the red points form an n-part partition (π_n) of l, zero parts being permitted. We call the pair $((\pi_m), (\pi_n))$ the *bipartition* of the bicoloured graph. Two bicoloured graphs are said to be *chromatically isomorphic* if there is a colour-preserving isomorphism between them.

We denote by $(\pi)_u$ the monomial symmetric function corresponding to the partition (π) in a set of variables $u_1, u_2, \ldots, u_n$; see **(5)**. The term of the symmetric function $(\pi)_u$ in which the powers of the variables $u_1, u_2, \ldots, u_n$ are in descending order of magnitude is called the *leading term* of $(\pi)_u$. The leading term of the symmetric function product $(\pi_m)_u(\pi_n)_v$ is the product of

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the leading terms of $(\pi_m)_u$ and $(\pi_n)_v$. The parts of a partition being arranged in the usual descending order of magnitude, we say of partitions

$$(\pi^1) = (p_1 p_2 \dots p_n)$$

and $(\pi^2) = (q_1 q_2 \dots q_n)$ with an equal number of parts that (π^1) *precedes* (π^2), written $(\pi^1) \succ (\pi^2)$ if there exists an index N such that $p_i = q_i$ for $i = 1, 2, \dots, N$ and $p_{N+1} > q_{N+1}$.

The *direct product, Cartesian product, composition,* and *exponentiation* of two permutation groups are defined as in **(3)** or **(4)**.

3. Polya's theorem. We use the following generalized form of Polya's theorem given by de Bruijn **(1)**.

Let D and R be finite sets and K be a permutation group acting on the elements of D. Let $\mathfrak{f}$ be the collection of all functions from D into R. Let $f_1 \in \mathfrak{f}$ and $f_2 \in \mathfrak{f}$ be called equivalent if and only if there exists a $k \in K$ such that $f_1(d) = f_2(kd)$ for all $d \in D$. Let the collection of equivalence classes in $\mathfrak{f}$ under this equivalence relation be denoted by $\mathfrak{F}$. Define a weight function $W(f)$ for the functions $f \in \mathfrak{f}$ such that it has a constant value $W(F)$ for each f belonging to the same equivalence class $F \in \mathfrak{F}$. Then the sum of the weights over equivalence classes of functions is given by

$$\sum_{F \in \mathfrak{F}} W(F) = \frac{1}{|K|} \sum_{k \in K} \sum_{f}^{(k)} W(f)$$

where $|K|$ is the order of K and $\sum_{f}^{(k)}$ indicates that we sum $W(f)$ over those f which are left invariant by k, that is, for which $f(d) = f(kd)$ for all $d \in D$.

4. Ordinary graphs. Let

$$D = \{(x_i, x_j) : x_i \in X, x_j \in X, i < j\}$$

where X is a set of n points. Let $R = \{0, 1\}$. Then there is a one-to-one correspondence between functions f from D into R and the ordinary graphs on the point set X. If further we take for the permutation group K the full pair group acting on the points of X (that is, the line group $\Gamma_1(K_n)$ of the complete graph K_n), the equivalence classes F induced in the set $\mathfrak{f}$ of all functions from D into R correspond to the isomorphism classes of graphs on X.

We define a weight function $W(f)$ on the functions $f \in \mathfrak{f}$ as follows:

$$W(f) = OW_0(f) = O \prod_{\substack{x_i \in X \\ i<j}} \prod_{x_j \in X} (u_i u_j)^{f(x_i, x_j)},$$

where O is an unordering operator which replaces $W_0(f)$ by the leading term of the symmetric function $(\pi_n)_u$ corresponding to the partition (π_n) of the graph presented by f. For example,

$$O(u_1^4 u_2 u_3^2 u_4^2 u_5 u_6^3) = O(u_1 u_2^4 u_3^3 u_4 u_5^3 u_6^2) = u_1^4 u_2^3 u_3^3 u_4^2 u_5 u_6.$$

This definition of the weight function clearly satisfies the condition of Polya's theorem, and it is easily seen that the sum of the weights over equivalence classes of functions serves as a generating function for the number of non-isomorphic graphs with the partition (π_n). Hence the generating function is given by Polya's theorem as

$$\sum_{F\in\mathfrak{F}} W(F) = \frac{1}{n!}\sum_{k\in\Gamma_1(K_n)} \sum_{f}^{(k)} O \prod_{x_i\in X} \prod_{\substack{x_j\in X\\ i<j}} (u_i u_j)^{f(x_i,x_j)}.$$

To simplify the generating function we compute the expression

$$\chi(k) = \sum_{f}^{(k)} \prod_{x_i\in X} \prod_{\substack{x_j\in X\\ i<j}} (u_i u_j)^{f(x_i,x_j)}$$

for a given $k \in \Gamma_1(K_n)$. Now every permutation $k \in \Gamma_1(K_n)$ is induced by a unique permutation $\kappa \in S_n$. We compute $\chi(k)$ by considering pairs (A, B) of cycles A and B of κ. The computation is analogous to that of the cycle index of $\Gamma_1(K_n)$ given by Harary in **(2)**. If κ has cycle structure $(\lambda) = 1^{\lambda_1} 2^{\lambda_2} \ldots n^{\lambda_n}$ and $U_{pi} = u_{pi1} u_{pi2} \ldots u_{pip}$ denotes the product of the u-terms corresponding to the ith cycle of length p in κ, then for the corresponding k we get

$$\begin{aligned}(1)\quad \chi(k) = &\prod_{p=1}^{n} \prod_{q=1}^{n} \prod_{i=1}^{\lambda_p} \prod_{j=1}^{\lambda_q} (1 + U_{pi}^{q/(p,q)} U_{qj}^{p/(p,q)})^{(p,q)} \\ &\times \prod_{p=1}^{n} \prod_{i=1}^{\lambda_p} \prod_{\substack{j=1\\ i<j}}^{\lambda_p} (1 + U_{pi} U_{pj})^{p} \\ &\times \prod_{p \text{ odd}} \prod_{i=1}^{\lambda_p} (1 + U_{pi}^2)^{(p-1)/2} \prod_{p \text{ even}} \prod_{i=1}^{\lambda} (1 + U_{pi})(1 + U_{pi}^2)^{(p-2)/2}\end{aligned}$$

where (p, q) is the greatest common divisor of p and q. Thus we have

THEOREM 1. *The generating function for ordinary graphs on n unlabelled points with given partition is*

$$\frac{1}{n!} O \sum_{k\in\Gamma_1(K_n)} \chi(k)$$

where $\chi(k)$ is given by (1).

A further simplification in the generating function is obtained when we sum over the expressions $\chi(k)$ corresponding to all κ's with the same cycle structure (λ). Thus, if we write

$$X(\lambda) = \sum_{\{k:\kappa\in(\lambda)\}} \chi(k)$$

$X(\lambda)$ turns out to be a linear function of monomial symmetric functions of the form $(\pi_n)_u$. In fact, if $X'(\lambda)$ represents the function obtained from $X(\lambda)$ by

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multiplying each symmetric function by the number of terms in the symmetric function, the generating function may be taken as

$$\frac{1}{n!}\sum_{(\lambda)} X'(\lambda)$$

where the coefficient of the symmetric function $(\pi_n)_u$ as such gives the number of non-isomorphic graphs with the partition (π_n).

Using the above method, the following generating function was obtained for graphs on five points with at most five lines:

$$1 + (1^2) + \{(21^2) + (1^4)\} + \{(31^3) + (2^3) + (21^4) + (2^21^2)\}$$
$$+ \{(41^4) + (32^21) + (321^3) + (2^4) + 2(2^31^2)\}$$
$$+ \{(42^21^2) + 2(32^31) + (3^221^2) + (2^5) + (3^22^2)\}.$$

The corresponding graphs with their partitions are shown in Figure 1. Graphs with more than five lines may be obtained from these graphs by complementation.

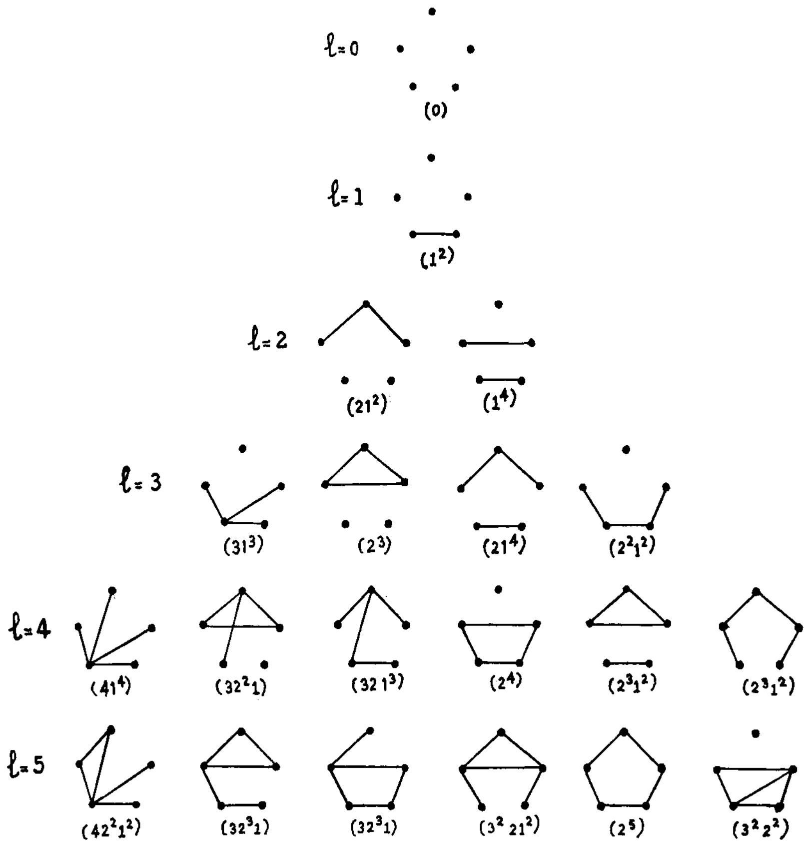

FIGURE 1. Ordinary graphs on five points with at most five lines.

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5. Bicoloured graphs on m blue points and n red points $(m \neq n)$. Let X and Y be the sets of blue and red points. Let $D = X \times Y$ be the Cartesian product of X and Y and $R = \{0, 1\}$. Then there is a one-to-one correspondence between the functions from D into R and the bicoloured graphs on $X \cup Y$. For the permutation group K acting on D we take $S_m \times S_n$, the Cartesian product of the symmetric groups S_m and S_n acting on X and Y. The chromatically isomorphic bicoloured graphs on $X \cup Y$ correspond to equivalent functions from D into R; see **(3)**.

We define the following weight function for functions f from D into R:

$$W(f) = OW_0(f) = O \prod_{x_i \in X} \prod_{y_j \in Y} (u_i v_j)^{f(x_i, y_i)}$$

where the unordering operator O replaces $W_0(f)$ by the leading term of the symmetric function product $(\pi_m)_u(\pi_n)_v$ corresponding to the bipartition $((\pi_m), (\pi_n))$ of the bicoloured graph represented by f. It is easily verified that this satisfies the condition of Polya's theorem.

The generating function for the bicoloured graphs is then given by Polya's theorem as

$$\sum_{F \in \mathfrak{F}} W(F) = \frac{1}{m!\, n!} \sum_{k \in (S_m \times S_n)} \sum_{f}^{(k)} O \prod_{x_i \in X} \prod_{y_j \in Y} (u_i v_j)^{f(x_i, y_j)}.$$

Now each $k \in (S_m \times S_n)$ corresponds to a pair (g, h) where $g \in S_m$ and $h \in S_n$ are permutations acting on X and Y respectively. To reduce the generating function we compute the expression

$$\phi(g, h) = \sum_{f}^{(k)} \prod_{x_i \in X} \prod_{y_j \in Y} (u_i v_j)^{f(x_i, y_j)}.$$

The computation proceeds on lines similar to those indicated in the previous section, by considering pairs (A, B) of cycles $A \in g$ and $B \in h$. If g has cycle structure $(\lambda) = 1^{\lambda_1} 2^{\lambda_2} \ldots m^{\lambda_m}$ and h has cycle structure $(\mu) = 1^{\mu_1} 2^{\mu_2} \ldots n^{\mu_n}$ and if $U_{pi} = u_{pi1} u_{pi2} \ldots u_{pip}$ and $V_{qj} = v_{qj1} v_{qj2} \ldots v_{qjq}$ are, respectively, the products of the u-terms occurring in the ith cycle of length p in g and the v-terms occurring in the jth cycle of length q in h, then we obtain

$$\text{(2)} \qquad \phi(g, h) = \prod_{p=1}^{m} \prod_{q=1}^{n} \prod_{i=1}^{\lambda_p} \prod_{j=1}^{\mu_q} (1 + U_{pi}^{q/(p,q)} V_{qj}^{p/(p,q)})^{(p,q)}.$$

Thus we have

THEOREM 2. *The generating function for bicoloured graphs with given bipartition on m unlabelled blue nodes and n unlabelled red nodes $(m \neq n)$ is*

$$\frac{1}{m!\, n!} O \sum_{g \in S_m} \sum_{h \in S_n} \prod_{p=1}^{m} \prod_{q=1}^{n} \prod_{i=1}^{\lambda_p} \prod_{j=1}^{\mu_q} (1 + U_{pi}^{q/(p,q)} V_{qj}^{p/(p,q)})^{(p,q)}.$$

A symmetric function representation for this generating function is obtained by using the fact that the expressions

$$\text{(3)} \qquad \Phi((\lambda), (\mu)) = \sum_{g \in (\lambda)} \sum_{h \in (\mu)} \phi(g, h)$$

are linear functions of symmetric function products of the form $(\pi_m)_u(\pi_n)_v$. In fact, denoting by $\Phi'((\lambda), (\mu))$ the expression obtained by multiplying the coefficient of each symmetric function product in $\Phi((\lambda), (\mu))$ by the number of terms in the product, we may take the generating function to be

$$\frac{1}{m!\,n!}\sum_{(\lambda)}\sum_{(\mu)}\Phi'((\lambda), (\mu)).$$

In this form of the generating function, the number of bicoloured graphs with the bipartition $((\pi_m), (\pi_n))$ appears as the coefficient of the symmetric function product $(\pi_m)_u(\pi_n)_v$.

6. Bicoloured graphs on n points of either colour. The changes required for this case from the previous one are in the weight function and the permutation group acting on the domain set $D = X \times Y$.

We define the weight function by

$$W(f) = \bar{O}W_0(f) = \bar{O}\prod_{x_i \in X}\prod_{y_j \in Y}(u_i v_j)^{f(x_i, y_j)}$$

where the effect of the operator $\bar{O}$ is to replace $W_0(f)$ by the leading term of the symmetric function product $(\pi_n^1)_u(\pi_n^2)_v$ corresponding to the bipartition $((\pi_n^1), (\pi_n^2))$ of the graph represented by f if $(\pi_n^1) \succ (\pi_n^2)$; if, however, $(\pi_n^2) \succ (\pi_n^1)$, $\bar{O}$ replaces $W_0(f)$ by the leading term of $(\pi_n^2)_u(\pi_n^1)_v$. Here (π_n^1) and (π_n^2) denote two n-part partitions of the same number l of lines.

Harary **(3)** has shown that the permutation group K appropriate for this case is the exponentiation group $S_n^{S_2}$ which is the line group of the complete bipartite graph K_{nn}. The point-group or automorphism group of K_{nn} is the composition group $S_2[S_n]$. In Harary's notation these groups can be written as

$$S_2[S_n] = (S_n \cdot S_n) \cup r(S_n \cdot S_n),$$
$$S_n^{S_2} = (S_n \times S_n) \cup \rho(S_n \times S_n).$$

Here $S_n \cdot S_n$ and $S_n \times S_n$ are the direct product and Cartesian product of two copies of S_n acting on X and Y. The set $r(S_n \cdot S_n)$ consists of $(n!)^2$ permutations on $X \cup Y$ of the form $r(g, h)$ where $g \in S_n$ and $h \in S_n$ and the effect of r is to interchange corresponding elements x_i and y_i of X and Y. Each of these permutations is obtained by interposing the elements of two permutations of the two copies of S_n. Thus, corresponding to each permutation $g \in S_n$ with cycle structure $(\lambda) = 1^{\lambda_1}2^{\lambda_2}\ldots n^{\lambda_n}$, there are $n!$ permutations of $r(S_n \cdot S_n)$ with cycle structure $(2\lambda) = 2^{\lambda_1}4^{\lambda_2}\ldots(2n)^{\lambda_n}$. Because of this law of formation, $r(S_n \cdot S_n)$ is also written as $S_n(2)$. The set of permutations $\rho(S_n \times S_n)$ are those induced on the elements of $D = X \times Y$ by the members of $S_n(2)$.

Using Polya's theorem with the above specifications, we obtain the number of isomorphism classes of bicoloured graphs:

$$(4) \qquad \sum_{F \in \mathfrak{F}} W(F) = \frac{1}{2(n!)^2}\bar{O}\left\{\sum_{k \in (S_n \times S_n)}\sum_{f}^{(k)} W_0(f) + \sum_{k \in \rho(S_n \times S_n)}\sum_{f}^{(k)} W_0(f)\right\}.$$

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The first term within the brackets in (4) is obtained as in §5. To compute the second term, we denote the contribution to the sum from a given $k \in \rho(S_n \times S_n)$ induced by a $\kappa \in S_n(2)$ by

$$\psi(\kappa) = \sum_f^{(k)} W_0(f) = \sum_f^{(k)} \prod_{x_i \in X} \prod_{y_j \in Y} (u_i v_j)^{f(x_i, y_j)}.$$

If κ is generated by a $g \in S_n$ with cycle structure (λ), then κ has cycle structure (2λ). Let U_{pi} and V_{pi} denote the product of the u-variables and v-variables corresponding to the x and y letters occurring in the ith cycle of length $2p$ in κ. Then straightforward computations give us

$$(5) \qquad \psi(\kappa) = \prod_{p=1}^{n} \prod_{q=1}^{n} \prod_{i=1}^{\lambda_p} \prod_{j=1}^{\lambda_q} \{1 + (U_{pi} V_{pi})^{q/(p,q)} (U_{qj} V_{qj})^{p/(p,q)}\}^{(p,q)}$$
$$\times \prod_{p=1}^{n} \prod_{\substack{i=1 \\ i<j}}^{\lambda_p} \prod_{j=1}^{\lambda_p} (1 + U_{pi} V_{pi} U_{pj} V_{pj})^p \prod_{p \text{ even}} \prod_{i=1}^{\lambda_p} (1 + U_{pi}^2 V_{pi}^2)^{p/2}$$
$$\times \prod_{p \text{ odd}} \prod_{i=1}^{\lambda} (1 + U_{pi} V_{pi})(1 + U_{pi}^2 V_{pi}^2)^{(p-1)/2}.$$

Thus we have

THEOREM 3. *The generating function for bicoloured graphs with given bipartitions on n unlabelled nodes of either colour is*

$$\frac{1}{2(n!)^2} \bar{O} \left\{ \sum_{g \in S_n} \sum_{h \in S_n} \phi(g, h) + \sum_{\kappa \in S_n(2)} \psi(\kappa) \right\},$$

where $\phi(g, h)$ *is obtained from* (2) *by putting* $m = n$ *and* $\psi(\kappa)$ *is given by* (5).

Here also we have a symmetric function representation of the generating function. Let

$$\Psi(\lambda) = \sum_{\kappa \in (2\lambda)} \psi(\kappa) \quad \text{and} \quad \Phi((\lambda), (\mu))$$

be the expression obtained from (3) by putting $m = n$. Then it is observed that these are linear functions in monomial symmetric function products of the form $(\pi_n^1)_u (\pi_n^2)_v$ and the generating function reduces to

$$\frac{1}{2(n!)^2} \bar{O} \left\{ \sum_{(\lambda)} \sum_{(\mu)} \Phi((\lambda), (\mu)) + \sum_{(\lambda)} \Psi(\lambda) \right\}.$$

When each symmetric function product is multiplied by the number of terms in it and after the interchange operation of rearranging the factors of the symmetric function products $(\pi_n^1)_u (\pi_n^2)_v$ such that $(\pi_n^1) > (\pi_n^2)$ has been performed, the above expression is reduced to

$$\frac{1}{2(n!)^2} \left\{ \sum_{(\lambda)} \sum_{(\mu)} \Phi'((\lambda), (\mu)) + \sum_{(\lambda)} \Psi'(\lambda) \right\}.$$

In this form of the generating function, the coefficient of the symmetric function product $(\pi_n^1)_u(\pi_n^2)_v$ gives the number of bicoloured graphs with the bipartition $((\pi_n^1), (\pi_n^2))$.

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