ON RING PROPERTIES OF INJECTIVE HULLS[1)]

B. L. Osofsky

(received November 7, 1963)

1. Introduction. Several authors have investigated "rings of quotients" of a given ring R. Johnson showed that if R has zero right singular ideal, then the injective hull of R_R may be made into a right self injective, regular (in the sense of von Neumann) ring (see [7] and [12]). In articles by Utumi [10], Findlay and Lambek [6], and Bourbaki [2], various structures which correspond to sub-modules of the injective hull of R are made into rings in a natural manner. In [8], Lambek points out that in each of these cases the rings constructed are subrings of Utumi's maximal ring of right quotients, which is the maximal rational extension of R in its injective hull. Lambek also shows that Utumi's ring is canonically isomorphic to the bicommutator of the injective hull of R_R if R has 1. It thus appears that a "natural" definition of the injective hull of R_R as a ring extending module multiplication by R has been carried out only in the case that the injective hull is a rational extension of R. (See [12], [10], or [6] for various definitions of this concept.)

The purpose of this note is to study what may happen if one tries to make the entire injective hull of a ring R into a ring extending module multiplication, rather than stopping at

1) This note is taken from the author's doctoral dissertation being written at Rutgers University under the direction of Professor Carl Faith.

The author gratefully acknowledges support from the National Science Foundation.

Canad. Math. Bull. vol. 7, no. 3, July 1964.

https://doi.org/10.4153/CMB-1964-039-3 Published online by Cambridge University Press

Utumi's ring of quotients. The author first exhibits an example which shows that it may be impossible to do so. Then a ring is constructed whose injective hull may be made into a ring, although this ring properly contains Utumi's ring of quotients. Finally some information about such a ring extension is derived.

In what follows, R will denote an associative ring with identity. M_R will signify that M is a unital right R module, and $\hat{M}$ will denote its injective hull. $\hat{M}$ is a maximal essential extension and minimal injective extension of M (see [5]). Much use will be made of the fact that M_R is injective if and only if for every right ideal I of R and every $f \in \mathrm{Hom}_R(I, M)$, there is an $m \in M$ such that $f(x) = mx$ for all $x \in I$ (see [1], or [3] p. 8). Such an element m will be said to induce f.

If $\{a, b, \ldots\} \subseteq M_R$, $|a, b, \ldots)$ will denote the sub-module of M_R, and $\langle a, b, \ldots \rangle$ the subgroup of $(M, +)$ generated by $\{a, b, \ldots\}$. Z will denote the ring of rational integers, and Z_n will denote Z/nZ for $n \in Z$.

2. <u>An example where $\hat{R}$ is not a ring.</u>[2] Let R be the ring

$$\begin{bmatrix} Z_4 & 2Z_4 \\ 0 & Z_4 \end{bmatrix}$$

under usual matrix addition and multiplication.

Let

$$I = \left| \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \right) , \qquad J = \left| \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \right) .$$

2) For further examples where $\hat{R}$ may fail to be a ring, see the author's dissertation. In these other examples, the associative law rather than the distributive law fails.

https://doi.org/10.4153/CMB-1964-039-3 Published online by Cambridge University Press

One readily verifies that the map

$$f\left(\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\right) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

gives an R isomorphism between I and J. Then f extends to an isomorphism $\hat{f}:\hat{I} \to \hat{J}$. One also readily verifies that $\left|\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right.$ is an essential extension of I, so it is contained in some injective hull of I, say $\hat{I}$.

Since $\hat{I}_R$ is injective, the map $f^{-1} \in \mathrm{Hom}_R(J, \hat{I})$ is induced by an element $m \in \hat{I}$. Let $m' = m\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then

$$m'\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = m', \quad m'\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad m'\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0 .$$

Assume $2m' \neq 0$. Since $\hat{I}$ is an essential extension of I, $|(2m')R) \cap I \neq 0$; but

$$|(2m')R) = \langle m'(2R)\rangle = \langle m'\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\rangle = \langle 2m'\rangle ,$$

so that $2m' = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$, and

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 2m'\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 2m' .$$

This contradicts our assumption that $2m' \neq 0$.

Now assume $\hat{R}$ is a ring. Then, from the above,

$$0 = (2m')\hat{f}\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = (m')\hat{f}\left(\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\right) = m'\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} ,$$

a contradiction.

3. $\hat{R}$ a ring properly containing Utumi's ring of quotients.

Let R be an algebra over Z_2 with basis $\{1, x, y, xy\}$ and multiplication defined by:

1 is a two sided identity,

$$0 = x^2 = y^2 = (xy)^2 = yx = x(xy) = y(xy) = (xy)x = (xy)y .$$

R is associative, since any triple product not involving 1 is 0.

We observe that the socle of $R_R = |y) \oplus |xy)$. Hence $\hat{R} = \widehat{|y)} \oplus \widehat{|xy)}$ (see [9]). Moreover, since $\hat{R}$ is unital, $2\hat{R} = 0$.

By direct computation we obtain $\widehat{|y)} = \langle y, m, n, u\rangle$ where

$$mx = y , \quad my = 0 ,$$
$$nx = 0 , \quad ny = y ,$$
$$ux = n , \quad uy = 0 .$$

This may be easily verified by showing that every map from a right ideal of R into $\langle y, m, n, u\rangle$ is induced by some element thereof, and that we indeed have an essential extension of $|y)$.

Since $|xy)$ is isomorphic to $|y)$, $\widehat{|xy)}$ is isomorphic to $\widehat{|y)}$. We then get an injective hull of $|xy)$ by taking $\langle xy, \bar{m}, x, 1-n\rangle$ where

$$\bar{m}x = xy , \quad \bar{m}y = 0 .$$

Then a basis for $\hat{R}$ is $\{1, x, y, xy, m, n, u, \bar{m}\}$. We construct the following multiplication table for $\hat{R}$ as an algebra over Z_2.

	1	x	y	xy	m	n	u	$\bar{m}$
1	1	x	y	xy	m	n	u	$\bar{m}$
x	x	0	xy	0	$\bar{m}$	x	1-n	0
y	y	0	0	0	0	y	m	0
xy	xy	0	0	0	0	xy	$\bar{m}$	0
m	m	y	0	0	0	0	0	0
n	n	0	y	0	m	n	u	0
u	u	n	0	y	0	0	0	m
$\bar{m}$	$\bar{m}$	xy	0	0	0	0	0	0

That this multiplication is associative may be verified by actually computing triple products.[3] The author was unable to find a non-computational method for proving that $\hat{R}$ is a ring.

To prove that $\hat{R}$ is not Utumi's ring of quotients, we use the fact that Utumi's ring consists precisely of those elements of $\hat{R}$ which are annihilated by all $\lambda \in \mathrm{Hom}_R(\hat{R}, \hat{R})$ such that $\lambda(1) = 0$ (see Lambek [8]). It is easily verified that the following induce R homomorphisms of $\hat{R}$:

$$f(m) = y \; ; \;\; f(1) = f(\bar{m}) = f(u) = 0 \; ;$$

$$g(\bar{m}) = y \; ; \;\; g(1) = g(m) = g(u) = 0 \; ;$$

$$h(u) = m ; \;\; h(1) = h(m) = h(\bar{m}) = 0 \; .$$

Since each homomorphism is 0 on the identity and each element of $\langle m, n, u, \bar{m} \rangle$ is not sent into 0 by some one of $\{f, g, h\}$, we conclude that Utumi's ring of quotients is precisely R.

4. <u>$\hat{R}$ is a ring.</u> In this section we generalize a result of Lambek [8] to the case where $\hat{R}$ may be made into a ring, although that ring may properly contain Utumi's ring of quotients.

3) A table of these triple products may be found in the author's doctoral dissertation.

Assume that $(\hat{R}, +, \circ)$ is a ring, where $m \circ r = mr$ for all $m \in \hat{R}$, $r \in R$. Let $\Lambda = \mathrm{Hom}_R(\hat{R}, \hat{R})$. We first prove a standard embedding lemma.

LEMMA 1. $(\hat{R}, +, \circ)$ is isomorphic to a subring of Λ.

Proof. Define a map from $\hat{R}$ to Λ by $m \to \bar{m}$, where $\bar{m}(x) = m \circ x$ for all $m, x \in \hat{R}$. For all $m, n, x \in \hat{R}$,

$$(\overline{m+n})(x) = (m+n)\circ x = m\circ x + n\circ x = \bar{m}(x) + \bar{n}(x) = (\bar{m}+\bar{n})(x) ,$$

$$(\overline{m\circ n})(x) = (m\circ n)\circ x = m\circ(n\circ x) = \bar{m}(\bar{n}(x)) = (\bar{m}\,\bar{n})(x) ,$$

so this map is a ring homomorphism. If $\bar{m} = 0$

$$0 = \bar{m}(1) = m\circ 1 = m1 = m ,$$

so the map is one-to-one.

We will denote the image of R under this map by $\mathcal{R}$, and the image of $\hat{R}$ by $\hat{\mathcal{R}}$.

LEMMA 2. Λ is a unital $\hat{\mathcal{R}}$ module.

Proof. Let e be the identity of Λ. $\bar{1}\,\bar{1} = \overline{1\circ 1} = \bar{1}$, so $\bar{1}$ is an idempotent of Λ. Hence $e - \bar{1}$ is also idempotent, and $(e - \bar{1})(r) = r - r = 0$ for all $r \in R$.

Since Λ is the endomorphism ring of the injective module $\hat{R}_R$, the Jacobson radical of Λ consists precisely of those elements of Λ which annihilate an essential submodule of $\hat{R}_R$ (see [11], Lemma 8). Then $(e - \bar{1})$ is an idempotent in the Jacobson radical, so $e - \bar{1} = 0$. Thus, e actually belongs to $\hat{\mathcal{R}}$.

We wish to show that $\Lambda_{\hat{\mathcal{R}}}$ is an injective module. To do so we need some more information about the structure of Λ. Let $\mathcal{R}^{\perp} = \{\lambda \in \Lambda \mid \lambda(1) = 0\}$.

LEMMA 3. $\Lambda_{\mathcal{R}} = \hat{\mathcal{R}}_{\mathcal{R}} \oplus \mathcal{R}^{\perp}{}_{\mathcal{R}}$.

https://doi.org/10.4153/CMB-1964-039-3 Published online by Cambridge University Press

Proof. Let $\lambda, \mu \in \mathcal{R}^{\perp}$, $r \in R$. $(\lambda \pm \mu)(1) = 0$ so $\lambda \pm \mu \in \mathcal{R}^{\perp}$. $\lambda \bar{r}(1) = \lambda(r) = 0$ so $\lambda \bar{r} \in \mathcal{R}^{\perp}$. Thus $\mathcal{R}^{\perp}$ is an $\mathcal{R}$ module.

Let $\lambda \in \Lambda$, $m = \lambda(1)$. Then $(\lambda - \bar{m})(1) = m - m = 0$, so $\lambda - \bar{m} \in \mathcal{R}^{\perp}$ and $\Lambda = \hat{\mathcal{R}} + \mathcal{R}^{\perp}$. If $x \in \hat{\mathcal{R}} \cap \mathcal{R}^{\perp}$, let $m \in \hat{R}$ be such that $x = \bar{m}$. Then $0 = x(1) = \bar{m}(1) = m$, so $0 = \bar{m} = x$. Thus the sum is direct.

We are now ready to prove the theorem.

THEOREM. Λ is an injective $\hat{\mathcal{R}}$ module.

Proof. Since $\Lambda_{\hat{\mathcal{R}}}$ is unital by Lemma 2, to prove that Λ is injective we need only show that every $f \in \mathrm{Hom}_{\hat{\mathcal{R}}}(\mathcal{J}, \Lambda)$, for $\mathcal{J}$ a right ideal of $\hat{\mathcal{R}}$, is induced by some element $\theta \in \Lambda$.

For $\lambda \in \Lambda$, let $\Pi\lambda$ be the projection of λ onto $\hat{\mathcal{R}}$ with respect to $\mathcal{R}^{\perp}$. Then $\Pi \in \mathrm{Hom}_{\mathcal{R}}(\Lambda, \hat{\mathcal{R}})$. Let $\mathcal{J}$ be a right ideal of $\hat{\mathcal{R}}$, $f \in \mathrm{Hom}_{\hat{\mathcal{R}}}(\mathcal{J}, \Lambda)$. Then $\Pi f \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{J}, \hat{\mathcal{R}})$. Since $\hat{R}_R$ is injective, by the isomorphism of Lemma 1, $\hat{\mathcal{R}}_{\mathcal{R}}$ is injective. Then there exists $\bar{\theta} \in \mathrm{Hom}_{\mathcal{R}}(\hat{\mathcal{R}}, \hat{\mathcal{R}})$ such that $\bar{\theta}$ restricted to $\mathcal{J}$ is Πf. For all $m \in \hat{R}$, define $\theta \in \Lambda$ by $\theta(m) = [\bar{\theta}(\bar{m})](1)$. Then

$$\theta(mr) = [\bar{\theta}(\overline{mr})](1) = [\bar{\theta}(\bar{m})\bar{r}](1) = [\bar{\theta}(\bar{m})](r) = [\bar{\theta}(\bar{m})](1)r = \theta(m)r,$$

so θ is indeed an R homomorphism.

For all $\bar{x} \in \mathcal{J}$,

$$\begin{aligned}(f(\bar{x}) - \theta\bar{x})(1) &= f(\bar{x})(1) - \theta(x) = f(\bar{x})(1) - [\bar{\theta}(\bar{x})](1) \\ &= [f(\bar{x}) - \Pi f(\bar{x})](1) = 0.\end{aligned}$$

Hence $f(\bar{x}) - \theta\bar{x} = u_x \in \mathcal{R}^{\perp}$.

Let m be any element of $\hat{R}$.

$$u_x\bar{m} = (f(\bar{x}) - \theta\bar{x})\bar{m} = f(\bar{x})\bar{m} - (\theta\bar{x})\bar{m} = f(\bar{x}\bar{m}) - \theta\bar{x}\bar{m} = u_{x \circ m} \in \mathcal{R}^{\perp}.$$

https://doi.org/10.4153/CMB-1964-039-3 Published online by Cambridge University Press

Hence $u_x\bar{m}(1) = u_x(m) = 0$, so $u_x = 0$. Thus $f(\bar{x}) = \theta\bar{x}$ for all $\bar{x} \in \mathcal{S}$ and $\Lambda_{\hat{\mathcal{R}}}$ is injective.

COROLLARY. Let R be a ring with 1 such that the injective hull $\hat{R}$ of R_R is a rational extension of R_R. Then $\hat{R}_{\hat{R}}$ is injective.

Proof. In this case, $\hat{R}$ is Utumi's ring of quotients, and it is a ring isomorphic to Λ. Then $\hat{R}_{\hat{R}} = \Lambda_{\hat{R}}$ is injective by the theorem.

This corollary is just (2) $\Longrightarrow$ (6) in the proposition of section 5 of Lambek [8].

The author does not know whether $\hat{R}_{\hat{R}}$ must always be injective if $\hat{R}$ may be made into a ring. In the example of section 3, we do get a self injective ring. For there is only one irreducible left $\hat{R}$ module and one irreducible right $\hat{R}$ module, and they are the duals of each other. Hence $\hat{R}_{\hat{R}}$ is injective (see [4], section 58). Similarly, one may show that Λ_{Λ} is not injective in this example.

REFERENCES

1. R. Baer, Abelian groups that are direct summands of every containing Abelian group, Bull. Amer. Math. Soc., 46(1940), pp. 800-806.

2. N. Bourbaki, Eléments de Mathématique, Vol. 29, Paris, 1961.

3. H. Cartan and S. Eilenberg, Homological Algebra, Princeton University Press, 1956.

4. C. W. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Interscience Vol. 11, New York, 1962.

https://doi.org/10.4153/CMB-1964-039-3 Published online by Cambridge University Press

5. B. Eckmann and A. Schopf, Über injektive Moduln, Arch. Math., 4(1956), pp. 75-78.

6. G. D. Findlay and J. Lambek, A generalized ring of quotients, Can. Math. Bull., 1(1958), pp. 77-85, 155-167.

7. R. E. Johnson, Quotient rings of rings with zero singular ideal, Pacific J. Math., 11(1961), pp. 1385-1395.

8. J. Lambek, On Utumi's ring of quotients, Can. J. Math., 15 (1963), pp. 363-370.

9. K. Morita, Y. Kawada, and H. Tachikawa, On injective modules, Math. Zeit., 68(1957), pp. 217-226.

10. Y. Utumi, On quotient rings, Osaka Math. J., 8(1956), pp. 1-18.

11. Y. Utumi, On a theorem on modular lattices, Proc. Japan Acad., 35(1959), pp. 16-21.

12. E. T. Wong and R. E. Johnson, Self-injective rings, Can. Math. Bull., 2(1959), pp. 167-174.

Rutgers, The State University of New Jersey

https://doi.org/10.4153/CMB-1964-039-3 Published online by Cambridge University Press