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HIGHER MONOTONICITY PROPERTIES OF CERTAIN STURM-LIOUVILLE FUNCTIONS. IV

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1. Introduction. The Sturm-Liouville functions considered in this instalment are real (as are all other quantities discussed here) non-trivial solutions of the differential equation

$$(g(x)y')' + f(x)y = 0, \qquad g(x) > 0. \tag{1.1}$$

Higher monotonicity properties, as defined in § 2, are investigated for a number of sequences (finite or infinite) associated with these functions. One such sequence, discussed in detail later, has the kth term

$$M_k(W; \lambda) = \int_{x_k}^{x_{k+1}} W(x)|y(x)|^\lambda \, dx, \; k = 1, 2, \ldots, \tag{1.2}$$

where the constant $\lambda > -1$ (to assure convergence of each integral), $W(x)$ possesses higher monotonicity properties and, moreover, is such that, again, each integral converges, and $x_1, x_2, \ldots$ is a sequence (finite or infinite) of consecutive zeros of a solution of (1.1), which may or may not be linearly independent of $y(x)$, in the interval of definition of the functions under consideration.

In our earlier work [**7**; **8**; **9**] and that of Vosmanský [**18**; **19**], similar sequences were studied, associated only with the special differential equation (1.1) where $g(x) \equiv 1$. However, the Introductions to [**7**] and [**9**] discuss the background and motivation to the entire series of papers.

In § 2, the definition of higher monotonicity is recalled and some notations introduced. In § 3, new results are provided, extending earlier results (valid when $g(x) \equiv 1$) to the more general $g(x)$ involved here. These sections utilize hypotheses on the coefficient functions $f(x)$, $g(x)$ in (1.1) to deduce higher monotonicity properties of sequences such as (1.2). In § 5, similar inferences are drawn, but now from hypotheses based rather on the functions $p(x) = [y_1(x)]^2 + [y_2(x)]^2$ and $\mathscr{W}(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$, where $y_1(x)$, $y_2(x)$ are appropriate linearly independent solutions of (1.1).

The intervening § 4 provides results on the behaviour of the extrema, and the slopes at the zeros, of solutions of (1.1). These theorems are new even in the special case $g(x) \equiv 1$. A rather special case of either Theorem 4.1 or Theorem 4.3 may serve to suggest the flavour: *If* $g(x) \equiv 1, f(x) > 0$, *and* $f'(x)$ *is completely*

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monotonic, then the squares of the extrema of any solution on (a, ∞) *of* (1.1) *form a completely monotonic sequence.*

The proofs in § 4 are based on appropriate representations in terms of quantities resembling (1.2) to which the results of § 3 can then be applied. A common feature of these representations is that $\lambda = 2$ throughout, a special case of (1.2), etc., not employed before. Previous applications [**7**; **8**; **9**; **18**] involved only $\lambda = 0$, yielding information about the zeros of Sturm-Liouville functions, $\lambda = 1$, concerning the areas under consecutive arches of their graphs, and the limiting case [**9**, § 8] $\lambda \to -1^+$, leading to special sequences.

The limiting case is mentioned only in passing here (§ 6), unlike in [**9**, § 8] where considerable detail is supplied.

The final § 7 consists of applications of our general results to Bessel functions. Similar applications could be made to other special functions, such as the generalized Airy function, the Coulomb wave function, and the confluent hypergeometric function for various ranges of the parameters.

The scope of the present applications is considerably wider than the earlier ones, thanks to the availability in §§ 3–5 of results relating to the differential equation (1.1) so that we are no longer restricted to the special case $g(x) \equiv 1$.

The earlier applications [**7**; **8**; **9**; **18**; **19**] were primarily to Bessel functions of order ν with $|\nu| > \frac{1}{2}$, with a substantial sprinkling [**9**] of those valid for $|\nu| \geqq \frac{1}{3}$ and an occasional result [**9**] for other ν. Here, besides additional such results, we establish for the first time higher monotonicity properties valid for *all* ν (Theorem 7.2). Thus, *the squares of the extrema* (beginning with one with positive abscissa at least $|\nu|$) *of an arbitrary Bessel function of any order form a completely monotonic sequence. So, too, do the* (first) *differences of their consecutive abscissae.*

Finally, we note three corrections to [**9**]: On p. 1248, in line 16, replace N by $N - 1$. On p. 1256, in the right members of (5.17) and (5.18), replace the minus signs by plus signs (the absolute value signs are thus superfluous). On p. 1258, in line 6, replace ν (in the exponent of the first x) by n.

2. Definitions and notations. A function $\varphi(x)$ is said to be *n-times monotonic* (or monotonic of order n) on an interval I if

$$(2.1) \qquad (-1)^m \varphi^{(m)}(x) \geqq 0 \qquad (m = 0, 1, \ldots, n; x \in I).$$

For such a function we write $\varphi(x) \in \mathscr{M}_n(I)$ or $\varphi(x) \in \mathscr{M}_n(a, b)$ in case I is an open interval (a, b). In case strict inequality holds throughout (2.1) we write $\varphi(x) \in \mathscr{M}_n^*(I)$ or $\varphi(x) \in \mathscr{M}_n^*(a, b)$. We say that $\varphi(x)$ is *completely monotonic* on I if (2.1) holds for $n = \infty$. A sequence $\{\mu_k\}_{k=1}^{\infty}$, denoted simply by $\{\mu_k\}$, is said to be *n-times monotonic* if

$$(2.2) \qquad (-1)^m \Delta^m \mu_k \geqq 0 \qquad (m = 0, 1, \ldots, n; k = 0, 1, \ldots).$$

Here $\Delta \mu_k = \mu_{k+1} - \mu_k$, $\Delta^2 \mu_k = \Delta(\Delta \mu_k)$, etc. For such a sequence we write $\{\mu_k\} \in \mathscr{M}_n$. In case strict inequality holds throughout (2.2) we write $\{\mu_k\} \in \mathscr{M}_n^*$;

$\{\mu_k\}$ is called *completely monotonic* if (2.2) holds for $n = \infty$. As usual, we write $[a, b)$ to denote the interval $\{x|\, a \leqq x < b\}$.

3. New basic results. In this section we consider the equation (1.1) with $f(x)$ and $g(x)$ continuous, $g(x) > 0$, for $a < x < \infty$. The change of variable

$$\xi = \int_a^x [g(u)]^{-1}\, du, \tag{3.1}$$

where the integral is assumed convergent, transforms (1.1) into

$$\frac{d^2\eta}{d\xi^2} + \varphi(\xi)\eta = 0, \tag{3.2}$$

where $\eta(\xi) = y(x)$ and $\varphi(\xi) = f(x)g(x)$. Thus, some of the results of [**9**] can be applied to equation (3.2) to give information on solutions of (1.1).

Our first theorem is a generalization of [**9**, Theorem 3.3].

THEOREM 3.1. *Let* $y(x)$, $z(x)$ *be solutions of* (1.1) *on* (a, ∞) *where*

$$0 < \lim_{x\to\infty} f(x)g(x) \leqq \infty,$$

and suppose that $z(x)$ *has consecutive zeros at* $x_1, x_2, \ldots$ *on* $[a, \infty)$. *Suppose also that* $g(x)$, $D_x[f(x)g(x)]$, *and* $W(x)$ *are positive and belong to* $\mathscr{M}_n(a, \infty)$ *for some* $n \geqq 0$. *Then, for fixed* $\lambda > -1$,

$$\left\{ \int_{x_k}^{x_{k+1}} W(x)[g(x)]^{-1}|y(x)|^\lambda dx \right\} \in \mathscr{M}_n^*. \tag{3.3}$$

Remark. Hence, under the hypotheses of the theorem,

$$\left\{ \int_{x_k}^{x_{k+1}} W(x)|y(x)|^\lambda dx \right\} \in \mathscr{M}_n^* \tag{3.4}$$

because (3.3) is still valid when $W(x)$ is replaced by $W(x)g(x)$, since this last function belongs to $\mathscr{M}_n(a, \infty)$.

Proof. For $n \geqq 1$, $g(x)$ is non-increasing. Hence, the mapping (3.1) takes the x-interval (a, ∞) into the ξ-interval $(0, \infty)$. By hypothesis, $0 < \varphi(\infty) \leqq \infty$, since $\varphi(\xi) = f(x)g(x)$. This shows (in case $n \geqq 1$) that $z(x)$ does indeed have an infinite sequence of zeros on $[a, \infty)$. Using the change of variable (3.1) we get

$$\int_{x_k}^{x_{k+1}} W(x)[g(x)]^{-1}|y(x)|^\lambda dx = \int_{\xi_k}^{\xi_{k+1}} W(x(\xi))|\eta(\xi)|^\lambda d\xi,$$

where $\xi_1, \xi_2, \ldots$ are the zeros of $\zeta(\xi)$ corresponding, respectively, to the zeros $x_1, x_2, \ldots$ of $z(x)$. (Here $\zeta(\xi) = z(x)$.) In case $n \geqq 2$ and $x_1 > a$, the present theorem will follow from [**9**, Theorem 3.3] as applied to the equation (3.2), provided we show that

$$\varphi'(\xi) > 0,\ \varphi'(\xi) \in \mathscr{M}_n(0, \infty), \tag{3.5}$$

and that

$$W(x(\xi)) > 0,\ W(x(\xi)) \in \mathscr{M}_n(0, \infty). \tag{3.6}$$

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Now, $\varphi'(\xi) = D_x[f(x)g(x)]x'(\xi) = g(x)D_x[f(x)g(x)] > 0$. But, $g(x) \in \mathcal{M}_n(a, \infty)$, so that a slight modification of [**9**, Lemma 2.2] (in which $p'(x) \leqq 0$ replaces $p'(x) < 0$ and $\geqq$ replaces $>$ in (2.7)), implies that $x'(\xi) \in \mathcal{M}_n(0, \infty)$. Hence, in view of [**9**, Lemma 2.1], our hypotheses on $W(x)$ show that $W(x(\xi)) \in \mathcal{M}_n(0, \infty)$, and (3.6) holds. Since $D_x[\varphi(\xi)]$, considered as a function of x, belongs to $\mathcal{M}_n(a, \infty)$, and $x'(\xi) \in \mathcal{M}_n(0, \infty)$, [**9**, Lemma 2.1] shows that $D_\xi[\varphi(\xi)] \in \mathcal{M}_n(0, \infty)$. Hence, (3.5) holds and the proof of Theorem 3.1 is complete, in case $n \geqq 2$ and $x_1 > a$.

It remains to prove the theorem in the cases $n = 0$, $n = 1$ and (for all n) in the case where $x_1 = a$. It is clear from the above discussion that it will be sufficient to show that these extensions can be made in the case of [**9**, Theorem 3.3].

The case $n = 0$ is obvious. For the case $n = 1$ we use results of P. Hartman [**4**, Theorem 18.1_n in case $f(\infty) < \infty$, Theorem 20.1_n when $f(\infty) = \infty$; both with $n = 0$] to show that under the hypotheses of [**9**, Theorem 3.3] the equation $y'' + f(x)y = 0$ has linearly independent solutions $y_1(x)$ and $y_2(x)$ on (a, ∞) which are such that $p(x) = [y_1(x)]^2 + [y_2(x)]^2$ satisfies $p(x) > 0$, $p'(x) \leqq 0$.

This leads, as in the proof of [**9**, Theorem 3.1 and Remark (i), p. 1249], to the inequalities $M_k > 0$, $\Delta M_k \leqq 0$ $(k = 1, 2, \ldots)$, but we need to show that in fact $\Delta M_k < 0$.

In the notation of [**9**, p. 1245],

$$\Delta M_k = \int_{t_k}^{t_{k+1}} \{\Delta_\pi\{W[x(t)][x'(t)]^{1+\frac{1}{2}\lambda}\}\}|u(t)|^\lambda dt$$

$$= \int_{t_k}^{t_{k+1}} W[x(t+\pi)]\Delta_\pi\{[x'(t)]^{1+\frac{1}{2}\lambda}\}|u(t)|^\lambda dt$$

$$+ \int_{t_k}^{t_{k+1}} [x'(t)]^{1+\frac{1}{2}\lambda}\Delta_\pi\{W[x(t)]\}|u(t)|^\lambda dt.$$

It is clear from the proof of [**9**, Theorem 3.1] that both integrands here are non-positive; thus $\Delta M_k = 0$ would imply $\Delta_\pi\{[x'(t)]^{1+\frac{1}{2}\lambda}\} = 0$ for $t_k < t < t_{k+1}$ and so $\Delta_\pi[x'(t)] = 0$ for $t_k < t < t_{k+1}$. This would entail then that

$$\Delta^2 x_k = \Delta M_k(1; 0) = \int_{t_k}^{t_{k+1}} \Delta_\pi[x'(t)]dt = 0,$$

which is a contradiction to the result $\Delta^2 x_k < 0$ arising from the Sturm comparison theorem. Thus $\Delta M_k < 0$ $(k = 1, 2, \ldots)$ and the proof for the case $n = 1$ is complete.

The case $n = 1$ for the modified form of [**9**, Theorem 3.3] noted in its last sentence has already been discussed [**9**, Remark (iii), p. 1250].

Finally, we establish that [**9**, Theorem 3.3] can be extended to include a possible end-point zero $x_1 = a$. As observed in [**9**, Remark (ii), p. 1250] an end-point zero can be included, as, e.g., in [**9**, Theorem 5.4, (5.10)], since the function $p(x)$ is bounded away from zero as $x \to a^+$, being a positive non-increasing function, in case $n \geqq 1$. When $n = 0$ the result is trivial.

In case $x_1 = a$ the function $W(x)$ must be chosen in such a way that the integrals occurring in the statement of the theorem exist.

We have the following generalization of [**9**, Theorem 3.4] (which corresponds to the case $g(x) \equiv 1$, $n \geqq 4$, $x_1' > a$).

THEOREM 3.2. *Let $y(x)$, $z(x)$ be solutions of* (1.1) *on* (a, ∞) *where* $f(x)$ *and* $D_x[f(x)g(x)]$ *are positive and where* $g(x)$ *and* $D_x[f(x)g(x)]$ *belong to* $\mathcal{M}_n(a, \infty)$ *for some* $n \geqq 2$. *Let* $z'(x)$ *have consecutive zeros* $x_1', x_2', \ldots$ *on* $[a, \infty)$. *Let* $W(x) > 0$ *and* $W(x) \in \mathcal{M}_{n-2}(a, \infty)$. *Then, for fixed* $\lambda > -1$,

$$\left\{ \int_{x_k'}^{x'_{k+1}} W(x)[g(x)]^{-1}|[f(x)]^{-\frac{1}{2}}[g(x)]^{\frac{1}{2}}y'(x)|^{\lambda}dx \right\} \in \mathcal{M}_{n-2}^*.$$

Proof. The change of variable (3.1) yields, as in the proof of Theorem 3.1,

$$\int_{x_k'}^{x'_{k+1}} W(x)[g(x)]^{-1}|[f(x)]^{-\frac{1}{2}}[g(x)]^{\frac{1}{2}}y'(x)|^{\lambda}dx$$
$$= \int_{\xi_k'}^{\xi'_{k+1}} W(x(\xi))|[\varphi(\xi)]^{-\frac{1}{2}}\eta'(\xi)|^{\lambda}d\xi,$$

where $\xi_1', \xi_2', \ldots$ are the zeros of $\zeta'(\xi) = g(x)z'(x)$ corresponding to the zeros $x_1', x_2', \ldots$ of $z'(x)$. Now, $\varphi'(\xi) = D_x[f(x)g(x)]g(x) > 0$ for $0 < \xi < \infty$ and, as in the proof of Theorem 3.1, (3.5) and (3.6) hold with n replaced by $n - 2$. The theorem follows on applying [**9**, Theorem 3.4] to solutions of the equation (3.2). (We require an extended form of [**9**, Theorem 3.4] in which a possible end-point zero and the cases $n = 2, 3$ are included. This can be established in much the same way as was the extension of [**9**, Theorem 3.3].)

Again, the existence of an infinite sequence of zeros is a consequence of the hypotheses on $f(x)$ and $g(x)$.

We have a further result with a conclusion similar to that of Theorem 3.2 but with different hypotheses.

THEOREM 3.3. *Let* $y(x)$ *and* $z(x)$ *be solutions of* (1.1) *on* (a, ∞) *with* $f(x) > 0$. *Suppose that, for some* $n \geqq 2$, $f'(x) \in \mathcal{M}_n(a, \infty)$ *and that* $W(x)$ *and* $D_x\{f(x)/g(x)\}$ *belong to* $\mathcal{M}_{n-2}(a, \infty)$. *Suppose also that* $W(x) > 0$ *and that at least one of the two functions* $f'(x)$ *and* $D_x[f(x)/g(x)]$ *is positive on* (a, ∞). *Suppose that* $z'(x)$ *has consecutive zeros* $x_1', x_2', \ldots$ *on* $[a, \infty)$. *Then, for fixed* $\lambda > -1$,

$$\left\{ \int_{x_k'}^{x'_{k+1}} W(x)|y'(x)[f(x)]^{-\frac{1}{2}}g(x)|^{\lambda}dx \right\} \in \mathcal{M}_{n-2}^*.$$

Proof. The function $u(x) = y'(x)[f(x)]^{-\frac{1}{2}}g(x)$ satisfies

$$u'' + F(x)u = 0,$$

where

$$F(x) = -\frac{3}{4}\left[\frac{f'(x)}{f(x)}\right]^2 + \frac{1}{2}\frac{f''(x)}{f(x)} + \frac{f(x)}{g(x)}.$$

Thus, the present theorem will follow from Theorem 3.1 provided we show that

$$0 < \lim_{x\to\infty} F(x) \leqq \infty, \tag{3.7}$$

that $F'(x) > 0$, and that $F'(x) \in \mathscr{M}_{n-2}(a, \infty)$.

It follows from work of Vosmanský [**18**, p. 108] that

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} [f(x)/g(x)].$$

This gives (3.7), since $f(x)/g(x)$ is non-decreasing and positive.

We have

$$F'(x) = -2\frac{f'(x)f''(x)}{[f(x)]^2} + \frac{1}{2}\left[\frac{f'''(x)}{f(x)}\right]^3 + \frac{3}{2}\left[\frac{f'(x)}{f(x)}\right]^3 + D_x\left[\frac{f(x)}{g(x)}\right]. \tag{3.8}$$

From [**9**, Lemma 2.1], we see that $[1/f(x)] \in \mathscr{M}_n(a, \infty)$. Moreover, $-f''(x) \in \mathscr{M}_{n-1}(a, \infty)$ and $f'''(x) \in \mathscr{M}_{n-2}(a, \infty)$. Since the set $\mathscr{M}_{n-2}(a, \infty)$ is closed under pointwise multiplication of functions, each term on the right-hand-side of (3.8) belongs to $\mathscr{M}_{n-2}(a, \infty)$; hence so does $F'(x)$. The positivity of $F'(x)$ follows also from (3.8), since it has been assumed that at least one of the two functions $f'(x)$ and $D_x[f(x)/g(x)]$ is positive on (a, ∞). This completes the proof of Theorem 3.3.

Remark. It is possible to prove results analogous to those of this section concerning solutions $g(x)y'(x)$ and $g(x)z'(x)$ of the equation

$$\left(\frac{1}{f(x)}u'\right)' + \frac{1}{g(x)}u = 0, \qquad f(x) > 0, \qquad g(x) > 0. \tag{3.9}$$

4. Some analogues for higher monotonicity of the Sonin-Butlewski-Pólya theorem. Sonin's theorem (see [**10**, p. 168] and [**20**, p. 518]; for an extension to non-linear equations cf. [**1**]) states that if $z(x)$ is a solution of $y'' + f(x)y = 0$, where $f(x)$ is a positive continuous function, then the successive maxima of $[z(x)]^2$ form a decreasing or increasing sequence according as $f(x)$ is increasing or decreasing. An extension, due independently to Butlewski [**2**, Théorème I, p. 42] and to G. Pólya [**16**, footnote, p. 166], concerns the more general equation (1.1) with $f(x)$ and $g(x)$ continuous. Their result says that if $z(x)$ is a solution of (1.1), the relative maxima of $[z(x)]^2$ form an increasing or decreasing sequence according as $f(x)g(x)$ is decreasing or increasing when $f(x) > 0$ and $g(x) > 0$.

The hypotheses just suggested are somewhat lighter than those required in [**16**, footnote, p. 166]. That they are adequate can be seen by using transformation (3.1) to reduce equation (1.1) to the form (3.2) and then applying Watson's [**20**, p. 518] or Makai's [**10**, p. 168] approach.

Strictly speaking, Watson proves only weak inequality ($\geqq$), and nothing stronger is valid in general, although his result asserts strict inequality ($>$). However, in the present circumstances, his proof can be modified to yield strict inequality by noting that now an additional hypothesis is satisfied. In his notation, it can be expressed as "$I_1(x) > I_2(x)$ for some x in each open interval $(a, a + \epsilon)$, $\epsilon > 0$." This suffices to assure strict inequality in his conclusions.

In this section we consider the higher monotonicity behaviour of the sequences $\{z^2(x_k')\}$ and $\{[g(x_k)z'(x_k)]^2\}$ where $z(x)$ is a solution of (1.1) on (a, ∞) and $z(x)$ and $z'(x)$ have sequences of successive zeros $x_1, x_2, \ldots$ and $x_1', x_2', \ldots$ on $[a, \infty)$, respectively. The inclusion of an end-point zero or extremum, i.e., permitting $x_1 = a$ or $x_1' = a$, is an extension of both the Sonin and Sonin-Butlewski-Pólya theorems even in the case of ordinary monotonicity.

Theorems 4.1 and 4.3 are partial extensions to higher monotonicity corresponding to the hypothesis $f(x)g(x)$ increasing. Theorem 4.2 corresponds to the assumption that $f(x)g(x)$ is decreasing.

We need two lemmas.

LEMMA 4.1. *Let $z(x)$ be a solution of* (1.1), *for $a < x \leqq b$, with both $f(x)$ and $g(x)$ positive and differentiable for $a < x \leqq b$, and $D_x\{[f(x)g(x)]^{-1}\}$ integrable for $a < x \leqq b$. Suppose $z'(b) = 0$. Then, for $a < x < b$,*

$$z^2(b) - z^2(x) = g(x)[f(x)]^{-1}[z'(x)]^2 + \int_x^b [g(t)z'(t)]^2 D_t\{[f(t)g(t)]^{-1}\}dt. \tag{4.1}$$

If, in addition, $g(x)[f(x)]^{-1}[z'(x)]^2 \to l$ as $x \to a^+$, and if $z(a^+)$ exists, then

$$z^2(b) - z^2(a^+) = l + \int_a^b [g(t)z'(t)]^2 D_t\{[f(t)g(t)]^{-1}\}dt. \tag{4.2}$$

Proof. The differential equation (1.1) implies that

$$2g(x)z'(x)[g(x)z'(x)]' + g(x)f(x)[2z(x)z'(x)] = 0;$$

i.e.,

$$D_t\{[g(t)z'(t)]^2\} + f(t)g(t)D_t\{[z(t)]^2\} = 0.$$

Dividing by $f(t)g(t)$ and using integration by parts between x and b yields (4.1). The remaining assertions follow at once.

The other lemma is due essentially to Butlewski [**2**, p. 41]; see also [**17**]. In the special case $g(x) \equiv 1$ it was proved by Wiman [**21**].

LEMMA 4.2. *Let $z(x)$ be a solution of* (1.1), *for $a < x \leqq b$, where $f(x)$ and $g(x)$ are differentiable and $D_x\{f(x)g(x)\}$ is integrable on $a \leqq x \leqq b$. Let $z(b) = 0$. Then if $g(x)z'(x)$ is continuous at a^+ and if $f(x)g(x)z^2(x) \to 0$, $x \to a^+$,*

$$[g(b)z'(b)]^2 - [g(a)z'(a)]^2 = \int_a^b [z(t)]^2 D_t\{f(t)g(t)\}dt.$$

The first of the principal results of this section incorporates an analogue of the Sonin-Butlewski-Pólya theorem.

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THEOREM 4.1. *Let $y(x)$ and $z(x)$ be linearly independent solutions of* (1.1) *on* (a, ∞) *where*

$$0 < \lim_{x\to\infty} f(x)g(x) \leqq \infty, \tag{4.3}$$

$x_1' > a$, $x_1 \geqq a$, *and for some* $n \geqq 0$, $g(x)$ *and* $D_x[f(x)g(x)]$ *are positive and belong to* $\mathscr{M}_n(a, \infty)$. *If* $x_1 = a$, *let the hypotheses of Lemma* 4.2 *hold on* $[a, x_2]$. *Then*

$$\{[g(x_{k+1})z'(x_{k+1})]^2 - [g(x_k)z'(x_k)]^2\} \in \mathscr{M}_n^*, \tag{4.4}$$

and if $y(x)$ *is continuous at* x_1^+,

$$\{[y(x_{k+1})]^{-2} - [y(x_k)]^{-2}\} \in \mathscr{M}_n^*. \tag{4.5}$$

If, in addition, $f(x) > 0$ *on* (a, ∞) *and* $n \geqq 1$, *then*

$$\{z^2(x_k')\} \in \mathscr{M}_{n-1}^*. \tag{4.6}$$

Proof. For $n \geqq 1$, Lemma 4.2 asserts that

$$[g(x_{k+1})z'(x_{k+1})]^2 - [g(x_k)z'(x_k)]^2 = \int_{x_k}^{x_{k+1}} [z(t)]^2 D_t\{f(t)g(t)\}dt.$$

Hence, (4.4) follows from Theorem 3.1 with $y(x) = z(x)$, $\lambda = 2$, and

$$W(x) = g(x)D_x[f(x)g(x)].$$

Abel's formula for the Wronskian shows that

$$y(x)z'(x) - y'(x)z(x) = c/g(x),$$

where c is a non-zero constant. Hence, for $k = 1, 2, \ldots$, $[g(x_k)z'(x_k)]^2 = c^2[y(x_k)]^{-2}$ and (4.5) follows from (4.4).

In case $n = 1$, (4.6) is obvious. For $n \geqq 2$, Lemma 4.1 implies

$$-[z^2(x_{k+1}') - z^2(x_k')] = -\int_{x_k'}^{x_{k+1}'} [g(t)z'(t)]^2 D_t\{[f(t)g(t)]^{-1}\}dt,$$

the number l of Lemma 4.1 being 0 here. Thus,

$$-[z^2(x_{k+1}') - z^2(x_k')] = \int_{x_k'}^{x_{k+1}'} \frac{1}{f(t)g(t)} D_t[f(t)g(t)]|[f(t)]^{-\frac{1}{2}}[g(t)]^{\frac{1}{2}}z'(t)|^2 dt.$$

Hence, Theorem 3.2 implies that

$$\{-[z^2(x_{k+1}') - z^2(x_k')]\} \in \mathscr{M}_{n-2}^*,$$

so that (4.6) holds, provided we can show that

$$\frac{1}{f(x)} D_x[f(x)g(x)] \in \mathscr{M}_{n-2}(a, \infty).$$

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However, because of the hypotheses on $g(x)$ and $f(x)g(x)$ and because of [**9**, Lemma 2.1] we even have

$$\frac{1}{f(x)} D_x\{f(x)g(x)\} \in \mathscr{M}_n(a, \infty).$$

This completes the proof of Theorem 4.1, for $n \geqq 1$.

When $n = 0$, the result can be reduced to Makai's [**10**, p. 168] or Watson's version [**20**, p. 518] of Sonin's theorem by applying a transformation of the type (3.1) to the equation (3.9) satisfied by $g(x)z'(x)$.

Remarks. (i) In case $z'(a^+) = 0$ and *either*

$$(4.7) \qquad 0 < \lim_{x \to a^+} [f(x)/g(x)] \leqq \infty,$$

or

$$(4.8) \qquad 0 < \lim_{x \to a^+} D_x[f(x)/g(x)] \leqq \infty, \qquad \lim_{x \to a^+} z'(x)z''(x) = 0,$$

we could include the case $x_1' = a$ in the statement of our theorem, because the number l in Lemma 4.1 would exist and equal zero, from l'Hospital's rule.

(ii) When $n = 0$, (4.5) is included in [**17**].

An application of Theorem 4.1 to solutions $g(x)y'(x)$ and $g(x)z'(x)$ of the equation (3.9) gives the following analogue of the Sonin-Butlewski-Pólya theorem for the case in which $f(x)g(x)$ is decreasing.

THEOREM 4.2. *Let $y(x), z(x)$ be linearly independent solutions of* (1.1) *on* (a, ∞) *where $1/f(x)$ and $D_x\{[f(x)g(x)]^{-1}\}$ are positive and belong to $\mathscr{M}_n(a, \infty)$ for some $n \geqq 0$, $x_1 > a$ and $x_1' > a$. Then*

$$(4.9) \qquad \{z^2(x_{k+1}') - z^2(x_k')\} \in \mathscr{M}_n^*.$$

If $n \geqq 1$, then

$$(4.10) \qquad \{[g(x_k)z'(x_k)]^2\} \in \mathscr{M}_{n-1}^*,$$

and

$$(4.11) \qquad \{[y(x_k)]^{-2}\} \in \mathscr{M}_{n-1}^*.$$

Remark. A possible zero of $z'(x)$ occurring at the end-point $x = a$ may be included in Theorem 4.2 under the same circumstances as those given in the remark following Theorem 4.1. In analogous circumstances x_1 could equal 2.

THEOREM 4.3. *Let $z(x)$ be a solution of* (1.1) *on* (a, ∞) *with $f(x) > 0$. For some $n \geqq 2$ suppose that $f'(x) \in \mathscr{M}_n(a, \infty)$, $1/g(x) \in \mathscr{M}_{n-1}(a, \infty)$ and*

$$D_x[f(x)/g(x)] \in \mathscr{M}_{n-2}(a, \infty).$$

Suppose, further, that at least one of the two functions $f'(x)$ and $D_x[f(x)/g(x)]$ is positive on (a, ∞), and $x_1' > a$. Then

$$(4.12) \qquad \{z^2(x_k')\} \in \mathscr{M}_{n-1}^*.$$

Proof. Using Lemma 4.1, we have

$$-[z^2(x_{k+1}') - z^2(x_k')] = -\int_{x_k'}^{x_{k+1}'} [g(t)z'(t)]^2 D_t\{[f(t)g(t)]^{-1}\}dt$$
$$= \int_{x_k'}^{x_{k+1}'} W(t)|z'(t)[f(t)]^{-\frac{1}{2}}g(t)|^2 dt,$$

where $W(x) = -f(x)D_x\{[f(x)g(x)]^{-1}\}$. The result follows from Theorem 3.3 once it is shown that $W(x) > 0$ and $W(x) \in \mathscr{M}_{n-2}(a, \infty)$. A simple calculation shows that

$$W(x) = \frac{f'(x)}{f(x)}\frac{1}{g(x)} + \frac{g'(x)}{[g(x)]^2}. \tag{4.13}$$

Because of [**9**, Lemma 2.1], we have $f'(x)/f(x) \in \mathscr{M}_n(a, \infty)$ and so the hypothesis on $g(x)$ shows that the first term on the right-hand-side of (4.13) belongs to $\mathscr{M}_{n-1}(a, \infty)$. Also,

$$g'(x)[g(x)]^{-2} = -D_x\{[g(x)]^{-1}\} \in \mathscr{M}_{n-2}(a, \infty),$$

and so $W(x) \in \mathscr{M}_{n-2}(a, \infty)$.

It remains to show that $W(x) > 0$. If $f'(x) > 0$, this is clear from (4.13). In case $D_x[f(x)/g(x)] > 0$, the positivity of $W(x)$ follows from

$$f(x)W(x) = D_x[f(x)/g(x)] + 2f(x)g'(x)[g(x)]^{-2}.$$

This completes the proof of Theorem 4.3.

Remark. As in the case of Theorems 4.1 and 4.2, the assertion of Theorem 4.3 may be extended to include an end point zero of $z'(x)$ occurring at the point $x = a$, provided we have either (4.7) or (4.8). In particular, if $n \geqq 3$ in the present theorem, one of these conditions is satisfied. To see this, we recall that $D_x[f(x)/g(x)]$ is non-increasing. Hence (4.8) holds unless $f(x)$ is a constant multiple of $g(x)$. In the latter case, (4.7) holds.

5. Hypotheses involving a specific pair of solutions. Here $y_1(x)$, $y_2(x)$ are linearly independent solutions of (1.1) on an open interval (a, b), not necessarily (a, ∞), and as usual, $p(x) = [y_1(x)]^2 + [y_2(x)]^2$, where $p^{(N)}(x)$ exists in (a, b) for some positive integer N. The function $y(x)$ is an arbitrary non-trivial solution of (1.1) on (a, b) and $\{x_1, x_2, \ldots\}$ denotes any finite or infinite increasing sequence of consecutive zeros on (a, b) of a nontrivial solution $z(x)$ of (1.1). We put

$$\mathscr{W}(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x), \tag{5.1}$$

where the solutions $y_1(x)$, $y_2(x)$ are normalized so that $\mathscr{W}(x) > 0$.

In §§ 3 and 4 of the present work, the results were inferred from hypotheses on the coefficient functions $f(x)$ and $g(x)$. In this section, we give some corresponding results in which hypotheses are placed instead on an appropriate pair of solution functions $y_1(x)$ and $y_2(x)$.

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THEOREM 5.1. *Suppose that for some positive integer N there exists a function $W(x)$ and a pair of linearly independent solutions $y_1(x)$, $y_2(x)$ such that*

$$(5.2)\qquad \begin{cases} (-1)^n p^{(n)}(x) > 0 & (n = 0, 1), \\ (-1)^n p^{(n)}(x) \geqq 0 & (n = 2, \ldots, N), \end{cases}$$

$$(5.3)\qquad W(x) > 0,\ (-1)^n W^{(n)}(x) \geqq 0 \qquad (n = 1, \ldots, N),$$

and

$$(5.4)\qquad (-1)^n D_x^n\{[\mathscr{W}(x)]^{-1}\} \geqq 0 \qquad (n = 0, 1, \ldots, N).$$

Then, for fixed $\lambda > -1$,

$$(5.5)\qquad (-1)^n \Delta^n \left[\int_{x_k}^{x_{k+1}} W(x)\mathscr{W}(x)|y(x)|^\lambda dx \right] > 0 \qquad (n = 0, 1, \ldots, N;\ k = 1, 2, \ldots).$$

All of the above remains true if the factor $(-1)^n$ *is deleted simultaneously from* (5.2), (5.3), (5.4) *and* (5.5).

Proof. Abel's formula for the Wronskian shows that $\mathscr{W}(x) = c[g(x)]^{-1}$, where c is a positive constant. As in the proof of Theorem 3.1, we make the change of variable (3.1)† so that equation (1.1) becomes (3.2) and we find that $W(x(\xi)) \in \mathscr{M}_N(\xi(a), \xi(b))$ and $x'(\xi) \in \mathscr{M}_N(\xi(a), \xi(b))$. If we put $\eta_1(\xi) = y_1(x)$, $\eta_2(\xi) = y_2(x)$ we find that $\pi(\xi) = \eta_1^2(\xi) + \eta_2^2(\xi)$ is an N-times monotonic function of ξ on $(\xi(a), \xi(b))$. Also $\pi(\xi) > 0$ and $\pi'(\xi) = p'(x)g(x) > 0$. This means that [**9**, Theorem 3.1] may be applied to solutions of equation (3.2). Thus

$$\int_{x_k}^{x_{k+1}} W(x)\mathscr{W}(x)|y(x)|^\lambda dx = c\int_{\xi_k}^{\xi_{k+1}} W(x(\xi))|\eta(\xi)|^\lambda d\xi,$$

where $\xi_1, \xi_2, \ldots$ are the zeros of the solution $\zeta(\xi)$ $(=z(x))$ of (3.2). Applying [**9**, Theorem 3.1] to this last expression yields the desired result.

The last sentence follows on making obvious changes in the above proof.

THEOREM 5.2. *Suppose that the solutions $y(x)$ and $z(x)$ of* (1.1) *are linearly independent and that, for some positive integer N, there exists a pair of linearly independent solutions $y_1(x)$, $y_2(x)$ for which*

$$(5.6)\qquad \begin{cases} (-1)^n p^{(n)}(x) > 0 & (n = 0, 1), \\ (-1)^n p^{(n)}(x) \geqq 0 & (n = 2, \ldots, N), \end{cases}$$

$$(5.7)\qquad (-1)^n D_x^n[1/\mathscr{W}(x)] \geqq 0 \qquad (n = 0, 1, \ldots, N),$$

and, for $a < x < b$,

$$(5.8)\qquad W(x) > 0,\ (-1)^n W^{(n)}(x) \geqq 0 \qquad (n = 1, \ldots, N).$$

Then, for any $\alpha > 0$,

$$(5.9)\qquad (-1)^n \Delta^n W(x_k)|y(x_k)|^\alpha > 0 \qquad (n = 0, 1, \ldots, N;\ k = 1, 2, \ldots)$$

†Increasing a slightly (here and elsewhere), if necessary to assure convergence in (3.1).

and

(5.10) $(-1)^n\Delta^n W(x_k)|\mathscr{W}(x_k)/z'(x_k)|^\alpha > 0$ $(n = 0, 1, \ldots, N; k = 1, 2, \ldots)$.

All of the above remains true if the factor $(-1)^n$ *is deleted simultaneously from* (5.6), (5.7), (5.8), (5.9) *and* (5.10).

Proof. Making the change of variable (3.1) and using the same notation as in the proof of Theorem 3.1 we have

$$W(x)|y(x)|^\alpha = V(\xi)|\eta(\xi)|^\alpha$$

where $V(\xi) = W[x(\xi)]$ is an N-times monotonic function of ξ. The differential equation now has the form (3.2). The further changes of variable [**7**, Lemma 2.3]

$$\xi'(t) = \pi(\xi), \qquad \eta(\xi) = [\pi(\xi)]^{\frac{1}{2}}u(t)$$

reduce this to the form $u''(t) + u(t) = 0$. Thus we have

$$W(x)|y(x)|^\alpha = V(\xi)[\pi(\xi)]^{\alpha/2}|u(t+B)|^\alpha,$$
$$W(x_k)|y(x_k)|^\alpha = CV[\xi(t_k)]\{\pi[\xi(t_k)]\}^{\alpha/2},$$

where C is a constant independent of k. As in the proof of Theorem 5.1, we have $\pi(\xi) > 0$, $\pi'(\xi) < 0$, $\pi(\xi) \in \mathscr{M}_N(\xi(a), \xi(b))$, $V(\xi) > 0$, $V(\xi) \in \mathscr{M}_N(\xi(a), \xi(b))$. Hence, if we write $q(\xi) = V(\xi)[\pi(\xi)]^{\alpha/2}$, we get, using [**9**, Lemma 2.2],

(5.11) $$(-1)^n D_t^n[q(\xi)] > 0 \qquad (n = 0, 1, \ldots, N).$$

We have for $n = 0, 1, \ldots, N$,

$$(-1)^n\Delta^n W(x_k)|y(x_k)|^\alpha = (-1)^n C\Delta_\pi^n q(\xi(t_k))$$
$$= (-1)^n C D_t^n q[\xi(t_k + \theta n\pi)], \qquad 0 < \theta = \theta(t) < 1,$$

on using a mean value theorem for higher derivatives and differences [**12**, no. 98, p. 55]. Hence, in view of (5.11), we get the desired (5.9).

The result (5.10) follows easily from (5.9) on noticing that the Wronksian of $y(x)$ and $z(x)$ is a constant non-zero multiple of $\mathscr{W}(x)$.

Again, the last sentence in the statement of the theorem follows on making obvious changes in the above proof.

COROLLARY 5.1. *Under the hypotheses of Theorem* 5.2 *we have*

(5.12) $(-1)^n\Delta^n \log|y(x_k)| \geqq 0$ $\qquad (n = 1, 2, \ldots, N; k = 1, 2, \ldots)$,

and

(5.13) $(-1)^n\Delta^n \log|\mathscr{W}(x_k)/z'(x_k)| \geqq 0$ $\qquad (n = 1, 2, \ldots, N; k = 1, 2, \ldots)$.

These results remain true if the factor $(-1)^n$ *is deleted simultaneously from* (5.6), (5.7), (5.8), (5.12) *and* (5.13).

Proof. From Theorem 5.2 we have, for each $\alpha > 0$,

$$(-1)^n \Delta^n \frac{\{|y(x_k)|^\alpha - 1\}}{\alpha} > 0 \qquad (n = 1, 2, \ldots, N; k = 1, 2, \ldots).$$

Taking the limit as $\alpha \to 0^+$, we get (5.12). Similarly, (5.13) follows from (5.10).

The inequalities (5.12) and (5.13) can be shown to be strict in certain cases by employing results of [**6**] or of [**11**].

The following corollary may be proved in the same way as was Corollary 5.1. The remarks on the possibility of strict inequality apply again.

COROLLARY 5.2. *If the hypotheses of Theorem* 5.2 *hold and if, in addition,*

$$(5.14) \qquad (-1)^n D_x^n\{[W(x)]^\alpha\} \geqq 0 \qquad (a < x < b; n = 1, 2, \ldots, N)$$

for each $\alpha > 0$, *then*

$$(5.15) \quad (-1)^n \Delta^n \log\{W(x_k)|y(x_k)|\} \geqq 0 \qquad (n = 1, 2, \ldots, N; k = 1, 2, \ldots)$$

and

$$(5.16) \quad (-1)^n \Delta^n \log|W(x_k)\mathscr{W}(x_k)/z'(x_k)| \geqq 0 \qquad (n = 1, 2, \ldots, N; k = 1, 2, \ldots).$$

The results remain true if the factor $(-1)^n$ *is deleted simultaneously from* (5.6), (5.7), (5.8), (5.14), (5.15) *and* (5.16).

6. The case $\lambda \to -1^+$. The sequence with kth term

$$(6.1) \qquad \lim_{\lambda \to -1^+} (1 + \lambda) M_k(W; \lambda), \qquad k = 1, 2, \ldots,$$

possesses essentially the same higher monotonicity properties as does the sequence $\{M_k(W; \lambda)\}$, $k = 1, 2, \ldots$. This observation is explored in detail in [**9**, § 8] where specific sequences arising thus are enumerated. The same basic lemmas and approach can be applied in the present more general setting.

Without going into detail or offering particular examples in terms, say, of Bessel functions, we remark only that the following sequences arise on applying [**9**, Lemmas 8.1 and 8.2] to the respective theorems indicated below. Each sequence displayed will possess higher monotonicity properties as stated essentially in the theorem referred to, *provided that the hypotheses of that theorem are satisfied. Moreover, in each case, the* $\{x_k\}$ *and* $\{x_k'\}$ *are the zeros of* $y(x)$ *and* $y'(x)$, *respectively.*

(i) From Theorem 3.1:

$$(6.2) \qquad \left\{\left|\frac{W(x_k)}{g(x_k)y'(x_k)}\right|\right\}, \qquad k = 1, 2, \ldots.$$

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(ii) From Theorem 3.2:

$$\left\{\left|\frac{W(x_k')}{[f(x_k')g(x_k')]^{\frac{1}{4}}y(x_k')}\right|\right\}, \qquad k = 1, 2, \ldots . \tag{6.3}$$

(iii) From Theorem 3.3:

$$\left\{\left|\frac{W(x_k')}{[f(x_k')]^{\frac{1}{4}}y(x_k')}\right|\right\}, \qquad k = 1, 2, \ldots . \tag{6.4}$$

7. Applications to Bessel (cylinder) functions. Throughout this section we write, as is customary,

$$\mathscr{C}_\nu(x) = AJ_\nu(x) + BY_\nu(x),$$

where $J_\nu(x)$, $Y_\nu(x)$ are the usual Bessel functions of first and second kind, respectively, and A, B are real constants, independent also of the order ν. This last is relevant to Theorem 7.3 in which different orders are present. The successive positive zeros of $\mathscr{C}_\nu(x)$ are denoted by $\{c_{\nu k}\}$, those of $\mathscr{C}_\nu'(x)$, by $\{c_{\nu k}'\}$, $k = 1, 2, \ldots$, those of $J_\nu(x)$ by $j_{\nu k}$, of $Y_\nu(x)$ by $y_{\nu k}$, etc., $k = 1, 2, \ldots$.

The first four theorems are consequences of §§ 3, 4 in which hypotheses are placed solely on the coefficient functions $f(x)$, $g(x)$ in the differential equation (1.1). Theorem 7.5, on the other hand, follows from § 5 which requires some knowledge of the behaviour of

$$p(x) = \tfrac{1}{2}\pi x\{[J_\nu(x)]^2 + [Y_\nu(x)]^2\}.$$

A number of the results of this section hold when the range of the order ν exceeds the usual one, $|\nu| > \frac{1}{2}$, to which most results on higher monotonicity have been restricted [**7**; **8**; **9**; **18**; **19**]. In particular, Theorem 7.2 reveals complete monotonicity properties which hold for *all* ν. However, the first result is not in this category.

THEOREM 7.1. *Let the zeros of* $D_x\{x^{\frac{1}{2}}\mathscr{C}_\nu(x)\}$ *on the interval* $[(\nu^2 - \frac{1}{4})^{\frac{1}{2}}, \infty)$ *be* $x_1', x_2', \ldots$. *Then, for* $|\nu| > \frac{1}{2}$ *and* $0 < \alpha \leqq 1$,

$$\{(x_{k+1}')^\alpha - (x_k')^\alpha\} \in \mathscr{M}_\infty^*, \tag{7.1}$$

$$\{\log(x_{k+1}'/x_k')\} \in \mathscr{M}_\infty^*, \tag{7.2}$$

$$\{x_k'\mathscr{C}_\nu^2(x_k')\} \in \mathscr{M}_\infty^*, \tag{7.3}$$

and

$$\{c_{\nu,k+1}[\mathscr{C}_\nu'(c_{\nu,k+1})]^2 - c_{\nu k}[\mathscr{C}_\nu'(c_{\nu k})]^2\}_{k=m}^{\infty} \in \mathscr{M}_\infty^*, \tag{7.4}$$

where m *is the smallest positive integer such that* $c_{\nu m} \geqq (\nu^2 - \frac{1}{4})^{\frac{1}{2}}$.

Proof. Assertion (7.1) is an immediate consequence of Theorem 3.2, with $\lambda = 0$, $W(x) = x^{\alpha-1}$, $n = \infty$, applied to the equation (1.1) with

$$f(x) = 1 - (\nu^2 - \tfrac{1}{4})x^{-2}$$

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and $g(x) = 1$; i.e. to

$$y'' + \left\{1 - \frac{\nu^2 - \frac{1}{4}}{x^2}\right\} y = 0,$$

satisfied by $y(x) = x^{\frac{1}{2}}\mathscr{C}_\nu(x)$.

Assertions (7.3) and (7.4) follow from an application of Theorem 4.1 to the same equation. In case $x_1' = (\nu^2 - \frac{1}{4})^{\frac{1}{2}}$, it is necessary in connection with (7.3) to observe also that the number l in (4.2) exists and equals zero, from l'Hospital's rule, since $f'([\nu^2 - \frac{1}{4}]^{\frac{1}{2}}) > 0$, so (4.8) holds.

The remaining conclusion (7.2) follows from (7.1) on using l'Hospital's rule: First, we have

$$\begin{aligned} 0 &\leqq \lim_{\alpha \to 0+} \frac{1}{\alpha} (-1)^n \Delta^n \{ (x_{k+1}')^\alpha - (x_k')^\alpha \} \\ &= (-1)^n \Delta^n \lim_{\alpha \to 0+} \frac{(x_{k+1}')^\alpha - (x_k')^\alpha}{\alpha} \\ &= (-1)^n \Delta^n \log \frac{x_{k+1}'}{x_k'} \end{aligned}$$

$$(n = 0, 1, \ldots; k = 1, 2, \ldots).$$

Now, if equality were ever to occur, then [6] (cf. [11] for extensions), for $k = 1, 2, \ldots,$

$$1 < \frac{x_3'}{x_2'} = \frac{x_{3+k}'}{x_{2+k}'} < \frac{\xi_{2+k}}{\xi_k} \to 1 \qquad (\text{as } k \to \infty),$$

where ξ_k is the zero of $\mathscr{C}_\nu(x)$ immediately preceding x_{2+k}' and the limit is obtained from a familiar asymptotic formula [**20**, p. 506]. The resulting contradiction shows that equality is precluded and thus completes the proof of (7.2) and of the theorem.

Remarks. (i) In (7.4), $m \leqq 2$. It has been pointed out in [**5**] and in [**9**, § 7] that the interval $([\nu^2 - \frac{1}{4}]^{\frac{1}{2}}, \infty)$ contains all except possibly the first of the positive zeros of $D_x\{x^{\frac{1}{2}}\mathscr{C}_\nu(x)\}$. When $\nu > \frac{1}{2}$, it embraces *all* these zeros in case $\mathscr{C}_\nu(x)$ is either $J_\nu(x)$ [**5**] or $Y_\nu(x)$ [**9**, (7.2), p. 1259] and $c_{\nu 2} > \nu > (\nu^2 - \frac{1}{4})^{\frac{1}{2}}$. Otherwise, $\mathscr{C}_\nu(x)$ would vanish twice in $(0, \nu]$ and the Sturm separation theorem would imply $j_{\nu 1} < \nu$, contradicting the well-known result that $j_{\nu 1} > \nu$ [**20**, p. 485(1)].

(ii) From (7.3) it is clear that $\{x_k'\mathscr{C}_\nu^2(x_k')\}$ is a decreasing sequence for $|\nu| > \frac{1}{2}$. The same inference follows from an application of Sonin's theorem to the differential equation

$$y'' + \{1 - (\nu^2 - \tfrac{1}{4})x^{-2}\}y = 0.$$

Watson has shown [**20**, p. 488 (II)] that the analogous sequence $\{c_{\nu k}'\mathscr{C}_\nu^2(c_{\nu k}')\}$ is decreasing for $\nu > 3^{\frac{1}{2}}/2$ from a certain k on. Richard [**13**, p. 320] has noted that the range of Watson's result can be extended.

(iii) The device used to establish (7.2) can be applied also to some of the sequences discussed in [**9**, pp. 1254–1255, esp. (5.12), (5.12′)]. This implies, e.g.,

$$(-1)^{n+1}\Delta^n \log\log c_{\nu k} > 0 \qquad (n = 1, 2, \ldots; k = 2, 3, \ldots), \tag{7.5}$$

for $|\nu| \geqq \frac{1}{3}$, since $c_{\nu 2} > e$ when $|\nu| \geqq \frac{1}{3}$. (In [**9**, p. 1255] it is shown that $c_{\nu 2} > 1$ for all ν. This inequality can be improved as required for $|\nu| \geqq \frac{1}{3}$ by noting that $c_{\nu 2} > j_{\nu 1} \geqq j_{\frac{1}{3},1} > 2.9 > e$ when $\nu \geqq \frac{1}{3}$ and by making then obvious modifications in the earlier proof.)

In this context, l'Hospital's rule is applied to

$$(-1)^n \Delta^n \lim_{\alpha\to 0^+} \frac{(\log c_{\nu k})^{-\alpha} - 1}{\alpha} \qquad (k = 2, \ldots; |\nu| \geqq \tfrac{1}{3}).$$

These differences are non-negative for $n = 1, 2, \ldots$, but not for $n = 0$. This accounts for the special form of (7.5).

The next theorem, valid for *all* ν, follows from an application of our general results to a differential equation satisfied by $\mathscr{C}_\nu(x)$ rather than, as in the preceding theorem, to one satisfied by $x^{\frac{1}{2}}\mathscr{C}_\nu(x)$.

THEOREM 7.2. *For all* ν

$$\{(c_{\nu,k+1}')^\alpha - (c_{\nu k}')^\alpha\}_{k=m}^\infty \in \mathscr{M}_\infty^*, \qquad 0 < \alpha \leqq 1, \tag{7.6}$$

$$\{\log(c_{\nu,k+1}'/c_{\nu k}')\}_{k=m}^\infty \in \mathscr{M}_\infty^*, \tag{7.7}$$

and

$$\{\mathscr{C}_\nu^2(c_{\nu k}')\}_{k=m}^\infty \in \mathscr{M}_\infty^*, \tag{7.8}$$

where m is the smallest positive integer for which $c_{\nu m}' \geqq |\nu|$, *i.e., the squares of such extrema of an arbitrary Bessel function form a completely monotonic sequence.*

Proof. The function $y = \mathscr{C}_\nu(x)$ satisfies the differential equation

$$(xy')' + (x^2 - \nu^2)x^{-1}y = 0, \qquad 0 < x < \infty.$$

This equation satisfies the hypotheses of Theorem 3.3, with $n = \infty$, on the interval $(|\nu|, \infty)$. Taking $\lambda = 0$ and $W(x) = x^{\alpha-1}$ gives (7.6); the proof of (7.7) then parallels that of (7.2). A similar application of Theorem 4.3 gives (7.8).

Remarks. (i) Here, $m \leqq 2$, e.g., the squares of the extrema, beginning with the second, of an arbitrary Bessel function form a completely monotonic sequence. (This is the most that can be said, as may be seen by taking $\mathscr{C}_\nu(\frac{1}{2}\nu) = 1$, $\mathscr{C}_\nu'(\frac{1}{2}\nu) = 0$. The differential equation $x^2y'' + xy' + (x^2 - \nu^2)y = 0, y = \mathscr{C}_\nu(x)$, shows then that $x = \frac{1}{2}\nu$ yields a positive minimum.)

It suffices to prove that $c_{\nu 2}' \geqq |\nu|$. Indeed, if $c_{\nu 1} < c_{\nu 1}'$, then already $c_{\nu 1}' \geqq |\nu|$, as may be seen from the foregoing differential equation: If $\mathscr{C}_\nu(0) = 0$, then $\nu > 0$ and $\mathscr{C}_\nu(x) \equiv AJ_\nu(x)$, $A \neq 0$, so that $c_{\nu 1}' = j_{\nu 1}' > \nu$. Otherwise, $\mathscr{C}_\nu(x)$ can be normalized so that $\mathscr{C}_\nu(0) > 0$ (possibly $\mathscr{C}_\nu(0) = +\infty$). Then $x = c_{\nu 1}'$ yields either a negative minimum or a negative point of inflection, whence

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$y''(c_{\nu 1}') \geqq 0$, $y'(c_{\nu 1}') = 0$, $y(c_{\nu 1}') < 0$. With $x = c_{\nu 1}'$, the differential equation makes it obvious that $c_{\nu 1}' \geqq |\nu|$, as asserted.

If, on the other hand, $c_{\nu 1}' < |\nu|$, then $c_{\nu 1} > c_{\nu 1}'$, so that $x = c_{\nu 1}'$ yields a positive minimum and $x = c_{\nu 2}'$ provides either a positive maximum or a positive point of inflection. Substituting $x = c_{\nu 2}'$ in the differential equation as before shows that $c_{\nu 2}' \geqq |\nu|$.

This establishes that $m \leqq 2$ by proving $c_{\nu 2}' \geqq |\nu|$.

(ii) Somewhat more is true: $c_{\nu 2}' > |\nu|$. Suppose that $c_{\nu 2}' = |\nu|$. Then $y''(c_{\nu 2}') = y'(c_{\nu 2}') = 0$, while $y(c_{\nu 2}') > 0$. These values, when substituted in the differentiated differential equation, imply that $y'''(c_{\nu 2}') < 0$. Thus, $y'(c_{\nu 2}') = 0$ is a local maximum of $y'(x)$. Hence $y(x)$ decreases in $c_{\nu 2}' - \epsilon \leqq x \leqq c_{\nu 2}'$ for some $\epsilon > 0$. This is clearly impossible, since $c_{\nu 1}'$ is a positive minimum, and thus the proof is complete.

(iii) If $\nu \geqq 0$ and $\mathscr{C}_\nu(x)$ is either $J_\nu(x)$ or $Y_\nu(x)$, then $m = 1$ [**20**, p. 485 (1), p. 487 (10), p. 521].

Next we show that the sequence of squares of the extrema of the function $\mathscr{L}_\nu(x) = x^{-\nu}\mathscr{C}_\nu(x)$ is completely monotonic for $\nu > -\frac{1}{2}$. That this sequence is decreasing was shown essentially by Cheng [**3**, Lemma 3], Steinig [**14**, Lemma 1], and Szász [**15**, (I)], although all three formulated this only for $\mathscr{C}_\nu(x) \equiv J_\nu(x)$.

THEOREM 7.3. *For* $\nu > -\frac{1}{2}$,

$$\{(c_{\nu+1,k})^{-2\nu}\mathscr{C}_\nu^{\,2}(c_{\nu+1,k})\} \in \mathscr{M}_\infty^*;$$

i.e., the sequence of the squares of the extrema of the function $x^{-\nu}\mathscr{C}_\nu(x)$ *belongs to* $\mathscr{M}_\infty^*$.

Proof. If $y(x) = x^{-\nu}\mathscr{C}_\nu(x)$, then $y(x)$ satisfies the differential equation

$$(x^{2\nu+1}y')' + x^{2\nu+1}y = 0.$$

Thus, from Lemma 4.1 and the differentiation formula

$$D_x\{x^{-\nu}\mathscr{C}_\nu(x)\} = -x^{-\nu}\mathscr{C}_{\nu+1}(x)$$

(which shows that the extrema of $x^{-\nu}\mathscr{C}_\nu(x)$ occur at $x = c_{\nu+1,k}$, $k = 1, 2, \ldots$), it follows that

$$-\Delta\{(c_{\nu+1,k})^{-2\nu}\mathscr{C}_\nu^{\,2}(c_{\nu+1,k})\} = 2(2\nu+1)\int_{c_{\nu+1,k}}^{c_{\nu+1,k+1}} x^{-1}[\{x^{-\nu}\mathscr{C}_\nu(x)\}']^2 dx$$
$$= 2(2\nu+1)\int_{c_{\nu+1,k}}^{c_{\nu+1,k+1}} x^{-2\nu-2}|x^{\frac{1}{2}}\mathscr{C}_{\nu+1}(x)|^2 dx.$$

This sequence, with $k = 1, 2, \ldots$, belongs to $\mathscr{M}_\infty^*$, as may be seen from [**9**, Theorem 5.1] (with $c_{\nu k} = d_{\nu k}$, $W(x) = x^{-2\nu-2}$, and $\lambda = 2$) and the theorem now follows easily.

Remarks. (i) In [**9**, (5.15), p. 1256] it was shown that the sequence whose kth term is

$$|\Delta\{(c_{\nu+1,k})^{-\nu}\mathscr{C}_\nu(c_{\nu+1,k})\}|$$

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belongs to $\mathscr{M}_\infty^*$ when $\nu > -\frac{1}{2}$ and conjectured that the same may be true of

$$\{|(c_{\nu+1,k})^{-\nu}\mathscr{C}_\nu(c_{\nu+1,k})|\},$$

$k = 1, 2, \ldots$. If so, this would imply Theorem 7.3.

(ii) When $\mathscr{C}_\nu(x) \equiv J_\nu(x)$, the function $\mathscr{L}_\nu(x)$ is ordinarily written in the notation $[2^\nu\Gamma(\nu + 1)]^{-1}\Lambda_\nu(x)$.

(iii) If $\nu = -\frac{1}{2}$, the sequences in Theorem 7.3 and Remark (i) become trivial and belong to $\mathscr{M}_\infty$ but not to $\mathscr{M}_\infty^*$. For $\nu < -\frac{1}{2}$, Theorem 7.3 would be false, since the sequence involved is unbounded for such ν.

(iv) The function $x^{-\nu}J_\nu(x)$ has an extremum also at $x = 0$. When $\mathscr{C}_\nu(x) \equiv J_\nu(x)$ Theorem 7.3 can easily be extended to include the extremum at $x = 0$ as the first term in the sequence, as an examination of Lemma 4.1 and its consequences shows.

A similar result follows.

THEOREM 7.4. *If* $\nu > \frac{1}{2}$, *then*

$$\{c_{\nu k}^{2-2\nu}[\mathscr{C}_\nu'(c_{\nu k})]^2\} \in \mathscr{M}_\infty^*,$$

and, with $j_{\nu 0} = 0$,

$$\{j_{\nu k}^{2-2\nu}[J_\nu'(j_{\nu k})]^2\}_{k=0}^\infty \in \mathscr{M}_\infty^*.$$

Proof. The function $y(x) = x^\nu\mathscr{C}_\nu(x)$ satisfies the differential equation $(x^{1-2\nu}y')' + x^{1-2\nu}y = 0$.

From Lemma 4.2 and the fact that $\mathscr{C}_\nu(c_{\nu k}) = 0$, it follows that

$$\begin{aligned}\Delta\{[(c_{\nu k})^{1-2\nu}y'(c_{\nu k})]^2\} &= \Delta\{(c_{\nu k})^{2-2\nu}[\mathscr{C}_\nu'(c_{\nu k})]^2\}\\ &= 2(1-2\nu)\int_{c_{\nu k}}^{c_{\nu,k+1}} x^{-2\nu}|x^{\frac{1}{2}}\mathscr{C}_\nu(x)|^2dx.\end{aligned}$$

The theorem can now be inferred from [**9**, Theorems 5.1 and 5.2].

Remark. When $\nu = 1$, this shows that the sequence $\{[\mathscr{C}_1'(c_{1k})]^2\}$, $k = 1, 2, \ldots$, is completely monotonic, a result which should be compared with (7.4) according to which

$$\{\Delta[c_{1k}\{\mathscr{C}_1'(c_{1k})\}^2]\}_{k=2}^\infty \in \mathscr{M}_\infty^*.$$

The final theorem follows from Theorem 5.2 in view of [**7**, p. 62] where it is shown that $(-1)^n p^{(n)}x > 0$, $n = 0, 1, 2, \ldots$.

THEOREM 7.5. *If* $|\nu| > \frac{1}{2}$, $\alpha > 0$, $\gamma \leqq \frac{1}{2}$, $\delta \geqq \frac{1}{2}$, *and if* $\mathscr{C}_\nu(x)$ *and* $\mathscr{D}_\nu(x)$ *are linearly independent Bessel functions, then*

$$\{|c_{\nu k}^\gamma\mathscr{D}_\nu(c_{\nu k})|^\alpha\} \in \mathscr{M}_\infty^*,$$
$$\{|c_{\nu k}^\delta\mathscr{C}_\nu'(c_{\nu k})|^{-\alpha}\} \in \mathscr{M}_\infty^*,$$

and, for $n, k = 1, 2, \ldots$,

$$(-1)^n\Delta^n \log|c_{\nu k}^\gamma\mathscr{D}_\nu(c_{\nu k})| > 0,$$
$$(-1)^{n+1}\Delta^n \log|c_{\nu k}^\delta\mathscr{C}_\nu'(c_{\nu k})| > 0.$$

Remarks. (i) Watson showed [**20**, p. 446] that $p(x)$ decreases to 1 as $x \to \infty$ when $|\nu| > \frac{1}{2}$. This implies that $\{c_{\nu k}\mathscr{D}_\nu^2(c_{\nu k})\}$ $k = 1, 2, \ldots,$ is a decreasing sequence. The first assertion of Theorem 7.5 generalizes this consequence to complete monotonicity.

(ii) Watson showed also that $p(x)$ *increases* to 1 as $x \to \infty$ when $|\nu| < \frac{1}{2}$, so that

$$\{|c_{\nu k}^{\frac{1}{2}}\mathscr{D}_\nu(c_{\nu k})|^\alpha\}, \qquad \alpha > 0,$$

is an *increasing* sequence when $|\nu| < \frac{1}{2}$. We have no information to offer on the higher monotonicity properties of this sequence in this case.

(iii) A number of the complete monotonicity properties associated with $\mathscr{C}_\nu(x)$ for $|\nu| \geqq \frac{1}{3}$ established in this section and in [**9**, § 5] can be inferred from the differential equation $y'' + f(x)y = 0$ with

$$f(x) = x - \tfrac{1}{4}(9\nu^2 - 1)x^{-2}; \qquad y = x^{\frac{1}{2}}\mathscr{C}_\nu(\tfrac{2}{3}x^{3/2}).$$

An advantage of using this equation is that the results in question follow from it *in toto*, since $f'(x)$ is completely monotonic for $x > 0$ and $|\nu| \geqq \frac{1}{3}$, whereas our previous proofs required a separation into the cases $\frac{1}{3} \leqq |\nu| \leqq \frac{1}{2}$ (using the generalized Airy equation [**9**, (5.8), p. 1252; **7**, p. 63]), and $|\nu| > \frac{1}{2}$ (using the Bessel equation, where $f(x) = 1 - (\nu^2 - \frac{1}{4})x^{-2}$).

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