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SIGN-CHANGING SOLUTIONS FOR A CLASS OF NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract

Using variational methods, we obtain the existence of sign-changing solutions for a class of asymptotically linear Schrödinger equations with deepening potential well.

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1. Introduction

In this paper we are concerned with the following nonlinear Schrödinger equation

$$\begin{cases} -\Delta u + V_\lambda(x)u = f(x, u) & \text{in } \mathbb{R}^N, \quad N \geq 3 \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $V_\lambda(x) = 1 + \lambda g(x)$, λ is a positive parameter, the function g satisfies the condition (G):

(G) $g \in L^\infty(\mathbb{R}^N, \mathbb{R})$ and there exists a nonempty bounded smooth domain $\Omega \subset \mathbb{R}^N$ such that

$$g(x) \equiv 0 \quad \text{on } \bar{\Omega}, \qquad g(x) \in (0, 1] \quad \text{on } \mathbb{R}^N \setminus \bar{\Omega} \quad \text{and} \quad \lim_{|x|\to\infty} g(x) = 1.$$

We make the following assumptions on f:

(f_1) $f(x, t) \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $f(x, t)t \geq 0$ for almost every $x \in \mathbb{R}^N$, for all $t \in \mathbb{R}$;

(f_2) $\lim_{|t|\to 0}(f(x, t)/t) = 0$ uniformly with respect to $x \in \mathbb{R}^N$;

(f_3) there exists $\alpha \in (0, \infty)$ such that $\lim_{|t|\to\infty}(f(x, t)/t) = 1 + \alpha$ uniformly with respect to $x \in \overline{B}_R$ for all $R > 0$, and $f(x, t)/t$ is bounded on $\mathbb{R}^N \times (\mathbb{R} \setminus \{0\})$;

(f_4) $\lim_{|x|\to\infty} \sup_{|t|\leq r}(|f(x, t)|/|t|) = 0$ for every $r > 0$.

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In (f_3), B_R denotes the ball in $\mathbb{R}^N$ centered at zero with radius R. Let us remark that there exist functions satisfying the hypotheses $(f_1)-(f_4)$, for example,

$$f(x,t) = \frac{(1+\alpha)e^{-|x|^2}t^3}{1+e^{-|x|^2}t^2}.$$

Usually, if there exists some real number l such that:
(f_3') $\lim_{|t|\to\infty}(f(x,t)/t) = l$ uniformly with respect to $x \in \mathbb{R}^N$,
then we call f asymptotically linear at infinity. However, the condition '$\lim_{|t|\to\infty}(f(x,t)/t) = 1+\alpha$ uniformly with respect to $x \in \overline{B}_R$ for all $R > 0$' in the condition (f_3) is weaker than (f_3'), and we point out that the technical condition (f_4) will be mainly employed to prove the Palais–Smale condition ((PS) condition) in this paper (see Lemma 2.1).

When f is sublinear or superlinear at infinity, some existence results related to Problem (1.1) were obtained recently, see for example [1–4, 9] and the references therein. In this paper, we focus on the case of f asymptotically linear at infinity.

Let Ω be given by (G). We denote by ξ_1 the first eigenvalue of the following Dirichlet problem

$$\begin{cases} -\Delta u = \xi u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We also denote by $\lambda = \Lambda(\alpha)$ the principal eigenvalue of the following eigenvalue problem

$$-\Delta u - \alpha u + \lambda g(x) u = 0, \quad u \in H^1(\mathbb{R}^N), \alpha > 0. \tag{1.2}$$

It has been proved in [7] that $\Lambda(\alpha)$ always exists for any $\alpha \in (\Gamma, \xi_1)$ with

$$\Gamma = \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^2\,dx : u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N}(1-g)u^2\,dx = 1\right\}, \tag{1.3}$$

and the $\Lambda(\alpha)$-eigenfunction $u_{\Lambda(\alpha)}$ is the only eigenfunction which does not change sign. Moreover, $\Lambda(\alpha) > \alpha$, which is the largest eigenvalue among all other eigenvalues of (1.2).

Let us separate the first quadrant of the (λ, α)-plane into several parts (see Figure 1), then we see from recent papers [8, 10, 11] that the existence and nonexistence of signed solutions of Problem (1.1) have been completely discussed for these regions by using different methods. We summarize those results in the following Table 1.

In the existence cases of the preceding table (that is, regions apart from III, IV and the positive λ-axis), it is not hard to show the existence of solutions which do not change sign.

It is natural to ask whether sign-changing solutions for Problem (1.1) exist or not for the existence cases mentioned above? To the best of our knowledge, in any regions of the first quadrant of the (λ, α)-plane, the existence and nonexistence results of sign-changing solutions of Problem (1.1) have not been obtained.

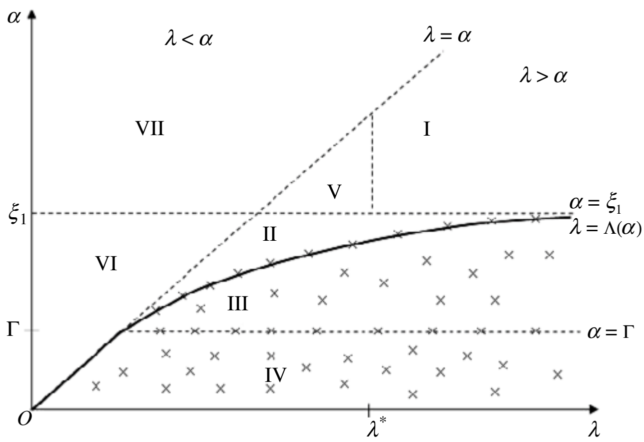

FIGURE 1. Separation of the first quadrant of the (λ, α)-plane into several parts.

TABLE 1. Summary of results.

Relations for λ and α	Corresponding regions	Obtained results	In papers
$\alpha > \xi_1, \lambda > \alpha$ large	I	A positive solution	[10]
$\alpha \in (\Gamma, \xi_1), \lambda \in (\alpha, \Lambda(\alpha))$	II	A positive solution and A negative solution	[8]
$\alpha \in (\Gamma, \xi_1), \lambda \geq \Lambda(\alpha)$	III	No positive solution	[8]
$\alpha \leq \Gamma, \lambda \geq \alpha$	IV	No positive solution	[8]
$\alpha = 0, \lambda \in [0, +\infty)$	The positive λ-axis	No positive solution	[11]
$\alpha > 0, \lambda = 0$	The positive α-axis	A positive solution	[11]
$\alpha > \Gamma, \lambda = \alpha$	The dotted diagonal line	A positive solution	[11]
$\alpha = \xi_1, \lambda \in (\alpha, +\infty)$	The dotted horizontal line	A positive solution	[11]
$\alpha > \xi_1, \lambda > 0$	I, V and VII	A positive solution	[11]
$\alpha > 0, \lambda \in (0, \alpha)$	VI and VII	A positive solution	[11]

In order to obtain sign-changing solutions of Problem (1.1), let us denote the operator $L_\lambda := -\Delta + \lambda g(x)$, $\Phi(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda g(x) u^2)\, dx$, and let λ be fixed in (1.2), then it follows from [6, 7] that the lowest eigenvalue of L_λ given by

$$\alpha_1(\lambda) = \inf\{\Phi(u) : u \in H^1(\mathbb{R}^N) \text{ and } \|u\|_2 = 1\}$$

is well defined, and there exists a positive eigenfunction $u_{\alpha_1(\lambda)}$ corresponding to $\alpha_1(\lambda)$. Moreover, $\alpha_1(\lambda)$ is simple in the sense of $\ker(L_\lambda - \alpha_1(\lambda)I) = \operatorname{span}\{u_{\alpha_1(\lambda)}\} := V_1$, and $\alpha_1(\lambda)$ increases from Γ (see (1.3) for the definition of Γ) to ξ_1 as λ increases from Γ to ∞.

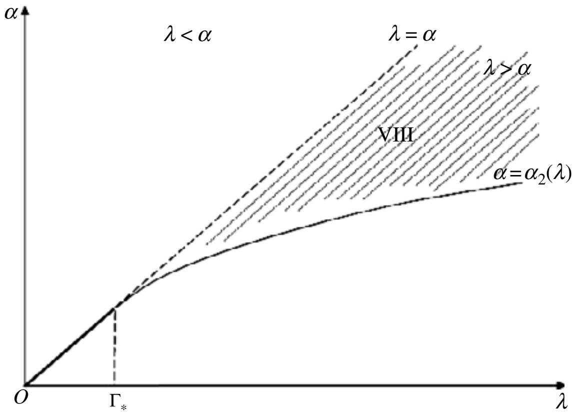

FIGURE 2. Sign-changing solution of Problem (1.1) in the region VIII $\setminus \sigma_p(L_\lambda)$.

Define

$$\Gamma_* = \inf_{u\in H^1(\mathbb{R}^N)\cap V_1^\perp} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx}{\int_{\mathbb{R}^N}(1-g)u^2\,dx}.$$

Now let $\lambda \in (\Gamma_*, \infty)$ be fixed; we define

$$\alpha_2(\lambda) = \inf\{\Phi(u) : u \in H^1(\mathbb{R}^N)\cap V_1^\perp \text{ and } \|u\|_2 = 1\}.$$

Similar to [11, Proposition 2.1], we can prove that there exists $u \in H^1(\mathbb{R}^N)\cap V_1^\perp$ such that $\Phi(u)$ archives $\alpha_2(\lambda)$, then the Lagrange-multipliers method implies that $\alpha_2(\lambda)$ is an eigenvalue of L_λ. By the simplicity of $\alpha_1(\lambda)$, we obtain that $\alpha_2(\lambda)$ is the second eigenvalue of L_λ. Moreover, it is easy to prove that $\alpha_2(\lambda)$ is increasing in λ.

In this paper, by using the variational method, we show that Problem (1.1) has at least a sign-changing solution in the region VIII $\setminus \sigma_p(L_\lambda)$ in Figure 2, where $\sigma_p(L_\lambda)$ denotes the point spectrum of L_λ.

Now we give the main result of this paper.

THEOREM 1.1. *Assume that f satisfies (f_1)–(f_4). If (λ, α) is in the region* VIII $\setminus$ $\sigma_p(L_\lambda)$ *in Figure 2, that is, $\alpha > \alpha_2(\lambda)$ and $\lambda > \alpha$, moreover, $\alpha \notin \sigma_p(L_\lambda)$, then Problem (1.1) has at least a sign-changing solution. In addition, Problem (1.1) has a positive solution and a negative solution.*

In the rest of the section, we list some preliminaries which we use later.

Recall that a functional I defined on a Banach space H is said to satisfy the (PS) condition if any sequence $\{u_n\} \subset H$ satisfying $|I(u_n)| \le c$ and $I'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence.

PROPOSITION 1.2 (Liu and Sun [5, Theorem 3.2]). *Let X be a Hilbert space and let f be a C^1 functional defined on X. Assume that f satisfies the (PS) condition on X*

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and $f'(u)$ has the expression $f'(u) = u - Au$ for $u \in X$. Assume that D_1 and D_2 are open convex subsets of X with the properties that $D_1 \cap D_2 \neq \varnothing$, $A(\partial D_1) \subset D_1$ and $A(\partial D_2) \subset D_2$. If there exists a path $h : [0, 1] \to X$ such that

$$h(0) \in D_1 \setminus D_2, \quad h(1) \in D_2 \setminus D_1$$

and

$$\inf_{u \in \overline{D_1} \cap \overline{D_2}} f(u) > \sup_{t \in [0,1]} f(h(t)),$$

then f has at least four critical points, one in $D_1 \cap D_2$, one in $D_1 \setminus \overline{D_2}$, one in $D_2 \setminus \overline{D_1}$, and one in $X \setminus (\overline{D_1} \cup \overline{D_2})$.

In a given Banach space, $\to$ and $\rightharpoonup$ denote the strong convergence and the weak convergence, respectively. Denote $\|u\|_s = (\int_{\mathbb{R}^N} |u|^s \, dx)^{1/s}$. We use

$$\langle u, v \rangle_\lambda = \int_{\mathbb{R}^N} (\nabla u \nabla v + V_\lambda(x) uv) \, dx$$

as the inner product in the Hilbert space

$$E_\lambda = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V_\lambda(x) u^2 \, dx < \infty \right\},$$

and induced norm $\|u\|_\lambda = \sqrt{\langle u, u \rangle_\lambda}$. By (G) the norm $\|u\|_\lambda$ is equivalent to the norm $\|u\| = (\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx)^{1/2}$.

Finally, $c, c_0, c_1, c_2, \ldots$ denote (possibly different) positive constants.

2. Proof of Theorem 1.1

By the assumptions on V_λ and f, the functional $I : E_\lambda \to \mathbb{R}$, corresponding to Problem (1.1), defined by

$$I(u) = \frac{1}{2} \|u\|_\lambda^2 - \int_{\mathbb{R}^N} F(x, u) \, dx$$

is continuously (Fréchet) differentiable, where $F(x, u) = \int_0^u f(x, t) \, dt$. Therefore, the critical points of the functional I correspond to solutions of Problem (1.1).

Let us write the gradient of I at u by

$$I'(u) = u - A(u), \quad A : E_\lambda \to E_\lambda, \quad A(u) = (-\Delta + V_\lambda)^{-1} f(x, u).$$

Then $\langle A(u), v \rangle = \int_{\mathbb{R}^N} f(x, u) v \, dx$ for all $v \in E_\lambda$. We consider the convex cones $P = \{u \in E_\lambda : u \geq 0\}$ and $-P = \{u \in E_\lambda : u \leq 0\}$. For $\varepsilon > 0$, we denote

$$P_\varepsilon = \{u \in E_\lambda : \operatorname{dist}(u, P) < \varepsilon\} \quad \text{and} \quad -P_\varepsilon = \{u \in E_\lambda : \operatorname{dist}(u, -P) < \varepsilon\}.$$

Note that P_ε and $-P_\varepsilon$ are open convex subsets of E_λ, therefore $E_\lambda \setminus (\overline{P_\varepsilon} \cup (\overline{-P_\varepsilon}))$ contains only sign-changing functions.

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LEMMA 2.1. *The functional I satisfies the (PS) condition.*

PROOF. Let $\{u_n\}$ be a sequence in E_λ such that $|I(u_n)| \le c$, $I'(u_n) \to 0$ as $n \to \infty$. We first prove that $\{u_n\}$ is bounded in E_λ. In fact, otherwise, we may suppose that $\|u_n\|_\lambda \to \infty$. Set $w_n = u_n/\|u_n\|_\lambda$. Obviously, $\{w_n\}$ is bounded in E_λ. Passing to a subsequence, still denoted by $\{w_n\}$, we may assume that there exists $w \in E_\lambda$ such that

$$\begin{aligned} w_n &\rightharpoonup w \quad \text{in } E_\lambda, \\ w_n &\to w \quad \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^N) \text{ for } 2 \le s < 2^*, \\ w_n(x) &\to w(x) \quad \text{almost everywhere on } \mathbb{R}^N. \end{aligned}$$

We claim that $w \not\equiv 0$ in $\mathbb{R}^N$. If $w = 0$, then $w_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. We show that for each $\lambda > 0$

$$\int_{\mathbb{R}^N} (V_\lambda(x) - (1+\lambda))w_n^2\,dx \to 0 \quad \text{as } n \to \infty. \tag{2.1}$$

Indeed, by (G), for every $\varepsilon' > 0$ there exists $R_* > 0$ such that $\bar{\Omega} \subset B_{R_*}$, and

$$|V_\lambda(x) - (1+\lambda)| < \varepsilon', \quad x \in \mathbb{R}^N \setminus B_{R_*}.$$

So, for all $n \in \mathbb{N}$, we obtain that there exists some positive constant c_1 such that

$$\left| \int_{\mathbb{R}^N \setminus B_{R_*}} (V_\lambda(x) - (1+\lambda))w_n^2\,dx \right| \le \varepsilon' \|w_n\|_2^2 \le c_1 \varepsilon'.$$

Also, since $w_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and note that $|V_\lambda(x) - (1+\lambda)| \le \lambda$, so there exists a constant $c_2 > 0$ such that

$$\left| \int_{B_{R_*}} (V_\lambda(x) - (1+\lambda))w_n^2\,dx \right| \le \lambda \int_{B_{R_*}} w_n^2\,dx < c_2 \varepsilon'$$

for n large enough. Thus,

$$\begin{aligned} \int_{\mathbb{R}^N} (V_\lambda(x) - (1+\lambda))w_n^2\,dx &= \int_{B_{R_*}} (V_\lambda(x) - (1+\lambda))w_n^2\,dx \\ &\quad + \int_{\mathbb{R}^N \setminus B_{R_*}} (V_\lambda(x) - (1+\lambda))w_n^2\,dx \to 0 \end{aligned}$$

as $n \to \infty$, which is (2.1). Thus, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left(|\nabla w_n|^2 + (1+\lambda)w_n^2\right) dx = \lim_{n\to\infty} \|w_n\|_\lambda^2 = 1$$

and then

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} w_n^2\,dx \le \frac{1}{1+\lambda}. \tag{2.2}$$

By (f_1) and (f_3), for any $\varepsilon' > 0$, there exists $M > 0$ such that

$$\frac{F(x,t)}{t^2} < \frac{1+\alpha}{2} + \frac{\varepsilon'}{4}, \quad \forall |t| \geq M, \ |x| \leq R$$

for all $R > 0$. Thus,

$$\begin{aligned} \int_{\{|x|\leq R,\ |u_n(x)|\geq M\}} \frac{F(x,u_n)}{u_n^2} w_n^2\,dx &\leq \frac{1+\alpha}{2}\int_{\mathbb{R}^N} w_n^2\,dx + \frac{\varepsilon'}{4}\|w_n\|_\lambda^2 \\ &= \frac{1+\alpha}{2}\int_{\mathbb{R}^N} w_n^2\,dx + \frac{\varepsilon'}{4} \end{aligned} \tag{2.3}$$

for all $n \in \mathbb{N}$.

It is clear that we may choose $R > 0$ large enough such that

$$\int_{\{|x|\geq R,\ |u_n(x)|\geq M\}} \frac{F(x,u_n)}{u_n^2} w_n^2\,dx \leq \frac{\varepsilon'}{4} \tag{2.4}$$

holds for all $n \in \mathbb{N}$. By (f_4),

$$\int_{\{|x|\geq R,\ |u_n(x)|\leq M\}} \frac{F(x,u_n)}{u_n^2} w_n^2\,dx \leq \sup_{|x|\geq R,\ |u_n(x)|\leq M} \frac{F(x,u_n)}{u_n^2}\|w_n\|_\lambda^2 \leq \frac{\varepsilon'}{4} \tag{2.5}$$

for all $n \in \mathbb{N}$. Note that $w_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ as $n \to \infty$. Then

$$\int_{\{|x|\leq R,\ |u_n(x)|\leq M\}} \frac{F(x,u_n)}{u_n^2} w_n^2\,dx \leq \frac{\varepsilon'}{4} \tag{2.6}$$

for n large enough. Combining (2.3)–(2.6) we conclude that for n large enough

$$\int_{\mathbb{R}^N} \frac{F(x,u_n)}{u_n^2} w_n^2\,dx \leq \frac{1+\alpha}{2}\int_{\mathbb{R}^N} w_n^2\,dx + \varepsilon'. \tag{2.7}$$

Then by $|I(u_n)| \leq c$, (2.2) and (2.7), we know that

$$\frac{1}{2} = \lim_{n\to\infty}\int_{\mathbb{R}^N} \frac{F(x,u_n)}{u_n^2} w_n^2\,dx \leq \frac{1+\alpha}{2(1+\lambda)},$$

which is impossible since $\lambda > \alpha$. So, we have shown that $w \not\equiv 0$.

By $I'(u_n) \to 0$ as $n \to \infty$, for all $\varphi \in E_\lambda$

$$\langle I'(u_n), \varphi\rangle = \int_{\mathbb{R}^N} (\nabla u_n \nabla\varphi + V_\lambda(x)u_n\varphi - f(x,u_n)\varphi)\,dx \to 0 \tag{2.8}$$

as $n \to \infty$. Dividing (2.8) by $\|u_n\|_\lambda$ we have

$$\int_{\mathbb{R}^N} \left(\nabla w_n \nabla\varphi + V_\lambda(x)w_n\varphi - \frac{f(x,u_n)}{u_n} w_n\varphi\right)dx = o(1). \tag{2.9}$$

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Since $u_n(x) = \|u_n\|_\lambda w_n(x) \to \infty$ for $x \notin \{x \in \mathbb{R}^N \mid w(x) = 0\}$, and $w_n(x) \to w(x)$ almost everywhere in $\mathbb{R}^N$, it implies by (f_3) that

$$\frac{f(x, u_n(x))}{u_n(x)} w_n(x) \to (1+\alpha)w(x) \quad \text{almost everywhere in } \mathbb{R}^N.$$

Also by (f_3), we know that $f(x,t)/t$ is bounded on $\mathbb{R}^N \times (\mathbb{R} \setminus \{0\})$. Then the sequence $\{(f(x, u_n(x))/u_n(x))w_n(x)\}$ is bounded in $L^2(\mathbb{R}^N)$, thus there exists a subsequence, still denoted by the same subscripts, such that

$$\frac{f(x, u_n)}{u_n} w_n \rightharpoonup (1+\alpha)w \quad \text{in } L^2(\mathbb{R}^N).$$

Therefore, by (2.9) and the weak convergence of $\{w_n\}$ in E_λ, we obtain

$$\int_{\mathbb{R}^N} (\nabla w \nabla \varphi + V_\lambda(x) w\varphi - (1+\alpha) w\varphi)\, dx = 0,$$

that is,

$$\int_{\mathbb{R}^N} (\nabla w \nabla \varphi + \lambda g(x) w\varphi)\, dx = \alpha \int_{\mathbb{R}^N} w\varphi \, dx.$$

From this we see that w satisfies

$$-\Delta w + \lambda g(x) w = \alpha w.$$

This is impossible since $\alpha \notin \sigma_p(L_\lambda)$. Therefore, $\{u_n\}$ is bounded in E_λ.

Next we prove that there exists $u \in E_\lambda$ such that $\|u_n\|_\lambda \to \|u\|_\lambda$ as $n \to \infty$.

Indeed, by the boundedness of $\{u_n\}$, passing to a subsequence, we may assume that $u_n \rightharpoonup u$ in E_λ and $u_n \to u$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$. Since $\langle I'(u_n), \varphi\rangle \to 0$ as $n \to \infty$, we let $\varphi = u_n - u$ and obtain

$$\begin{aligned} 0 \le \limsup_{n\to\infty}(\|u_n\|_\lambda^2 - \|u\|_\lambda^2) &= \limsup_{n\to\infty}\langle u_n, u_n - u\rangle \\ &= \limsup_{n\to\infty} \int_{\mathbb{R}^N} f(x, u_n)(u_n - u)\, dx. \end{aligned} \tag{2.10}$$

Clearly, by (f_3), $\lim_{|t|\to\infty} f(x,t)/t^{p-1} = 0$ for all fixed $p \in (2, 2^*)$. Then it follows from (f_1), (f_2) that for each $\varepsilon' > 0$ there exists $C_{\varepsilon'} > 0$ such that

$$|f(x,t)| \le \varepsilon'|t| + C_{\varepsilon'}|t|^{p-1}. \tag{2.11}$$

By (2.11) and the Hölder inequality, for $r \ge 1$

$$\begin{aligned} \int_{\{|u_n(x)|\ge r\}} f(x, u_n)(u_n - u)\, dx &\le 2C \int_{\{|u_n(x)|\ge r\}} |u_n|^{p-1}|u_n - u|\, dx \\ &= 2C \int_{\{|u_n(x)|\ge r\}} \frac{1}{|u_n|^{2^*-p}} |u_n|^{2^*-1}|u_n - u|\, dx \\ &\le 2C r^{p-2^*} \|u_n\|_{2^*}^{2^*-1} \|u_n - u\|_{2^*}, \end{aligned}$$

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where $C > 0$ is a constant. Since $p < 2^*$, we may choose r so large that for all $n \in \mathbb{N}$

$$\int_{\{|u_n(x)| \geq r\}} f(x, u_n)(u_n - u)\, dx \leq \frac{\varepsilon'}{3}. \tag{2.12}$$

Moreover, by (f_4), there exists $R > 0$ such that for all $n \in \mathbb{N}$

$$\int_{\{|x| \geq R,\ |u_n(x)| \leq r\}} f(x, u_n)(u_n - u)\, dx \leq \frac{\varepsilon'}{3}. \tag{2.13}$$

This implies by (2.11), the Hölder inequality and the compactness of the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^s_{\mathrm{loc}}(\mathbb{R}^N)$ that for n large enough

$$\int_{\{|x| \leq R,\ |u_n(x)| \leq r\}} f(x, u_n)(u_n - u)\, dx \leq \frac{\varepsilon'}{3}, \tag{2.14}$$

so by (2.12)–(2.14)

$$\int_{\mathbb{R}^N} f(x, u_n)(u_n - u)\, dx \leq \varepsilon'$$

for n large enough. From this and (2.10) we know that $\|u_n\|_\lambda \to \|u\|_\lambda$.

Finally, the locally uniform convexity of E_λ gives that $u_n \to u$ in E_λ as $n \to \infty$. The lemma is proved. $\square$

LEMMA 2.2. *Assume that (f_1)–(f_3) hold, then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon : 0 < \varepsilon \leq \varepsilon_0$ there holds*

$$A(\partial(\pm P_\varepsilon)) \subset \pm P_\varepsilon.$$

Moreover, if $u \in \pm P_\varepsilon$ is a solution of Problem (1.1), then $u \in \pm P$.

PROOF. Indeed, if $u \in E_\lambda$ and $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$, then

$$\mathrm{dist}(A(u), P) = \min_{w \in P} \|A(u) - w\|_\lambda = \min_{w \in P} \|A(u)^+ + A(u)^- - w\|_\lambda \leq \|A(u)^-\|_\lambda.$$

Let $s \in [2, 2^*)$, then by the continuity of the embedding $E_\lambda \hookrightarrow L^s(\mathbb{R}^N)$, there exists $C_s > 0$ such that

$$\|u^\mp\|_s = \min_{w \in \pm P} \|u - w\|_s \leq C_s \min_{w \in \pm P} \|u - w\|_\lambda = C_s\, \mathrm{dist}(u, \pm P). \tag{2.15}$$

We claim that $A(u) \in P_\varepsilon$ for any $u \in \partial P_\varepsilon$.

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In fact, it follows from (2.11), (2.15), (f_1) and the Hölder inequality that

$$\begin{aligned}\operatorname{dist}(A(u), P)\|A(u)^-\|_\lambda &\le \|A(u)^-\|_\lambda^2 = \langle A(u), A(u)^-\rangle \\ &= \int_{\mathbb{R}^N} f(x, u)A(u)^- \, dx \\ &\le \int_{\{u\ge 0\}} f(x, u)A(u)^- \, dx + \int_{\{u\le 0\}} f(x, u)A(u)^- \, dx \, dx \\ &\le \int_{\{u\le 0\}} f(x, u^-)A(u)^- \, dx \\ &\le \int_{\{u\le 0\}} (\varepsilon'|u^-| + C_{\varepsilon'}|u^-|^{p-1})A(u)^- \, dx \\ &\le \varepsilon'\tilde{c} \operatorname{dist}(u, P)\|A(u)^-\|_\lambda + \tilde{C}_{\varepsilon',p} \operatorname{dist}(u, P)^{p-1}\|A(u)^-\|_\lambda.\end{aligned}$$

Taking $\varepsilon' = 1/2\tilde{c}$, then we obtain

$$\operatorname{dist}(A(u), P) \le \tfrac{1}{2} \operatorname{dist}(u, P) + \tilde{C} \operatorname{dist}(u, P)^{p-1},$$

where $\tilde{C} > 0$ is a constant. Let $\varepsilon_0 = \big(1/4\tilde{C}\big)^{1/(p-2)}$, then for all ε with $0 < \varepsilon \le \varepsilon_0$, we have

$$\operatorname{dist}(A(u), P) \le \tfrac{3}{4} \operatorname{dist}(u, P) \tag{2.16}$$

for all $u \in P_\varepsilon$. Clearly, $\operatorname{dist}(A(u), P) \le \frac{3}{4}\varepsilon < \varepsilon$ for every $u \in \partial P_\varepsilon$, that is, $A(u) \in P_\varepsilon$ for all $u \in \partial P_\varepsilon$. Hence, $A(\partial P_\varepsilon) \subset P_\varepsilon$. In a similar way, $A(\partial(-P_\varepsilon)) \subset -P_\varepsilon$. If $u \in P_\varepsilon$ is a solution of Problem (1.1), then $I'(u) = u - A(u) = 0$, that is, $u = A(u)$, by (2.16), then $u \in P$. Similarly, if $u \in -P_\varepsilon$, then $u \in -P$. □

Lemma 2.3. *Assume* (f_1)–(f_3) *hold. Let* $0 < \varepsilon \le \varepsilon_0$, *then there exists* $C_* > -\infty$ *such that* $\inf_{\overline{P_\varepsilon}\cap(-\overline{P_\varepsilon})} I(u) = C_*$.

Proof. It follows from (2.11) that

$$\begin{aligned}I(u) &= \frac{1}{2}\|u\|_\lambda^2 - \int_{\mathbb{R}^N} F(x, u)\, dx \\ &\ge \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{2}\varepsilon' \int_{\mathbb{R}^N} u^2 \, dx - C_{\varepsilon',p} \int_{\mathbb{R}^N} |u|^p \, dx \\ &\ge -\frac{1}{2}\varepsilon'\|u\|_2^2 - C_{\varepsilon',p}\|u\|_p^p.\end{aligned}$$

By (2.15) we have $\|u^\pm\|_s \le C_s \operatorname{dist}(u, \mp P) \le C_s\varepsilon_0$ for every $u \in P_\varepsilon \cap (-P_\varepsilon)$. So there exists $C_* > -\infty$ such that $\inf_{\overline{P_\varepsilon}\cap(-\overline{P_\varepsilon})} I(u) = C_*$. Hence, the lemma is proved. □

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PROOF OF THEOREM 1.1. Let $u_{\alpha_2(\lambda)}$ be an eigenfunction corresponding to the eigenvalue $\alpha_2(\lambda)$, and set $W := \{u \in E_\lambda : \|u\|_\lambda = 1,\ u \in \operatorname{span}\{u_{\alpha_1(\lambda)}, u_{\alpha_2(\lambda)}\}\}$. Define a path $\gamma : [0, 1] \to W$

$$\gamma(t) = \cos(\pi t) u_{\alpha_1(\lambda)} + \sin(\pi t) u_{\alpha_2(\lambda)}$$

connecting $\gamma(0) = u_{\alpha_1(\lambda)}$ and $\gamma(1) = -u_{\alpha_1(\lambda)}$. Now let $h_R(t) = R\gamma(t)$. Since $0 \leq F(x, t)/t^2 \leq C$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}^N$, and

$$\lim_{R\to+\infty} \frac{F(x, R\gamma(t))}{R^2\gamma(t)^2} = \frac{1+\alpha}{2} \quad \text{for almost every } x \in \mathbb{R}^N,$$

thus, it follows by Lebesgue's theorem that

$$\begin{aligned}
\lim_{R\to+\infty} \frac{I(h_R(t))}{R^2} &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla\gamma(t)|^2 + \gamma(t)^2 + \lambda g(x)\gamma(t)^2)\,dx \\
&\quad - \lim_{R\to+\infty} \int_{\mathbb{R}^N} \frac{F(x, R\gamma(t))}{R^2\gamma(t)^2}\gamma(t)^2\,dx \\
&= \frac{1}{2} - \frac{1}{2}(1+\alpha)\int_{\mathbb{R}^N} (\cos^2(\pi t)u^2_{\alpha_1(\lambda)} + \sin^2(\pi t)u^2_{\alpha_2(\lambda)})\,dx \\
&= \frac{1}{2} - \frac{1}{2}(1+\alpha)\left(\frac{\cos^2(\pi t)}{\alpha_1(\lambda)+1} + \frac{\sin^2(\pi t)}{\alpha_2(\lambda)+1}\right) \\
&< \frac{1}{2}\left(1 - \frac{1+\alpha}{1+\alpha_2(\lambda)}\right) < 0.
\end{aligned}$$

So, this yields that there exists R_0 such that $I(h_{R_0}(t)) < C_* - 1$. Hence, we obtain

$$\inf_{\overline{P_\varepsilon}\cap(\overline{-P_\varepsilon})} I(u) > \sup_{t\in[0,1]} I(h_{R_0}(t)).$$

Obviously $h_{R_0}(0) \in P_\varepsilon \setminus (-P_\varepsilon)$, $h_{R_0}(1) \in (-P_\varepsilon) \setminus P_\varepsilon$. By using Lemmas 2.1–2.3 and Proposition 1.2, we can find a critical point in $E_\lambda \setminus (\overline{P_\varepsilon} \cup (\overline{-P_\varepsilon}))$, which is a sign-changing solution of Problem (1.1). Also we have a critical point in $P_\varepsilon \setminus (\overline{-P_\varepsilon})$ and a critical point in $(-P_\varepsilon) \setminus \overline{P_\varepsilon}$, which correspond to a positive solution and a negative solution of Problem (1.1). □

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