Proceedings of the Edinburgh Mathematical Society (2009) **52**, 569–581 ©
DOI:10.1017/S0013091506000897 Printed in the United Kingdom

MULTIPLICITY OF SOLUTIONS OF DIRICHLET PROBLEMS ASSOCIATED WITH SECOND-ORDER EQUATIONS IN $\mathbb{R}^2$

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(Received 20 July 2006)

Abstract We study the existence of multiple solutions for a two-point boundary-value problem associated with a planar system of second-order ordinary differential equations by using a shooting technique. We consider asymptotically linear nonlinearities satisfying suitable sign conditions. Multiplicity is ensured by assumptions involving the Morse indices of the linearizations at zero and at infinity.

Keywords: asymptotically linear; multiplicity; second order; planar systems; Morse index

2000 *Mathematics subject classification:* Primary 34B15

1. Introduction

In this paper we are interested in the existence of multiple solutions to the equation

$$x'' + A(t,x)x = 0, \tag{1.1}$$

$x \in \mathbb{R}^2$, $t \in (0,\pi)$, satisfying the Dirichlet boundary conditions $x(0) = x(\pi) = 0$. We will assume that $A : [0,\pi] \times \mathbb{R}^2 \to \mathrm{GL}_s(\mathbb{R}^2)$,

$$A(t,x) = \begin{bmatrix} a_{11}(t,x) & a_{12}(t,x) \\ a_{12}(t,x) & a_{22}(t,x) \end{bmatrix},$$

is a continuous function such that

$$\lim_{|x|\to 0} A(t,x) = A_0(t) \quad \text{uniformly in } t \in [0,\pi],$$
$$\lim_{|x|\to \infty} A(t,x) = A_\infty(t) \text{ uniformly in } t \in [0,\pi];$$

that is, we assume asymptotically linear conditions at the origin and at infinity.

There exists an extensive literature concerning the existence of solutions of boundary-value problems associated with asymptotically linear Hamiltonian systems. Regarding first-order Hamiltonian systems and existence of periodic solutions, we can mention, for

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example, [**3**, **4**, **9**, **11**] and, among others, the more recent works [**18**, **24**, **35**] (see also the references therein). In these papers, the existence of at least one solution, or, in some cases, at least two, is met when the Maslov-type indices [**2**] of the linearizations at zero and at infinity are different. When some additional conditions (like convexity, symmetry in the space variable, or for autonomous equations) are guaranteed, multiplicity results are obtained in [**1**, **14**, **17**, **20**, **21**, **28**, **29**]. For the particular case of $\mathbb{R}^2$, multiplicity results were obtained in [**27**] using no additional conditions by using the Poincaré–Birkhoff theorem.

The existence of solutions of Hamiltonian systems satisfying Dirichlet and Bolza boundary conditions was studied in [**7**] and [**16**], respectively.

The next set of references we wish to refer to deals with existence and multiplicity results for second-order asymptotically linear systems. Interesting contributions in the periodic setting can be found, among others, in [**5**, **25**, **33**], in which existence results are obtained, and in [**6**], where multiplicity of solutions is proved in the autonomous case. The literature is not so rich in contributions as far as Dirichlet problems associated with second-order systems are concerned. In this direction, we refer the reader to [**10**, **24**, **32**], containing existence results for systems of partial differential equations. Multiplicity results have recently been obtained under some extra assumptions in [**8**, **15**] for ordinary differential equations and in [**30**, **34**] for partial differential equations. It is worth noting that in these works it is shown that the bigger the gap between suitable indices associated with the linearizations of the problem at the origin and at infinity, the larger the number of solutions. In particular, in [**8**] the authors consider (1.1) in $\mathbb{R}^n$ and obtain multiplicity of solutions satisfying the Dirichlet boundary conditions under asymptotically linear growth conditions. The results are deduced via a generalized shooting approach using the notion of moments of verticality and phase angles. However, the number of solutions obtained depends on the cardinality of a suitable set which sometimes can be empty. On the other hand, in [**15**] the author proved the existence of multiple solutions to the Dirichlet problem associated with the equation $x''+V'(t,x)=0$ (which is a particular case of (1.1), as is shown in [**8**]) assuming asymptotically linear conditions and a symmetry condition on the potential V: that is, $V(t,x)\equiv V(t,-x)$.

In this paper we re-examine this problem in the case of $\mathbb{R}^2$ and prove the existence of multiple solutions of (1.1) satisfying Dirichlet boundary conditions. The aim of this paper is to try to generalize the results of [**15**] to a context where no symmetry assumptions are required. To reach this goal, we must assume some sign conditions for the matrix $A(t,x)$ (see Theorem 2.3). Under these conditions and whenever there is a gap equal to N between the indices of the linearizations at the origin and at infinity, we are able to guarantee the existence of $2N$ solutions to (1.1) satisfying Dirichlet boundary conditions. Our proof is developed in the framework of shooting methods. Multiplicity results follow by combining degree theory with some preliminary results about eigenvalues and eigenvectors of second-order Dirichlet problems, proved in Propositions 2.4 and 2.6.

In the following we denote by $\mathrm{GL}_s(\mathbb{R}^2)$ the group of 2×2 real symmetric matrices and by I_2 the identity matrix in that group. According to the notation of [**15**], for any $B_1, B_2 \in L^1([0,\pi];\mathrm{GL}_s(\mathbb{R}^2))$ we write $B_1 < B_2$ if $B_1(t) \leqslant B_2(t)$ for a.e. $t \in (0,\pi)$ and

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$B_1(t) < B_2(t)$ on a subset of $(0, \pi)$ with positive measure. Also, we denote by $\mathbb{R}^+$ the set of positive real numbers and, when no confusion arises, by 0 the origin in the plane.

2. Main result

Let us consider the two-point boundary-value problem

$$\left.\begin{aligned} x'' + A(t,x)x = 0, \quad x \in \mathbb{R}^2,\ t \in (0,\pi), \\ x(0) = x(\pi) = 0, \end{aligned}\right\} \tag{2.1}$$

where $A : [0, \pi] \times \mathbb{R}^2 \to \mathrm{GL}_s(\mathbb{R}^2)$,

$$A(t,x) = \begin{bmatrix} a_{11}(t,x) & a_{12}(t,x) \\ a_{12}(t,x) & a_{22}(t,x) \end{bmatrix},$$

is a continuous function such that uniqueness of solutions of Cauchy problems associated with system (2.1) is guaranteed. We will assume that

$$\lim_{|x|\to 0} A(t,x) = A_0(t) \quad \text{uniformly in } t \in [0,\pi], \tag{2.2}$$

$$\lim_{|x|\to \infty} A(t,x) = A_\infty(t) \text{ uniformly in } t \in [0,\pi]. \tag{2.3}$$

Under the condition (2.3) we conclude that A is bounded and hence the continuability of the solutions of Cauchy problems associated with system (2.1) is guaranteed.

In order to state our main result, we recall the definitions of index and of nullity of a path of symmetric matrices [**15**]. To do this, we first reformulate the proposition proved in [**15**].

Proposition 2.1. *Given* $B \in L^\infty([0, \pi]; \mathrm{GL}_s(\mathbb{R}^2))$*, consider the boundary-value problem*

$$\left.\begin{aligned} x'' + (B(t) + \lambda I_2)x = 0, \quad x \in \mathbb{R}^2,\ t \in (0,\pi), \\ x(0) = x(\pi) = 0. \end{aligned}\right\} \tag{2.4}$$

There exists a sequence of eigenvalues of problem (2.4), $\lambda_1(B) \leqslant \lambda_2(B) \leqslant \cdots$*, and* $\lambda_j(B) \to +\infty$ *as* $j \to +\infty$ *such that, for each* j*, there exists a space of dimension* 1 *of non-trivial solutions (eigenvectors of* B*) of the problem (2.4) with* $\lambda = \lambda_j(B)$*. Moreover,*

$$H_0^1([0,\pi];\mathbb{R}^2) := \{x : [0,\pi] \to \mathbb{R}^2 \mid x(\cdot) \text{ is continuous on } [0,\pi], \\ \text{satisfies } x(0) = 0 = x(\pi) \text{ and } x' \in L^2([0,\pi];\mathbb{R}^2)\}$$

admits a basis of eigenvectors of B*.*

Definition 2.2. Given $B \in L^\infty([0, \pi]; \mathrm{GL}_s(\mathbb{R}^2))$, its index $i(B)$ is defined as the number of negative eigenvalues of problem (2.4) and its nullity $\nu(B)$ is the number of zero eigenvalues of problem (2.4) (both counting their multiplicities).

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The index of $B \in L^\infty([0,\pi];\mathrm{GL}_s(\mathbb{R}^2))$ as we have just defined it coincides with the Morse index of the boundary-value problem $x'' + B(t)x = 0$, $x(0) = x(\pi) = 0$ in the non-degenerate case [**28**].

Note that in the sequence of the eigenvalues of problem (2.4) we cannot have the same value repeated more than twice. In the case when it is repeated twice, we say that the corresponding eigenvalue $\lambda(B) = \lambda_j(B) = \lambda_{j+1}(B)$, for some j, has a space of eigenvectors of dimension 2. Otherwise, we say that the space of eigenvectors has dimension 1.

Now we are in position to state the main result.

Theorem 2.3. *Assume that $A(t,x)$ satisfies (2.2) and (2.3). Suppose moreover that*

$$\left.\begin{array}{lll} a_{11}(t,x) < 0, & a_{22}(t,x) < 0 & \forall (t,x) \in [0,\pi]\times\mathbb{R}^2 \quad \textit{and} \\ \textit{either } a_{12}(t,x) \geqslant 0 \quad \textit{or} & a_{12}(t,x) \leqslant 0, & \forall (t,x) \in [0,\pi]\times\mathbb{R}^2. \end{array}\right\} \tag{2.5}$$

Then if $i(A_0) > i(A_\infty)$ and $\nu(A_\infty) = 0$ (or $i(A_0) < i(A_\infty)$ and $\nu(A_0) = 0$), the problem (2.1) has at least $2|i(A_0) - i(A_\infty)|$ non-trivial solutions.

Before proving the theorem, we need to state some preliminary results. First we present some results about eigenvalues and eigenvectors of problem (2.4) that will be useful in the proof of Theorem 2.3. Analogous results for the case of a second-order equation can be found in [**22, 23**].

Proposition 2.4. *For each $j = 1, 2, \dots$, $\lambda_j : L^\infty([0,\pi];\mathrm{GL}_s(\mathbb{R}^2)) \to \mathbb{R}$, $B \to \lambda_j(B)$ is continuous on $\{B \in L^\infty([0,\pi];\mathrm{GL}_s(\mathbb{R}^2)) : B < 0\}$ with respect to the topology induced by $L^1([0,\pi];\mathrm{GL}_s(\mathbb{R}^2))$ on $L^\infty([0,\pi];\mathrm{GL}_s(\mathbb{R}^2))$.*

Proof. According to [**13**], each eigenvalue $\lambda_j(B)$ satisfies $\lambda_j(B) = 1/\mu_j(B)$, where

$$\mu_j(B) = \sup_{F_j} \inf \left\{ \int_0^\pi |u|^2 : \|u\|_{a_B} = 1,\ u \in F_j \right\},$$

where F_j varies over all j-dimensional subspaces of $H_0^1([0,\pi];\mathbb{R}^2)$ and $\|\cdot\|_{a_B}$ is the norm associated with the inner product

$$(u,v)_{a_B} = \int_0^\pi [u'(t)\cdot v'(t) - B(t)u(t)\cdot v(t)]\,\mathrm{d}t.$$

The result follows from the fact that, given $\varepsilon > 0$, $B \in L^1([0,\pi];\mathrm{GL}_s(\mathbb{R}^2))$, $B < 0$ and $j \in \mathbb{N}$, there exists a positive constant $\delta = \delta(\varepsilon, B, j)$ such that, for each $B_1 \in L^1([0,\pi];\mathrm{GL}_s(\mathbb{R}^2))$ with $B_1 < 0$ and $\|B - B_1\|_{L^1} < \delta$, for each j-dimensional subspace F_j of $H_0^1([0,\pi];\mathbb{R}^2)$ and for each $u \in F_j$ with $\|u\|_{a_B} = 1$ (or $\|u\|_{a_{B_1}} = 1$), there exists $v \in F_j$ with $\|v\|_{a_{B_1}} = 1$ (respectively, $\|v\|_{a_B} = 1$) such that

$$\left| \int_0^\pi |u|^2 - \int_0^\pi |v|^2 \right| < \varepsilon.$$

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To prove this, for every F_j we can choose an orthonormal basis with respect to the new inner product $(\cdot,\cdot)_{a_B}$, ϕ_i, $i=1,\dots,j$. Taking into account the equivalence between $\|\cdot\|_{a_B}$ and the usual norm of the Hilbert space $H_0^1([0,\pi];\mathbb{R}^2)$ (see [**15**] and the references therein), it is easy to see that $|(\phi_i,\phi_k)_{a_{B_1}}-1|$ and $|(\phi_i,\phi_k)_{a_{B_1}}|$ are small if $\|B-B_1\|_{L^1}$ is small, whenever $i,k\in\{1,\dots,j\}$, $i\neq k$. Thus, for each $u=\sum_{i=1}^{j}c_i\phi_i$ we can choose

$$v=\sum_{i=1,\,i\neq k}^{j}c_i\phi_i+(c_k+\eta)\phi_k,$$

for an adequate k and a sufficiently small η. □

Corollary 2.5. *Let $j=1,2,\dots$. For a fixed $M>0$, $B\to\lambda_j(B)$ is continuous on $\{B\in L^\infty([0,\pi];\mathrm{GL}_s(\mathbb{R}^2)):\|B(t)\|<M$ for a.e. $t\in(0,\pi)\}$ with respect to the topology induced by $L^1([0,\pi];\mathrm{GL}_s(\mathbb{R}^2))$ on $L^\infty([0,\pi];\mathrm{GL}_s(\mathbb{R}^2))$.*

Proof. Consider $B\in L^1([0,\pi];\mathrm{GL}_s(\mathbb{R}^2))$ satisfying $\|B(t)\|<M$ for a.e. $t\in(0,\pi)$. It immediately follows that $|B(t)x\cdot x|\leqslant|B(t)x|\,|x|<M|x|^2$ for every $x\in\mathbb{R}^2$ and for a.e. $t\in(0,\pi)$. This implies that $B<MI_2$. According to [**13,15**], each eigenvalue can be expressed by the relation

$$\lambda_j(B)=\frac{1}{\tilde{\mu}_j(B)}-M,$$

where

$$\tilde{\mu}_j(B)=\mu_j(B^*)=\sup_{F_j}\inf\left\{\int_0^\pi|u|^2:\|u\|_{a_{B^*}}=1,\ u\in F_j\right\},$$

with $B^*(t)=B(t)-MI_2$. By combining the continuous dependence of $\mu_j(B^*)$ with respect to B^* ensured by the previous proposition with the continuity of the map $B\to B^*$ from $L^1([0,\pi];\mathrm{GL}_s(\mathbb{R}^2))$ into itself, we prove the claim. □

The next result concerns the possibility of considering continuous branches of eigenvectors when the equation depends continuously on a parameter. We state it in the case of the zero eigenvalue but it is still valid if we consider eigenvalues depending continuously on a parameter. A similar result can be found in [**22**]. In the statement we denote by S^1 the circle of centre at the origin and radius 1 in the plane.

Proposition 2.6. *Let $\mathcal{C}$ be a continuum of $\mathbb{R}^2$ and assume that $B:[0,\pi]\times\mathcal{C}\to\mathrm{GL}_s(\mathbb{R}^2)$ is continuous. Suppose that, for each $\alpha\in\mathcal{C}$, zero is an eigenvalue of problem*

$$\left.\begin{aligned}x''+(B(t,\alpha)+\lambda I_2)x=0,\quad x\in\mathbb{R}^2,\ t\in(0,\pi),\\ x(0)=x(\pi)=0,\end{aligned}\right\}\tag{2.6}$$

and that there exists $(a,b)\in S^1$ such that for each $\alpha\in\mathcal{C}$ the solution of $x''+B(t,\alpha)x=0$ which satisfies $x(0)=(0,0)$ and $x'(0)=(a,b)$ does not vanish at $t=\pi$. Then, we can choose a continuous function from $\mathcal{C}$ to $(C^1([0,\pi],\mathbb{R}^2))^2$, $\alpha\to(v_\alpha(\cdot),v'_\alpha(\cdot))$, such that, for each α, v_α is an eigenvector of (2.6) associated with the zero eigenvalue.

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Proof. Consider the solutions $U^i(\cdot\,,\alpha) : [0,\pi] \times \mathcal{C} \to \mathbb{R}^4$, $i = 1, 2$, of

$$\left.\begin{aligned} x' &= y, \\ y' &= -B(t,\alpha)x \end{aligned}\right\} \tag{2.7}$$

satisfying $U^1(0,\alpha) = (0,0,a,b)$ and $U^2(0,\alpha) = (0,0,-b,a)$. We will denote by $U^i_j(\cdot\,,\alpha)$ the jth component of $U^i(\cdot\,,\alpha)$.

We want to construct a continuous function $\alpha \to (v_\alpha(\cdot), v'_\alpha(\cdot))$ such that, for each α, $(v_\alpha(\cdot), v'_\alpha(\cdot))$ satisfies (2.7), $v_\alpha(\cdot)$ is not identically zero and $v_\alpha(0) = v_\alpha(\pi) = 0$. That is, for each α, $(v_\alpha(\cdot), v'_\alpha(\cdot))$ will be a non-zero linear combination of $U^i(\cdot\,,\alpha)$, $i = 1, 2$, satisfying $v_\alpha(\pi) = 0$.

Let us recall that, by assumption, $(U^1_3(0,\alpha), U^1_4(0,\alpha)) = (a,b)$ for each $\alpha \in \mathcal{C}$. This implies that $(U^1_1(\pi,\alpha), U^1_2(\pi,\alpha)) \neq (0,0)$ for each $\alpha \in \mathcal{C}$.

As a consequence of the theorems on continuous dependence on parameters (see, for example, [**19**]), the functions U^i are continuous in α.

We now choose

$$c_1(\alpha) := -\frac{U^1_1 U^2_1 + U^1_2 U^2_2}{(U^1_1)^2 + (U^1_2)^2}(\pi,\alpha) \quad \text{and} \quad c_2(\alpha) := 1.$$

By the remark above, c_1 and c_2 are well defined and are continuous on α.

Finally, let us set $(v_\alpha(t), v'_\alpha(t)) = c_1(\alpha)U^1(t,\alpha) + c_2(\alpha)U^2(t,\alpha)$.

Note that the continuity of $\alpha \to (v_\alpha(\cdot), v'_\alpha(\cdot))$ is guaranteed. Since, by assumption, zero is an eigenvalue of problem (2.6) for every $\alpha \in \mathcal{C}$, it follows that $(U^1_1 U^2_2)(\pi,\alpha) = (U^2_1 U^1_2)(\pi,\alpha)$, implying $v_\alpha(\pi) = 0$ for every $\alpha \in \mathcal{C}$. To complete the proof, it remains to show that $v_\alpha(\cdot)$ is not identically zero for each $\alpha \in \mathcal{C}$ or, equivalently, that $v'_\alpha(0)$ never vanishes. This is a consequence of the fact that $v'_\alpha(0) = c_1(\alpha)(a,b) + (-b,a)$ and that (a,b) and $(-b,a)$ are linearly independent. $\square$

Now we state two preliminary lemmas which will be important for the proof of the main result.

Lemma 2.7. *Consider the problem*

$$\left.\begin{aligned} x'' + B(t)x = 0, &\quad t \in (0,\pi), \\ x(0) = x(\pi) = 0, & \end{aligned}\right\} \tag{2.8}$$

where $B \in L^\infty([0,\pi]; \mathrm{GL}_s(\mathbb{R}^2))$ *and*

$$B(t) = \begin{bmatrix} b_{11}(t) & b_{12}(t) \\ b_{12}(t) & b_{22}(t) \end{bmatrix}.$$

Assume that $b_{11}(t) < 0$ *and* $b_{22}(t) < 0$ *for every* $t \in [0,\pi]$. *Then we have that if* $b_{12}(t) \leqslant 0$ *(or* $b_{12}(t) \geqslant 0$*) for every* $t \in [0,\pi]$, *there are no non-trivial solutions of the Dirichlet problem (2.8) such that* $x'(0)$ *lies in the first or the third (respectively, second or fourth) quadrant.*

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Proof. Assume that $b_{12}(t) \leqslant 0$ for every $t \in [0,\pi]$. We are first interested in proving the strict monotonicity of each component of every solution $x = (x_1, x_2)$ to the problem

$$\left.\begin{aligned} x_1'' &= -b_{11}(t)x_1 - b_{12}(t)x_2, \\ x_2'' &= -b_{12}(t)x_1 - b_{22}(t)x_2, \\ x(0) &= 0, \end{aligned}\right\} \tag{2.9}$$

whenever $x_1'(0)x_2'(0) > 0$.

Suppose that $x_i'(0) > 0$ for each $i \in \{1,2\}$. This implies that there exists $\delta > 0$ such that $x_i(t) > 0$ for each $t \in (0,\delta]$ and for each $i \in \{1,2\}$. By the sign assumption, it can immediately be seen that $x_i''(t)$ is positive and, consequently, $x_i'(t) > x_i'(0) > 0$ for every $t \in (0,\delta]$, $i \in \{1,2\}$. In particular, as long as x_1'' and x_2'' remain positive, x_1 and x_2 keep on increasing. This allows us to conclude that each component of x'' never vanishes in $\mathbb{R}^+$ if $x = (x_1, x_2)$ is a solution of (2.9) satisfying $x_i'(0) > 0$ for each $i \in \{1,2\}$. Thus, x_1 and x_2 are strictly increasing in $\mathbb{R}^+$.

Consider now a solution $x = (x_1, x_2)$ of (2.9) with $x_i'(0) < 0$ for each $i \in \{1,2\}$. As the problem (2.9) is linear, $-x$ is also a solution. By the previous step, it follows that $-x_i' > 0$ in $\mathbb{R}^+$ for each $i \in \{1,2\}$ and, consequently, x_1 and x_2 are strictly decreasing in $\mathbb{R}^+$.

We have therefore proved that the problem (2.8) does not admit any solution $x = (x_1, x_2)$ satisfying $x_1'(0)x_2'(0) > 0$.

Our next aim consists in showing that there are no non-trivial solutions x of the Dirichlet problem (2.8) with $x_h'(0) = 0$, h fixed in $\{1,2\}$. Let $\tilde{x}$ be the solution of (2.9) verifying $\tilde{x}_h'(0) = 0$, $\tilde{x}_k'(0) \neq 0$, with $h \neq k$, $h,k \in \{1,2\}$. We want to prove that $\tilde{x}(\pi) \neq 0$. By the linearity of the problem, it is not a restriction to assume that $\tilde{x}_k'(0) > 0$. Moreover, for every $\varepsilon > 0$, let us consider the solution $x_\varepsilon = (x_{\varepsilon,1}, x_{\varepsilon,2})$ to the Cauchy problem (2.9) with $x_{\varepsilon,h}'(0) = \varepsilon > 0$ and $x_{\varepsilon,k}'(0) = \tilde{x}_k'(0) > 0$. By the theorem of continuous dependence of the solutions to Cauchy problems with respect to the initial data, we can deduce that $(x_\varepsilon, x_\varepsilon')$ tends uniformly to $(\tilde{x}, \tilde{x}')$ on the interval $[0,\pi]$ as ε tends to 0. As by the previous step each component of x_ε' is positive in $[0,\pi]$, we deduce that $\tilde{x}_i'(t) \geqslant 0$ for every $t \in [0,\pi]$, $i \in \{1,2\}$. From the fact that $\tilde{x}_k'(0) > 0$, it follows that $\tilde{x}(\pi) \neq (0,0)$.

This completes the proof under the assumption $b_{12}(\cdot) \leqslant 0$ on $[0,\pi]$.

The case involving opposite inequalities can be treated in an analogous way. □

In order to state the other preliminary lemma, we consider the Cauchy problem

$$\begin{aligned} x'' + A(t,x)x &= 0, \\ x(0) &= 0, \\ x'(0) &= \alpha \end{aligned}$$

associated with the system in (2.1). For each $\alpha \in \mathbb{R}^2$, we denote by x_α its unique solution.

We now concentrate on the linear, parameter-dependent equation

$$x'' + A(t, x_\alpha(t))x(t) = 0 \tag{2.10}$$

with $\alpha \in \mathbb{R}^2 \setminus \{(0,0)\}$.

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In [**8**], where equation (2.10) is considered, a relation is established between the initial data α of the Cauchy problem and the behaviour of the parameter-dependent matrix introduced above, whenever the asymptotically linear assumptions (2.2) and (2.3) are verified. In particular, the following lemma holds.

Lemma 2.8 (Capietto *et al.* [8]). *Suppose that the continuous function $A : [0,\pi] \times \mathbb{R}^2 \to \mathrm{GL}_s(\mathbb{R}^2)$ satisfies assumptions (2.2) and (2.3). Then*

$$\begin{aligned} A(t, x_\alpha(t)) &\to A_\infty(t) \quad \text{in } L^1([0,\pi]) \text{ if } |\alpha| \to +\infty, \\ A(t, x_\alpha(t)) &\to A_0(t) \quad \text{in } L^1([0,\pi]) \text{ if } |\alpha| \to 0. \end{aligned}$$

Note that the above lemma was used in [**8**] in order to obtain multiplicity of solutions to asymptotically linear vectorial problems.

Proof of Theorem 2.3. Let us assume that $i(A_0) > i(A_\infty)$; the other case can be treated similarly. By the definition of index there are exactly $i(A_0)$ negative eigenvalues $\lambda_l(A_0)$, $l \in \{1, \dots, i(A_0)\}$. Also there are exactly $i(A_\infty)$ negative eigenvalues $\lambda_j(A_\infty)$, $j \in \{1, \dots, i(A_\infty)\}$. Moreover, from the further assumption that $\nu(A_\infty) = 0$ we obtain that $\lambda_j(A_\infty)$ is positive for every $j \in \mathbb{N}$ with $j \geqslant i(A_\infty) + 1$.

Consider now $h \in \mathbb{N}$ satisfying $i(A_0) \geqslant h \geqslant i(A_\infty) + 1$. From the monotonicity properties of the sequence of eigenvalues, we immediately deduce that

$$\lambda_h(A_0) < 0 < \lambda_h(A_\infty). \tag{2.11}$$

We now concentrate on the study of the parameter-dependent problem

$$\left.\begin{aligned} x'' + A(t, x_\alpha(t))x(t) = 0, \\ x(0) = x(\pi) = 0. \end{aligned}\right\} \tag{2.12}$$

Assume that $a_{12}(t,x) \geqslant 0$ for every $(t,x) \in [0,\pi] \times \mathbb{R}^2$. Lemma 2.7 ensures that there are no solutions of the Dirichlet problem (2.12) such that $x'(0)$ lies in the second or the fourth quadrant. Let $\mathcal{Q}_1 := [0,+\infty) \times [0,+\infty)$ and $\mathcal{Q}_3 := (-\infty,0] \times (-\infty,0]$ denote the first and the third quadrant, respectively.

Our next aim consists in proving the existence of $\alpha_{i,h} \in \mathcal{Q}_i \setminus \{(0,0)\}$, $i \in \{1,3\}$, such that $\lambda_h(A(\cdot\,, x_{\alpha_{i,h}}(\cdot))) = 0$ and $x_{\alpha_{i,h}}(\pi) = 0$. We will focus on the search for $\alpha_{1,h} \in \mathcal{Q}_1 \setminus \{(0,0)\}$; the case $\alpha_{3,h} \in \mathcal{Q}_3 \setminus \{(0,0)\}$ can be treated analogously.

By combining Lemma 2.8 and Corollary 2.5 with the inequalities (2.11), we obtain

$$\lim_{|\alpha| \to 0} \lambda_h(A(\cdot\,, x_\alpha(\cdot))) < 0 < \lim_{|\alpha| \to +\infty} \lambda_h(A(\cdot\,, x_\alpha(\cdot))). \tag{2.13}$$

Hence, we can choose $0 < R_1 < R_2$ such that $\lambda_h(A(\cdot\,, x_\alpha(\cdot))) < 0$ for every $\alpha \in \mathcal{Q}_1$ with $|\alpha| = R_1$ and $\lambda_h(A(\cdot\,, x_\alpha(\cdot))) > 0$ for every $\alpha \in \mathcal{Q}_1$ with $|\alpha| = R_2$.

From an application of the theorems on continuous dependence on initial data, we can deduce the continuity of the map $\gamma : \mathbb{R}^2 \to C([0,\pi], \mathrm{GL}_s(\mathbb{R}^2))$, defined by $\gamma(\alpha) := A(\cdot\,, x_\alpha(\cdot))$. Therefore, also taking into account Corollary 2.5, we have that $g : [R_1, R_2] \times$

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$[0,\frac{1}{2}\pi]\to\mathbb{R}$ defined by $g(r,\theta)=\lambda_h(A(\cdot\,,x_{(r\cos(\theta),r\sin(\theta))}(\cdot)))$ is a continuous function. As, for each θ, $g(R_1,\theta)<0<g(R_2,\theta)$, we have that $\deg(g(\cdot\,,0),(R_1,R_2),0)\neq 0$, where we denote by 'deg' the Brower degree, and also that $g(r,\theta)\neq 0$ if $r=R_1$ or $r=R_2$. Hence, using the Leray–Schauder continuation theorem [**26**, Théorème Fondamental] we infer the existence of a closed connected set $\mathcal{C}^*\subset\{(r,\theta)\in(R_1,R_2)\times[0,\frac{1}{2}\pi]:g(r,\theta)=0\}$ such that $\mathcal{C}^*\cap([R_1,R_2]\times\{0\})\neq\emptyset$ and $\mathcal{C}^*\cap([R_1,R_2]\times\{\frac{1}{2}\pi\})\neq\emptyset$. Thus, we may infer the existence of a closed connected set $\mathcal{C}\subset\mathcal{Q}_1\setminus\{(0,0)\}$ such that $\mathcal{C}\cap(\{0\}\times\mathbb{R}^+)\neq\emptyset$, $\mathcal{C}\cap(\mathbb{R}^+\times\{0\})\neq\emptyset$ and

$$\lambda_h(A(\cdot\,,x_\alpha(\cdot)))=0\quad\forall\alpha\in\mathcal{C}.$$

Let us now prove that all the assumptions of Proposition 2.6, considering $B:[0,\pi]\times\mathcal{C}\to\mathrm{GL}_s(\mathbb{R}^2)$ defined by $B(t,\alpha)=A(t,x_\alpha(t))$, are satisfied. From the continuity of γ, it easily turns out that the map $B(t,\alpha)$ is continuous too. Finally, by combining Lemma 2.7 with assumptions (2.5) we deduce that for every $\alpha\in\mathbb{R}^2$ there are no solutions $\phi_\alpha=(\phi_{1,\alpha},\phi_{2,\alpha})$ to the Dirichlet problem (2.12) with $\phi'_{1,\alpha}(0)=0$. Note that this implies that all the solutions of $x''+A(t,x_\alpha(t))x(t)=0$ satisfying $x(0)=(0,0)$ and $x'(0)=(0,1)$ do not vanish at $t=\pi$ (and hence the space of eigenvectors associated with a zero eigenvalue has dimension 1). We can now apply Proposition 2.6 and conclude the existence of a continuous function defined on $\mathcal{C}$, $\alpha\to(v_\alpha(\cdot),v'_\alpha(\cdot))$, such that, for each α, v_α is an eigenvector of $x''+(A(\cdot\,,x_\alpha(\cdot))+\lambda I_2)x=0$, $x(0)=x(\pi)=0$, associated with the zero eigenvalue.

For each $\alpha\in\mathcal{C}$ we can set $\beta(\alpha):=v'_\alpha(0)\in\mathbb{R}^2\setminus\{(0,0)\}$; hence, v_α is a non-trivial solution of the system

$$\left.\begin{aligned}x''+A(t,x_\alpha(t))x=0,\\ x(0)=x(\pi)=0\end{aligned}\right\}\tag{2.14}$$

satisfying $x'(0)=\beta(\alpha)$.

Taking into account Lemma 2.7 and the fact that $a_{12}(t,x)\geqslant 0$ for every $(t,x)\in[0,\pi]\times\mathbb{R}^2$, we note that $\beta(\alpha)=(\beta_1(\alpha),\beta_2(\alpha))\in\mathcal{Q}_1\cup\mathcal{Q}_3$ and $\beta_1(\alpha)\beta_2(\alpha)\neq 0$. Since the problem (2.14) is linear, we can restrict ourselves to the case when $\beta(\alpha)\in\mathcal{Q}_1\setminus\{0,0\}$.

Now we prove that for some $\bar\alpha\in\mathcal{C}$ there exists $C>0$ such that $\beta(\bar\alpha)=C\bar\alpha$, from which we obtain $x_{\bar\alpha}=x_{\beta(\bar\alpha)}/C$ and, consequently, $x_{\bar\alpha}(\pi)=0$. In particular, we can choose $\alpha_{1,h}=\bar\alpha$.

Consider γ in the polar coordinates (ϑ,ρ) in the plane: $\gamma_1=\rho\cos\vartheta$, $\gamma_2=\rho\sin\vartheta$. Since the function $\alpha\mapsto\beta(\alpha)$ is continuous from $\mathcal{C}\subset\mathcal{Q}_1\setminus\{(0,0)\}$ to $\mathcal{Q}_1\setminus\{(0,0)\}$, the function $\alpha\mapsto\vartheta(\beta(\alpha))-\vartheta(\alpha)$ from $\mathcal{C}$ to $(-\frac{1}{2}\pi,\frac{1}{2}\pi)$ is also continuous.

There exist $\tilde\alpha=(0,\tilde\alpha_2),\hat\alpha=(\hat\alpha_1,0)\in\mathcal{C}$. Observe that $\vartheta(\beta(\tilde\alpha))-\vartheta(\tilde\alpha)<0$ and $\vartheta(\beta(\hat\alpha))-\vartheta(\hat\alpha)>0$. Hence, recalling that $\mathcal{C}$ is a connected set, we may infer the existence of $\bar\alpha\in\mathcal{C}$ such that $\vartheta(\beta(\bar\alpha))=\vartheta(\bar\alpha)$.

Arguing as above in the third quadrant, at the end we find $\alpha_{i,h}\in\mathcal{Q}_i\setminus\{(0,0)\}$ such that $\lambda_h(A(\cdot\,,x_{\alpha_{i,h}}(\cdot)))=0$ and $x_{\alpha_{i,h}}(\pi)=0$ for every $i\in\{1,3\}$. In particular, for each $i\in\{1,3\}$, $x_{\alpha_{i,h}}$ is a non-trivial solution of the Dirichlet problem (2.1), satisfying $\lambda_h(A(\cdot\,,x_{\alpha_{i,h}}(\cdot)))=0$, where h is an arbitrary natural number with $i(A_0)\geqslant h\geqslant i(A_\infty)+1$.

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To complete the proof of the case when $a_{12}(\cdot) \geqslant 0$ on $[0,\pi] \times \mathbb{R}^2$, it remains to show that all the values $\alpha_{i,h}$ that we found above are pairwise different, or, equivalently, that all the solutions of the form $x_{\alpha_{i,h}}$ are mutually different.

Assume, by contradiction, that there exist two natural numbers $h, k \in [i(A_\infty)+1, i(A_0)]$ with $h \neq k$ such that $\alpha_{i,h} = \alpha_{i,k}$. Let us set $\tilde{\alpha} := \alpha_{i,h} = \alpha_{i,k}$. In this case $\lambda_h(A(\cdot\,, x_{\tilde{\alpha}}(\cdot))) = \lambda_k(A(\cdot\,, x_{\tilde{\alpha}}(\cdot))) = 0$ and this contradicts the fact that under our assumptions the space of eigenvectors associated with the zero eigenvalue has dimension 1.

Since the case when $a_{12}(t,x) \leqslant 0$ for every $(t,x) \in [0,\pi] \times \mathbb{R}^2$ is similar to that above, we omit the corresponding proof. $\square$

Remark 2.9. By using arguments analogous to those used in this paper, the existence of multiple solutions can also be obtained for the scalar, Dirichlet problem

$$x''(t) + A(t, x(t))x(t) = 0, \quad x(0) = 0 = x(\pi),$$

where $A : [0,\pi] \times \mathbb{R} \to \mathbb{R}$ is continuous, satisfies the asymptotically linear conditions at the origin (2.2) and at infinity (2.3) and is such that uniqueness of solutions of Cauchy problems associated with the above equation is guaranteed. The multiplicity results which we are able to obtain in this scalar setting coincide with well-known results concerning asymptotically linear Dirichlet scalar problems (see, for example, [**12**, **31**] and references therein). We point out that in the literature more general nonlinearities have been studied and multiplicity of solutions has been proved also without uniqueness assumptions on the solutions of the initial-value problems.

The following remarks are devoted to possible extensions of Theorem 2.3 to more general contexts. Both the generalizations stated below can easily be proved by following procedures analogous to that used to prove our main result.

Remark 2.10. In Theorem 2.3, instead of condition (2.5) we could have imposed other kinds of condition which guarantee the result of Lemma 2.7.

In particular, the conclusion of Theorem 2.3 holds true if we replace the condition (2.5) with

$$a_{11}(t,x) \leqslant 0, \quad a_{22}(t,x) \leqslant 0 \quad \text{and} \quad a_{12}(t,x) \neq 0 \quad \forall (t,x) \in [0,\pi] \times \mathbb{R}^2.$$

Remark 2.11. Note that by removing the assumption $\nu(A_\infty) = 0$ (or $\nu(A_0) = 0$) in the statement of Theorem 2.3 we can prove the existence of at least $2|i(A_0) - i(A_\infty)| - 4$ non-trivial solutions to problem (2.1), provided that we assume the positivity of the value $|i(A_0) - i(A_\infty)| - 2$.

Remark 2.12. Assume that there exists $(a,b) \in S^1$ such that for every continuous function $g : [0,\pi] \to \mathbb{R}^2$ there are no solutions of the Dirichlet problem

$$\begin{gathered} x'' + A(t, g(t))x = 0, \quad t \in (0,\pi), \\ x(0) = x(\pi) = 0, \end{gathered}$$

satisfying $x'(0) = (a,b)$. Then the conclusion of Theorem 2.3 still holds.

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Theorem 2.3 holds true if we generalize the asymptotically linear conditions (2.2) and (2.3) by assuming the existence of $A_1, A_2, B_1, B_2 \in C^0([0,\pi]; \mathrm{GL}_s(\mathbb{R}^2))$ such that

$$B_1(t)z \cdot z \leqslant \liminf_{|x|\to 0} A(t,x)z \cdot z \leqslant \limsup_{|x|\to 0} A(t,x)z \cdot z \leqslant B_2(t)z \cdot z, \tag{2.15}$$

$$A_1(t)z \cdot z \leqslant \liminf_{|x|\to \infty} A(t,x)z \cdot z \leqslant \limsup_{|x|\to \infty} A(t,x)z \cdot z \leqslant A_2(t)z \cdot z \tag{2.16}$$

uniformly in $t \in [0,\pi]$ and $z \in \mathbb{R}^2$. In particular, the statement of Theorem 2.3 can be extended into the following.

Corollary 2.13. *Assume that $A(t,x)$ satisfies (2.5), (2.15) and (2.16).*

Then if $i(B_1) > i(A_2)$ and $\nu(A_2) = 0$ (or $i(A_1) > i(B_2)$ and $\nu(B_2) = 0$), the problem (2.1) has at least $2(i(B_1) - i(A_2))$ (or $2(i(A_1) - i(B_2))$) non-trivial solutions.

Sketch of the proof. The first step of the proof consists in generalizing Lemma 2.8 by adopting arguments analogous to that used in the proof of [**8**, Proposition 4.4] and by taking into account the fact that, under the assumptions of Corollary 2.13, $\|A\|$ is bounded. More precisely, we prove that, for every sequence $\alpha_n \in \mathbb{R}^2$ satisfying $\lim_{n\to+\infty} |\alpha_n| = +\infty$ and for a.e. $t \in [0,\pi]$, the inequalities

$$A_1(t)z \cdot z \leqslant \liminf_{n\to+\infty} A(t, x_{\alpha_n}(t))z \cdot z \leqslant \limsup_{n\to+\infty} A(t, x_{\alpha_n}(t))z \cdot z \leqslant A_2(t)z \cdot z \tag{2.17}$$

hold uniformly in $z \in \mathbb{R}^2$. By using the Fatou lemma, Lebesgue's dominated convergence theorem and the boundedness of the matrix $A(t,x)$, one can pass from (2.17) to integral inequalities. More precisely, for every sequence $z_n \in L^\infty([0,\pi]; \mathbb{R}^2)$ with $\lim_{n\to+\infty} z_n = z_0$ in $\|\cdot\|_\infty$, we get

$$\int_0^\pi A_1(t)z_0(t) \cdot z_0(t)\,\mathrm{d}t \leqslant \liminf_{n\to+\infty} \int_0^\pi A(t, x_{\alpha_n}(t))z_n(t) \cdot z_n(t)\,\mathrm{d}t$$

and

$$\limsup_{n\to+\infty} \int_0^\pi A(t, x_{\alpha_n}(t))z_n(t) \cdot z_n(t)\,\mathrm{d}t \leqslant \int_0^\pi A_2(t)z_0(t) \cdot z_0(t)\,\mathrm{d}t.$$

By using the same procedure, from (2.15) it is possible to deduce integral inequalities analogous to the one above, in which α_n is replaced by $\beta_n \to 0$ as $n \to +\infty$ and where A_i is replaced by B_i for each $i \in \{1,2\}$.

The final steps of the proof are based on a generalized Sturm comparison result contained in [**15**] and, in particular, on its Proposition 2.6, where it is proved that $i(B) \leqslant i(C)$ if $B(t) \leqslant C(t)$ for a.e. $t \in (0,\pi)$ and $i(B) + \nu(B) \leqslant i(C)$ if $B < C$, whenever $B, C \in L^\infty([0,\pi]; \mathrm{GL}_s(\mathbb{R}^2))$.

By combining this result with the continuity of the eigenvalues proved in Corollary 2.5, it is easy to show that

$$\exists \varepsilon_0 > 0 : \forall \varepsilon \in (0,\varepsilon_0],\ i(B - \varepsilon I_2) = i(B) \text{ and } i(B + \varepsilon I_2) = i(B) + \nu(B). \tag{2.18}$$

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Taking into account the techniques used to prove Proposition 2.6 in [**15**], the integral inequalities exhibited above and (2.18), we prove that

$$\left.\begin{aligned}\exists R>0:\forall\alpha\in\mathbb{R}^2,\ |\alpha|>R,\quad i(A_1)\leqslant i(A(\cdot\,,x_\alpha(\cdot)))\leqslant i(A_2),\\ \exists\delta>0:\forall\alpha\in\mathbb{R}^2,\ |\alpha|<\delta,\quad i(B_1)\leqslant i(A(\cdot\,,x_\alpha(\cdot)))\leqslant i(B_2).\end{aligned}\right\} \tag{2.19}$$

Let us now concentrate on the case in which $i(B_1)>i(A_2)$ and $\nu(A_2)=0$. Consider $h\in\mathbb{N}$ satisfying $i(B_1)\geqslant h\geqslant i(A_2)+1$. According to (2.19), it turns out that

$$\forall\alpha\in\mathbb{R}^2,\ |\alpha|>R:\lambda_h(A(\cdot\,,x_\alpha(\cdot)))>0\quad\text{and}\quad\forall\alpha\in\mathbb{R}^2,\ |\alpha|<\delta:\lambda_h(A(\cdot\,,x_\alpha(\cdot)))<0.$$

This relation recalls the relation (2.13) on which the proof of Theorem 2.3 is based. The claim follows by proceeding as in the proof of our main theorem. □

Acknowledgements. F.D. was supported by the FCT Grant no. SFRH-BPD-16830-2004 and C.R. was supported by FCT Project no. POCI/Mat/57258/2004.

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