J. Austral. Math. Soc. **23** (Series A) (1977), 499–503.

GROUPS WHICH ARE AN INFINITE CYCLIC EXTENSION OF A UNIQUE BASE GROUP

A. M. BRUNNER, JAMES McCOOL and ALFRED PIETROWSKI

(Received 6 July 1976; revised 7 October 1976)

Let G be a group which has exactly one normal subgroup N with G/N infinite cyclic; clearly such a group is an *HNN* group with base N and stable letter t, where t induces an automorphism θ_t of N under conjugation. We call such a group G a *unique base group* with automorphism θ_t and base N.

Theorem 1 of this paper shows that the isomorphism problem for unique base groups is equivalent to a conjugacy problem in the outer automorphism group of the base group.

Any one-relator group with a finitely generated commutator subgroup is a two-generator unique base group, with base a finitely generated free group (Moldavanskiĭ (1967)). In Theorem 2 we establish some criteria for the isomorphism of such groups. We apply this, in Theorem 3, to generalise a result of Strasser (1959). We determine all two-generator one-relator groups which are an extension of a free group of rank two by an infinite cyclic group.

Let w and v be elements of a group G; then w^v will denote the element $v^{-1}wv$, and $\langle w \rangle^G$ the least normal subgroup of G which contains w.

1. An isomorphism theorem

We denote by $A(N)$ the automorphism group of the group N, and by $I(N)$ the group of inner automorphisms of N; the quotient $A(N)/I(N)$ is $\mathrm{Out}(N)$ the outer automorphism group of N.

THEOREM 1. *Let C and E be unique base groups with stable letters c and e respectively and the same base N. Then C and E are isomorphic if, and only if, θ_c is conjugate to $\theta_e^{\pm 1}$ in* $\mathrm{Out}(N)$.

PROOF. Let φ be an isomorphism from C to E. Then φ induces an automorphism γ of N. Also, since C and E have stable letters c and e

Research partially supported by the C.S.I.R.O. of Australia and by a grant from the National Research Council of Canada.

respectively, $c\varphi = e^{\varepsilon}h$ where $\varepsilon = \pm 1$ and h belongs to N. We show that $\gamma^{-1}\theta_c\gamma = \theta_e^{\varepsilon}\tau$ where τ is the automorphism in $I(N)$ which corresponds to conjugation by h.

But, for each n in N

$$n(\gamma^{-1}\theta_c\gamma) = (c^{-1}(n\varphi^{-1})c)\gamma = (e^{\varepsilon}h)^{-1}n(e^{\varepsilon}h) = n\theta_e^{\varepsilon}\tau.$$

Hence θ_c is conjugate to θ_e^{ε} in Out(N).

Conversely, suppose θ_c and θ_e^{ε} are conjugate in Out(N). Then there is a τ in $I(N)$ and γ in $A(N)$ such that $\gamma^{-1}\theta_c\gamma = \theta_e^{\varepsilon}\tau$. Let τ correspond to conjugation by h in N.

Now $C = \langle c, N; \mathrm{rel}\, N, c^{-1}nc = n\theta_c\ (n \in N)\rangle$, and $E = \langle e, N; \mathrm{rel}\, N, e^{-1}ne = n\theta_e\ (n \in N)\rangle$. Moreover, every element in C can be uniquely expressed in the form $c^r n$, where r is an integer and n belongs to N. If φ is the map from C to E defined by $(c^r n)\varphi = (e^{\varepsilon}h)^r(n\gamma)$ then it is easy to check, using the above descriptions of C and E, that φ is an isomorphism. □

We will require the notion of Nielsen equivalence of presentations in the next section: two presentations of a group, given as factor groups F/R and F/S, of a free group F are said to be Nielsen equivalent if there is an automorphism α of F with $R\alpha = S$.

Suppose G is a unique base group with automorphism θ_t and base N. If N' denotes the derived group of N, it is clear that G/N' is also a unique base group with automorphism $\tilde{\theta}_t$ (induced by θ_t on N/N') and base N/N'. When G is two-generator, G/N' has one T-system of generating pairs (Brunner (1974), Theorem 2.4), and thus G/N' has one Nielsen class of two-generator presentations.

2. Infinite cyclic extensions of free groups

Let $\underline{\underline{X}}(r)$ stand for the class of two-generator one-relator groups which are an extension of a free group of finite rank r by an infinite cyclic group.

As we shall see there is a close connection between the two-generator one-relator presentations of groups in $\underline{\underline{X}}(r)$ and certain "diagonal" presentations:

$$\Delta = \langle a, b; a^{b^r} = ua^{\varepsilon}u'\rangle,$$

where $\varepsilon = \pm 1$, and u, u' belong to the (free) group generated by $a^b, \cdots, a^{b^{r-1}}$.

LEMMA 1. *Let G be a group with a diagonal presentation Δ. Then $\langle a\rangle^G$ is a free group of rank r, and G belongs to $\underline{\underline{X}}(r)$.*

PROOF. Considerations such as those involved in Case 2 of Theorem 4.10

of Magnus, Karrass, Solitar (1966), show that $a, a^b, \cdots, a^{b^{r-1}}$ generate freely a free subgroup of $\langle a\rangle^G$. But, since $a^{b^r} = ua^\varepsilon u'$, it follows that $\langle a\rangle^G$ is actually generated by $a, a^b, \cdots, a^{b^{r-1}}$ and the assertion follows.

LEMMA 2. *Any two-generator one-relator presentation of a group in* $\underset{=}{X}(r)$ *is Nielsen equivalent to a diagonal one.*

This is a more general form of Lemma 1 of Strasser (1959), and its proof is exactly analogous to the one given there.

It follows that the isomorphism problem for two-generator one-relator presentations of groups in $\underset{=}{X}(r)$ reduces to the problem of determining when diagonal presentations define isomorphic groups.

Let G be a group with a diagonal presentation Δ; let $\sigma_i = \sigma_i(a, b)$ denote the exponent sum of $a^{-b^r}ua^\varepsilon u'$ on $a^{b^{i-1}}$ when $a^{-b^r}ua^\varepsilon u'$ is considered as a word in $a, a^b, \cdots, a^{b^r}$. Clearly G/G' is the direct product of an infinite cyclic group generated by bG', and (if $s \neq 0$) a cyclic group of order s generated by aG'; we call s the torsion number of G.

LEMMA 3. *Let G be a group with a diagonal presentation Δ and non-zero torsion number s. Then G is a unique base group with base $H = \langle a\rangle^G$ and automorphism θ_b.*

PROOF. Suppose M is a normal subgroup of G with G/M infinite cyclic. Then G' is contained in M; hence a belongs to M, since a^s belongs to G' and G/M has no elements of finite order. It follows that H is contained in M. But $G/M \cong (G/H)/(M/H)$ is an infinite cyclic group, as is G/H; therefore $M = H$. □

It is evident that G is also a unique base group with base $H = \langle a\rangle^G$ and automorphism $\theta_{b^{-1}}$; moreover $\theta_{b^{-1}} = (\theta_b)^{-1}$.

Now H/H' is a free abelian group, freely generated by $aH', a^bH', \cdots, a^{b^{r-1}}H'$. Also, if $\tilde{\theta}_b$ denotes the automorphism of H/H' induced by θ_b, then

$$(a^{b^i}H')\tilde{\theta}_b = \begin{cases} a^{b^{i+1}}H' & \text{if} \quad 0 \leqq i \leqq r-1 \\ a^{\sigma_1}a^{\sigma_2 b}\cdots a^{\sigma_r b^r}H' & \text{if} \quad i = r. \end{cases}$$

It follows that $\tilde{\theta}_b$, when considered as an element of $PSL(r, Z)$, has the characteristic polynomial

$$(-1)^r[\zeta^r - \sigma_r\zeta^{r-1} - \sigma_{r-1}\zeta^{r-2} - \cdots - \sigma_1].$$

THEOREM 2. *Let C and E be isomorphic groups in $\underset{=}{X}(r)$ with non-zero torsion number. If (d, c) and (f, e) are generating pairs of C and E, respectively, associated with diagonal presentations Δ, then $\sigma_k(d, c) = \sigma_k(f, e^\varepsilon)$ for $k = 1, 2, \cdots, r$ where $\varepsilon = \pm 1$.*

PROOF. Since C and E have non-zero torsion numbers they are unique base groups, by Lemma 3; as they are isomorphic we may assume they are given with base N and automorphisms θ_c and θ_e respectively.

Now N/N' is a free abelian group of rank r, and clearly $\mathrm{Out}(N/N') \cong A(N/N') \cong PSL(r, Z)$. Thus, if $\tilde{\theta}_c$ and $\tilde{\theta}_e$ denote the automorphisms induced on N/N' then, by Theorem 1, $\tilde{\theta}_c$ and $\tilde{\theta}_e^{\pm 1}$ are conjugate. Hence they have the same characteristic polynomial, and the result follows.

3. Infinite cyclic extensions of free groups of rank two

In this section we completely determine the one-relator presentations of groups in $\underset{=}{X}(2)$. This generalises work of Strasser (1959).

Let $P(k, \varepsilon)$ denote the presentation

$$P(k, \varepsilon) = \langle x, y ; x^{y^2} = x^{\varepsilon} x^{ky} \rangle,$$

where $\varepsilon = \pm 1$ and k is an integer.

LEMMA 4. *The group C with presentation $P(0, 1)$ is not isomorphic to the group E having presentation $P(2, -1)$.*

PROOF. Let V_1 and V_2 denote the verbal subgroups of C and E, respectively, generated by the third powers of the elements. Then, as can easily be checked, C/V_1 is the direct product of two cyclic groups of order 3, while E/V_2 is the free two-generator group of exponent 3 (which is non-abelian of order 27).

THEOREM 3. *Let G be a group in $\underset{=}{X}(2)$ with torsion number s. Then any two-generator one-relator presentation of G is Nielsen equivalent to one of $P(s, 1)$, $P(2+s, -1)$ or $P(2-s, -1)$. When s is positive these three groups are non-isomorphic; when $s = 0$ the last two coincide.*

PROOF. By Lemma 2 a two-generator one-relator presentation is Nielsen equivalent to

$$P = \langle x, y ; x^{y^2} = x^{ky} x^{\varepsilon} x^{ly} \rangle, \text{ where } \varepsilon = \pm 1 \text{ and } k, l \text{ are integers.}$$

Replacing y by $x^k y$ and leaving x fixed, we see that P is Nielsen equivalent to $P(n, \varepsilon)$ where $n = k + l$.

Suppose now that a group C with presentation $P(n_1, \varepsilon_1)$ is isomorphic to a group E with presentation $P(n_2, \varepsilon_2)$. Then, in particular, they have the same torsion number, so $n_1 + \varepsilon_1 - 1 = \pm(n_2 + \varepsilon_2 - 1)$. We dispense first with the case that $n_1 + \varepsilon_1 - 1 = n_2 + \varepsilon_2 - 1 = 0$: when $\varepsilon_1 = 1$ we have $n_1 = 0$; when $\varepsilon_1 = -1$ we have $n_1 = 2$. Moreover, by Lemma 4, $P(0, 1)$ and $P(2, -1)$ do not define isomorphic groups.

Suppose now that $n_1+\varepsilon_1-1=\pm(n_2+\varepsilon_2-1)\neq 0$. By Theorem 2 there are two cases to be considered:

(a) $\varepsilon_1=\sigma_1(d,c)=\sigma_1(f,e)=\varepsilon_2$ and $n_1=\sigma_2(d,c)=\sigma_2(f,e)=n_2$;

(b) $\varepsilon_1=\sigma_1(d,c)=\sigma_1(f^{-1},e)=\varepsilon_2$ and $n_1=\sigma_2(d,c)=\sigma_2(f^{-1},e)=-n_2\varepsilon_2$.

Consider case (b). If $n_1+\varepsilon_1-1=n_2+\varepsilon_2-1$ then, as $\varepsilon_1=\varepsilon_2$, we have $n_1=n_2$. If $n_1+\varepsilon_1-1=-(n_2+\varepsilon_2-1)$, as $\varepsilon_1=\varepsilon_2$ and $n_1=-n_2\varepsilon_2$, in the one case $\varepsilon_1=\varepsilon_2=-1$ implies $n_1=n_2$, and in the other, $\varepsilon_1=\varepsilon_2=1$ implies $n_1=-n_2$.

We conclude that either $\varepsilon_1=\varepsilon_2$ and $n_1=n_2$, or $\varepsilon_1=\varepsilon_2=1$ and $n_1=-n_2$. This later possibility may be neglected because $P(n,1)$ and $P(-n,1)$ are Nielsen equivalent under the transformation which replaces y by y^{-1} and fixes x. Thus, for a given $s=\pm(n+\varepsilon-1)$ there are three solutions; $\varepsilon=1$ and $n=s$, $\varepsilon=-1$ and $n=2+s$, $\varepsilon=-1$ and $n=2-s$.

References

A. M. Brunner (1974), 'Transitivity-systems of certain one-relator groups', *Proc. Second Internat. Conf. Theory of Groups*, Canberra 1973 (Lecture Notes in Mathematics, 372, 131–140. Springer-Verlag, Berlin, Heidelberg, New York, 1974).

Wilhelm Magnus, Abraham Karrass, Donald Solitar (1966), *Combinatorial Group Theory* (Pure and Appl. Math. 13. Interscience [John Wiley & Sons], New York, London, Sydney, 1966).

D. I. Moldavanskiĭ (1967), 'Certain subgroups of groups with one defining relation', *Siberian Math. J.* **8**, 1039–1048 (1968).

Elvira Strasser Rapaport (1959), 'Note on Nielsen transformations', *Proc. Amer. Math. Soc.* **10**, 228–235.

Department of Mathematics,
York University,
Downsview,
Ontario, Canada.

Department of Mathematics,
University of Toronto,
Toronto,
Canada.

Department of Mathematics,
York University,
Downsview,
Ontario, Canada.