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SCALAR ACTIONS

A. LEBOW AND M. SCHREIBER

1. Introduction. The subject of this paper arises from the familiar process whereby an automorphism of a field generates new representations from old. One may think of that process spatially, as a change of vector space structure in the representation space by means of the automorphism. The operators of the representation acting in the "new" space then constitute the new representation. This point of view makes visible an algebraic structure we call a *scalar action*. A scalar action $\mathbf{f}$ of a ring R (with unity) in an abelian group V is a ring homomorphism $\mathbf{f}:R \to \mathrm{End}(V)$ taking the unity element of R to the identity operator in $\mathrm{End}(V)$. If $\mathbf{f}$ is a scalar action of a field F and ϕ is an automorphism of F then $\mathbf{f} \circ \phi$ is another scalar action of F, and it is this construction which is used to define the "new" representation space mentioned above. But the variety of scalar actions goes rather beyond that construction. Indeed, we show here that the set of all scalar actions of a field is a union of homogeneous spaces determined by certain linear groups.

In a companion note [**4**] we apply the idea of scalar actions to a question in representation theory. The present paper is devoted to two topics, the extension problem for scalar actions of fields, and the structure of commuting pairs of scalar actions of fields. The extension problem asks for conditions that a scalar action in a group V of a field F be extendable to a scalar action in V of an extension field $K \supset F$. There is in fact just one condition, necessary and sufficient, namely that $[K:F]$ divide $[V:F]$, generalizing the fact that $\mathbf{R}^n$ admits complex structures only for even n. Following this out for arbitrary cardinals one gets a fully general description of the set of extensions to K. Since the prime fields can act in only one way, that description yields as well a description of the set of all actions of K.

Actions $\mathbf{g}$ and $\mathbf{h}$ in a group V are said to commute if the set of endomorphisms of V given by $\mathbf{g}$ all commute with all those given by $\mathbf{h}$. The set of commuting pairs of actions which are extensions to a field K of a given action in V of a subfield F is bijective with the set of all actions in V of the ring $K \otimes_F K$, and when K is separably finite over F, so that $K \otimes_F K$

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has a decomposition by primitive idempotents, we can compare the two actions on certain mutually invariant subgroups determined by the idempotents. This requires some information about the idempotents which may be new.

The comparison of actions on a mutually invariant subgroup suggests the comparison of two actions on a common "line" (orbit of a point under the action). This leads to the surprising result that two actions are related by composition with an automorphism of the acting field if and only if they commute and determine identical sets of lines. There is a resemblance between this result and the fundamental theorem of projective geometry as formulated by Artin. We hope to discuss this in a later note.

We use the following standard notation: $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$ for the rational, real, and complex fields; K^+ for the additive group of the field K; and $M_n(R)$ for the nxn matrices over $\mathbf{R}$. We write $\mathbf{id}$ for the identity map in all contexts.

We are greatly indebted to H. Bass for showing us the general form of our former version of the extension result, and particularly for pointing out the connection between commuting pairs of actions and the tensor square of the field. We thank also P. A. Griffiths for helpful conversations.

2. Scalar actions. Let R be a ring with unit, and V an additive abelian group. By a *scalar action* of R in V we mean a ring homomorphism $\mathbf{f}:R \rightarrow \operatorname{End}(V)$ taking $1 \in R$ to $I \in \operatorname{End}(V)$. We write $\mathbf{f}_c$ for the endomorphism corresponding to $c \in R$. In this notation the defining properties of a scalar action are

$$(1) \qquad \mathbf{f}_{c+d} = \mathbf{f}_c + \mathbf{f}_d, \mathbf{f}_{cd} = \mathbf{f}_c\mathbf{f}_d, \quad f_1 = I.$$

Whereas usually one views an R-module structure on V as a map $R \times X \rightarrow V$ with appropriate properties, we here view it as the corresponding map $R \rightarrow V^V$. This is merely a change of emphasis. We write $\mathbf{S}(V, R)$ for the set of all scalar actions of R in V. If R is a field then each $f \in \mathbf{S}(V, R)$ endows V with the structure of an R-vector space. We denote this vector space by $V_\mathbf{f}$. The vector group R^n admits the R-scalar action of scalar multiplication, which we denote by $\mathbf{sm}$. Thus $R^n_{\mathbf{sm}}$ is in this notation the standard n-tuple vector space over R. The set of all $T \in \operatorname{End}(V)$ which commute with f in the natural sense that $T\mathbf{f}_c = \mathbf{f}_cT$ for all $c \in R$ constitutes the space $L(V_\mathbf{f})$ of all linear operators of $V_\mathbf{f}$.

Let K be a field, let $\mathbf{f} \in \mathbf{S}(V, K)$ be given, and let $F \subset K$ be a subfield. Then $\mathbf{f}$ is also a scalar action of F in V, and the dimensions of the two spaces $V_\mathbf{f}/K$ and $V_\mathbf{f}/F$ satisfy

(2.1) $\quad [V_{\mathbf{f}}:K] \cdot [K:F] = [V_{\mathbf{f}}:F].$

This is an adaptation to the present circumstances of the theorem in field theory that if $L \supset K \supset F$ are fields then $[L:K][K:F] = [L:F]$. That theorem uses the product in L of elements of L and K, and in K of elements of K and F. Here we replace the former product by the $\mathbf{f}$-scalar "product" of vectors in V by elements of K, and then we may proceed as for fields.

Scalar actions $\mathbf{f}, \mathbf{g} \in \mathbf{S}(V, R)$ are said to *commute* if the subring of $\mathrm{End}(V)$ they together generate is commutative, which is to say $\mathbf{f}_c\mathbf{g}_d = \mathbf{g}_d\mathbf{f}_c$ for all $c, d \in R$. We abbreviate this relation by the expression $[\mathbf{f}, \mathbf{g}] = 0$.

$$\begin{tikzcd} K \arrow[dr, "\mathbf{f}"] \arrow[d, "\phi"'] & \\ L \arrow[r, "\mathbf{g}"'] & \mathrm{End}(V) \end{tikzcd}$$

Scalar actions $\mathbf{f} \in \mathbf{S}(V, K)$ and $\mathbf{g} \in \mathbf{S}(V, L)$ of rings K and L in V are said to be *equivalent* if there exists an isomorphism $\phi:K \to L$ such that $\mathbf{f} = \mathbf{g} \circ \phi$, which is to say, the diagram shown is commutative. It is clear that this is an equivalence relation. We denote it by $\mathbf{f} \simeq \mathbf{g}$.

Let E and K be fields, let $\mathbf{f} \in \mathbf{S}(V, K)$ be given, and suppose $\phi \in K^E$ is an arbitrary map of E to K. We claim that $\mathbf{f} \circ \phi$ is a scalar action of E in V $\mathbf{f} \circ \phi \in \mathbf{S}(V, E)$, if and only if ϕ is an isomorphism (not assumed surjective) of E to K. This follows from the fact that a scalar action of a field is injective (because its kernel is an ideal in the field). Thus if $\mathbf{f} \circ \phi$ is a scalar action then ϕ must be injective else $\mathbf{f} \circ \phi$ would not be, and ϕ must preserve sum and product else $\mathbf{f} \circ \varphi$ would not, by the injectivity of $\mathbf{f}$. This proves half the assertion, and the other half is trivial.

A scalar action $\mathbf{f} \in \mathbf{S}(V, K)$ can be restricted to a subfield $F \subset K$ or to an invariant subgroup $W \subset V$. We denote these restrictions by the upper and lower displays $\mathbf{f}|^F$ and $\mathbf{f}|_W$ respectively.

3. Extensions. Let V be an abelian group admitting scalar action by a field F, and let $K \supset F$ be an extension field. Given $f \in \mathbf{S}(V, F)$, we ask whether there exist actions $\mathbf{g} \in \mathbf{S}(V, K)$ of K in V which extend $\mathbf{f}$. This is the extension problem.

If $\mathbf{f} \in \mathbf{S}(V, F)$ has an extension $\mathbf{g} \in S(V, K)$ to $K \supset F$ then

(3.1) $\quad [V_{\mathbf{g}}:K]\cdot[K:F] = [V_{\mathbf{f}}:F]$

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by (2.1) where now the given action in (2.1) is $\mathbf{g}$, and its restriction to F is $\mathbf{f}$, by assumption. If $\mathbf{f}$ is a finite dimensional action, $[V_{\mathbf{f}}:F] < \infty$, then by (3.1) it is a necessary condition for the existence of extensions to K that $[K:F]$ divide $[V_{\mathbf{f}}:F]$.

On the other hand, consider a finite dimension action $\mathbf{f} \in \mathbf{S}(V, F)$, say $[V_{\mathbf{f}}:F] = n$, and a field extension $K \supset F$ such that $[K:F] = d$ divides n. Then K as an F-space must fit F-isomorphically into $V_{\mathbf{f}}$ with multiplicity $m = n/d$. That is, there exists a group isomorphism $\phi: V \to K^m$ preserving the F-vector space structures, namely that given by $\mathbf{f}$ in V, and the restriction to F of $\mathbf{sm}$ in K^m. This means that, for all $c \in F$ and $v \in V$, $\phi(f_c(v)) = \mathbf{sm}_c\phi(v)$, whence

$$f_c(v) = \phi^{-1}\mathbf{sm}_c\phi(v).$$

If x ranges over K then evidently the map $x \to \phi^{-1}\mathbf{sm}_x \phi$ defines a scalar action of K in V, and we have just seen that the restriction of this action to F is $\mathbf{f}$. Thus the divisibility condition is also sufficient for the solution of the extension problem. We state this as

PROPOSITION 1. *A finite dimensional scalar action* $\mathbf{f}$ *of a field F in a vector group V has extensions to a finite extension field $K \supset F$ if and only if* $[K:F]$ *divides* $[V_{\mathbf{f}}:K]$.

This generalizes the fact that $\mathbf{R}^n$ admits complex structures if and only if n is even.

These considerations hold also for arbitrary (infinite) cardinal numbers. Namely, if $\mathbf{f} \in \mathbf{S}(V, F)$ is given, with $[V_{\mathbf{f}}:F] = \nu$, ν any cardinal number, and $[K:F] = \delta$, δ any cardinal number, then for every μ such that $\mu\delta = \nu$ we have an F-isomorphism $\phi: V \to K^\mu$ and an extension $\mathbf{g} \in \mathbf{S}(V, K)$ to K of $\mathbf{f}$, namely $\mathbf{g} = \phi^{-1}\mathbf{sm}\phi$, as before. From the properties of the correspondence this defines between isomorphisms and extensions one can develop a general description of the set of all extensions. For this purpose we introduce some notation.

There is in the case of infinite cardinality the possibility that several values of μ satisfy $\mu\delta = \nu$, the different values of μ giving different isomorphisms, and hence different extensions. Let us write $\mathrm{Ext}_\mu(\mathbf{f})$ for the set of extensions of the form $\phi^{-1}\mathbf{sm}\,\phi$ associated with a given value μ, and $\mathrm{Iso}_F(V_{\mathbf{f}}, K^\mu)$ for the isomorphisms ϕ associated with the same value μ. We argue now that the set Ext $(\mathbf{f})$ of all extensions is the disjoint union

$$\text{(3.2)} \qquad \mathrm{Ext}(\mathbf{f}) = \bigcup_{\mu\delta=\nu} \mathrm{Ext}_\mu(\mathbf{f})$$

of the several sets $\mathrm{Ext}_\mu(\mathbf{f})$. For, given $\mathbf{f} \in \mathbf{S}(V, F)$ with $[V_{\mathbf{f}}:F] = \nu$ and $K \supset$

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F with $[K:F] = \delta$, if $\mathbf{g} \in \mathrm{Ext}(\mathbf{f})$ then we may choose a $\mathbf{g}$-basis of V and get thereby an isomorphism $\phi: V_{\mathbf{g}} \to K^{\mu}_{\mathbf{sm}}$ for some value of μ such that $\mu\delta = \nu$. Thus $\phi \in \mathrm{Iso}_F(V_{\mathbf{f}}, K^{\mu})$ in particular. With this ϕ we have the action $\phi^{-1}\mathbf{sm}\phi$ of K in V, and as before this action is an extension of $\mathbf{f}$. In particular $\phi^{-1}\mathbf{sm}\phi \in \mathrm{Ext}_{\mu}(\mathbf{f})$ and (3.2) is proved.

We denote the map $\tau \to \tau^{-1}\mathbf{sm}\tau$ of $\mathrm{Iso}_F(V_{\mathbf{f}}, K^{\mu})$ to $\mathrm{Ext}_{\mu}(\mathbf{f})$ by Φ:

$$\Phi(\tau) = \tau^{-1}\mathbf{sm}\tau.$$

$$\begin{array}{ccc} \mathrm{Iso}_F(V_{\mathbf{f}}, K^{\mu}) & \xrightarrow{R} & \mathrm{Iso}_F(V_{\mathbf{f}}, K^{\mu}) \\ \downarrow \Phi & & \downarrow \Phi \\ \mathrm{Ext}_{\mu}(\mathbf{f}) & \xrightarrow{C} & \mathrm{Ext}_{\mu}(\mathbf{f}) \end{array}$$

The group $GL(V_{\mathbf{f}})$ of invertible elements of $L(V_{\mathbf{f}})$ acts on $\mathrm{Iso}_F(V_{\mathbf{f}}, K^{\mu})$ by right multiplication. We write this action as $R: R_{\sigma}\tau = \tau\sigma$ for $\tau \in \mathrm{Iso}$, $\sigma \in GL(V_{\mathbf{f}})$. This action is transitive (given $\tau_1, \tau_2 \in \mathrm{Iso}$, $\sigma = \tau_2\tau_1^{-1} \in GL(V_{\mathbf{f}})$ and $\tau_2\sigma = \tau_2\tau_2^{-1}\tau_1 = \tau_1$) and isotropically trivial (if $R_{\sigma}\tau = \tau$ then $\tau(v) = \tau\sigma(v)$, all $v \in V$, but τ is an isomorphism, so $v = \sigma(v)$, all $v \in V$, and so $\sigma = \mathbf{id}$). The map Φ is surjective, as we have noted, but not injective. Indeed, for $\tau \in \mathrm{Iso}$, $\sigma \in GL(V_{\mathbf{f}})$, we have $\Phi\tau\sigma = \sigma^{-1}(\Phi\tau)\sigma$, so $\Phi\tau\sigma = \Phi\tau$ if and only if $\sigma \in GL(V_{\Phi\tau})$, the latter being a subgroup of $GL(V_{\mathbf{f}})$ because $\Phi\tau$ is an extension of $\mathbf{f}$. The group $GL(V_{\mathbf{f}})$ acts by conjugation on $\mathrm{Ext}_{\mu}(f)$ (if $\sigma \in GL(V_{\mathbf{f}})$ and $\mathbf{g} \in \mathrm{Ext}_{\mu}(\mathbf{f})$ then for $c \in F$ we have $\sigma^{-1}\mathbf{g}_c\sigma = \sigma^{-1}\mathbf{f}_c\sigma = \mathbf{f}_c$ because $\sigma \in GL(V_{\mathbf{f}})$). We denote this action by $C: C_{\sigma}\mathbf{g} = \sigma^{-1}\mathbf{g}\,\sigma$. For any $\tau \in \mathrm{Iso}$ and $\sigma \in GL(V_{\mathbf{f}})$ we have

$$C_{\sigma}\Phi\tau = C_{\sigma}\tau^{-1}\mathbf{sm}\,\tau = \sigma^{-1}\tau^{-1}\mathbf{sm}\,\tau\sigma = (\tau\sigma)^{-1}\mathbf{sm}\,(\tau\sigma) = \Phi R_{\sigma}\tau,$$

which is to say

$$C_{\sigma} = \Phi R_{\sigma}\Phi^{-1}$$

is the action induced via Φ in $\mathrm{Ext}_{\mu}(\mathbf{f})$ by the action R on Iso. Since R is transitive on Iso, C is transitive on Ext_{μ} (indeed, for any pair $\mathbf{g}$, $\mathbf{h}$ of extensions let $\tau, \theta \in \mathrm{Iso}$ be such that $\Phi\tau = \mathbf{g}$, $\Phi\theta = \mathbf{h}$; there exists $\sigma \in GL(V_{\mathbf{f}})$ such that $\theta = R_{\sigma}\tau$; then

$$\mathbf{h} = \Phi\theta = \Phi\tau\sigma = \sigma^{-1}(\Phi\tau)\sigma = C_{\sigma}\Phi\tau = C_{\sigma}\mathbf{g},$$

showing our claim). Let us calculate the isotropy group $\mathrm{Is}(\mathbf{g})$ of an extension $\mathbf{g}$. We have

$$\mathrm{Is}(\mathbf{g}) = \{\sigma \in GL(V_{\mathbf{f}}): \sigma^{-1}\mathbf{g}\sigma = \mathbf{g}\} = GL(V_{\mathbf{f}}) \cap L(V_{\mathbf{g}})$$

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because σ is invertible and commutes with $\mathbf{g}$. Putting $GL(V_{\mathbf{g}})$ for $L(V_{\mathbf{g}})$ because σ is invertible we now have

$$\mathrm{Is}(\mathbf{g}) = GL(V_{\mathbf{f}}) \cap GL(V_{\mathbf{g}}).$$

We have already noted that $GL(V_{\mathbf{g}})$ is a subgroup of $GL(V_{\mathbf{f}})$ (because $\mathbf{g}$ is an extension of $\mathbf{f}$), so finally $\mathrm{Is}(\mathbf{g}) = GL(V_{\mathbf{g}})$. Thus $\mathrm{Ext}_{\mu}(\mathbf{f})$ is bijective with the homogeneous space $GL(V_{\mathbf{f}})/GL(V_{\mathbf{g}})$ for any $\mathbf{g} \in \mathrm{Ext}_{\mu}(\mathbf{f})$, and $\mathrm{Ext}(\mathbf{f})$ is the disjoint union of these spaces. If we identify the elements of $\mathrm{Iso}_F(V_{\mathbf{f}}, K^{\mu})$ which have the same image under Φ then we can express $\mathrm{Ext}(\mathbf{f})$ as the disjoint union of these identification spaces. We collect these results in

THEOREM 1. *Let* $\mathbf{f} \in \mathbf{S}(V, F)$ *be a scalar action of a field* F *in an additive abelian group* V, *and let* $K \supset F$ *be an extension field. Let* ν *and* δ *denote the (possibly infinite) dimensions* $[V_{\mathbf{f}}:F] = \nu$, $[K:F] = \delta$. *The set* $\mathrm{Ext}(\mathbf{f})$ *of all extensions to* K *of* $\mathbf{f}$ *is bijective with the disjoint union, over all cardinals* μ *such that* $\mu\delta = \nu$, *of equivalence classes of* F-*isomorphisms of* $V_{\mathbf{f}}$ *with* $K^{\mu} = K \times K \times \ldots \times K$, *two such isomorphisms* τ, θ *being equivalent if they determine the same extension* $\tau^{-1}\mathbf{sm}\tau = \theta^{-1}\mathbf{sm}\,\theta$ *of* $\mathbf{f}$. *The set of these equivalence classes is bijective with the set* $\mathrm{Ext}_{\mu}(\mathbf{f})$ *of those extensions of* $\mathbf{f}$ *under which* V *has dimension* μ *over* K. *The group* $GL(V_{\mathbf{f}})$ *acts transitively by conjugation on* $\mathrm{Ext}_{\mu}(\mathbf{f})$ *with isotropy* $GL(V_{\mathbf{g}})$ *for any* $g \in \mathrm{Ext}_{\mu}(\mathbf{f})$, *and so* $\mathrm{Ext}(\mathbf{f})$ *can be expressed as the disjoint union of homogeneous spaces*

$$GL(V_{\mathbf{f}})/GL(V_{\mathbf{g}_{\mu}})$$

over a set of representatives $\mathbf{g}_{\mu} \in \mathrm{Ext}_{\mu}(\mathbf{f})$.

It is clear that the prime fields $\mathbf{Q}$ and $\mathbf{F}_p$ have at most one scalar action in a given V. Therefore if V admits a scalar action $\mathbf{f}$ by a prime field F and $K \supset F$ is an extension field then $\mathrm{Ext}(\mathbf{f})$ constitutes the set of all scalar actions of K in V.

Since the general description of extensions is rather abstract it may be well to give an explicit construction in a (somewhat general) special case. To this end, let F be an arbitrary ground field and $K \supset F$ a separably finite extension field. Let $\mathbf{a}$ be a primitive element for the extension, $K = F(\mathbf{a})$, and let $p(x) \in F[x]$ be the minimal polynominal of $\mathbf{a}$. Suppose we are given an action $\mathbf{f} \in \mathbf{S}(V, F)$ such that $[V_{\mathbf{f}}:F] < \infty$ and $[K:F] \mid [V_{\mathbf{f}}:F]$. Then the direct sum $T = \oplus C(p)$ of suitably many copies of the companion matrix $C(p)$ of p is the matrix of an F-endomorphism of V. T thus defines an element of $L(V_{\mathbf{f}})$, which we denote also by T. Clearly T is algebraic over the image field $\mathbf{f}(F) \subset \mathrm{End}(V)$ isomorphic to F. (A scalar action $\mathbf{f}$ of a field F in V imbeds F isomorphically into $\mathrm{End}(V)$ because $\ker(\mathbf{f})$ is an ideal in F.) Namely, $\sum \mathbf{f}_{p_i}T^i = 0$, where $p(x) = \sum p_i x^i$. The adjunction to $\mathbf{f}(F)$ of T therefore results in a field isomorphic to $K = F(\mathbf{a})$, the correspondence being

$$\sum c_j \mathbf{a}^j \to \sum f_{c_j} T^J \quad \text{for } c_j \in F.$$

If we define $\mathbf{g}:K \to \mathrm{End}(V)$ by

$$\mathbf{g}_{(\sum c_j a^j)} = \sum \mathbf{f}_{c_j} T^j,$$

which is to say we define $g_{\mathbf{a}}$ as T and extend $\mathbf{g}$ to all of K in the natural way, then we have a scalar action of K in V which extends f.

The companion matrix construction yields a picturesque description of the set of complex structures on $\mathbf{R}^2$, as follows. The matrix

$$C(p) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

the companion matrix of the minimal polynomial $p(x) = x^2 + 1$ of i over $\mathbf{R}$, determines one extension $\mathbf{g} \in \mathbf{S}(R^2, \mathbf{C})$ to $\mathbf{C}$ of the natural action of R on R^2, which is to say, a complex structure. All other extensions are conjugate in $GL(R^2)$ to $\mathbf{g}$. Therefore to find them all it suffices to determine all $A \in M_2(\mathbf{R})$ similar to $C(p)$. Since $p(x)$ is irreducible it is a complete similarity invariant, so the matrices sought are those A with $\mathrm{tr}(A) = 0$ and $\det(A) = 1$. These are easily seen to be

$$A = \begin{pmatrix} x & y \\ -(x^2+1)/y & -x \end{pmatrix}, \quad y \neq 0.$$

Thus the set of complex structures can be viewed as the plane with real axis removed. Evidently this construction can be generalized in various ways.

4. Commuting pairs. With commuting linear operators in mind we inquire whether there are geometric consequences of commutation for scalar actions. Without loss of generality one may view a pair of scalar actions of a field K as extensions to K of one given action of a subfield $F \subset K$. Indeed, the pair of actions of K are extensions of their common restriction to the prime field of K. If $\mathbf{g}$, $\mathbf{h} \in \mathbf{S}(V, K)$ are commuting extensions to K of an action of a subfield $F \subset K$ then we can define an action of the ring $K \otimes_F K$ in V, which we call the *product action* and denote by $\mathbf{g} \times \mathbf{h}$, as follows:

$$(\mathbf{g} \times \mathbf{h})_{a \otimes b} = \mathbf{g}_a \mathbf{h}_b.$$

$\mathbf{g} \times \mathbf{h}$ extends additively to all of $K \otimes_F K$, clearly, and the commutativity

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$[\mathbf{g}, \mathbf{h}] = 0$ is required in order that $\mathbf{g} \times \mathbf{h}$ preserve the multiplication in $K \otimes_F K$. The following commutations are evident:

$$\mathbf{g}_c(\mathbf{g} \times \mathbf{h})_\tau = (\mathbf{g} \times \mathbf{h})_\tau g_c, \ \mathbf{h}_c(\mathbf{g} \times \mathbf{h})\tau = (\mathbf{g} \times \mathbf{h})_\tau \mathbf{h}_c \tag{4.1}$$

for $c \in K$, $\tau \in K \otimes_F K$.

PROPOSITION 4.1. *Every commuting pair* $\mathbf{g}, \mathbf{h} \in \mathbf{S}(V, K)$ *of extensions to* K *of an action in* V *of a subfield* F *determines an action of* $K \otimes_F K$ *in* V, *and every action of* $K \otimes_F K$ *arises in this way.*

Proof. Any pair of actions with the specified properties determines their product action of $K \otimes_F K$, as noted. If $\phi \in \mathbf{S}(V, K \otimes_F K)$ is given, we put $\mathbf{g}_c = \phi_{c\otimes 1}$, $\mathbf{h}_c = \phi_{1\otimes c}$ for $c \in K$. Then $\mathbf{g}$ and $\mathbf{h}$ are commuting actions of K, $\mathbf{g}_d = \phi_{d\otimes 1} = \phi_{1\otimes d} = \mathbf{h}_d$ for $d \in F$, and $\phi = \mathbf{g} \times \mathbf{h}$.

The proposition says in fact that $\mathbf{g}$ and $\mathbf{h}$ can be recovered from their product action by the formulae

$$\mathbf{g}_c = (\mathbf{g} \times \mathbf{h})_{c\otimes 1}, \quad \mathbf{h}_c = (\mathbf{g} \times \mathbf{h})_{1\otimes c}.$$

Using the trivial isomorphisms of K with $K \otimes 1$ and $1 \otimes K$ we have from these formulae the pair of equivalences

$$\mathbf{g} \simeq \mathbf{g} \times \mathbf{h}|^{K\otimes 1}, \quad \mathbf{h} \simeq \mathbf{g} \times \mathbf{h}|^{1\otimes K}, \tag{4.2}$$

which is to say $\mathbf{g}$ and $\mathbf{h}$ are equivalent to the restrictions to a pair of isomorphic subfields of an action (namely $\mathbf{g} \times \mathbf{h}$) of an extension ring (namely $K \otimes_F K$). This describes the most general pair of commuting scalar field actions. It involves an essentially arbitrary field extension. By specializing to separably finite extensions, and subsequently to Galois extensions, we shall have successively more refined structural results.

If K is separably finite over F then $K \otimes_F K$ is semi-simple ([**3**], page 460), and so has a decomposition

$$K \otimes_F K = \Sigma(K \otimes_F K)e_j \tag{4.3}$$

as the direct sum of uniquely determined minimal ideals, each of which is principal, generated by an idempotent e_j. These idempotents, uniquely determined by their ideals, are primitive in the sense that they cannot be expressed as sums $e = \alpha + \beta$ of idempotents α, β such that $\alpha\beta = 0$, and it follows that $e_je_k = \delta_{jk}e_k$, $\sum e_j = 1 \otimes 1$ ([**1**], Chapter 4). We note that the number of primitive idempotents in $K \otimes_F K$ does not exceed $[K:F]$. Indeed, each ideal $(K \otimes_F K)e_j$ is a field ([**1**], Theorem 5.4A), and

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$$(K \otimes_F K)e_j \supset (K \otimes_F 1)e_j \simeq K,$$

so

$$[(K \otimes_F K)e_j : F] \geqq [K:F].$$

Then

$$[K:F]^2 = [K \otimes_F K:F] = \sum [(K \otimes_F K)e_j:F] \geqq r \cdot [K:F]$$

where r is the number of primitive idempotents.

Let $\mathbf{g}, \mathbf{h} \in \mathbf{S}(V, K)$ be commuting extensions to K of an action of $F \subset K$ in V, K being separably finite over F. We then have the product action $\mathbf{g} \times \mathbf{h}$ and the images under $\mathbf{g} \times \mathbf{h}$ of the idempotents e_j. Put $P_{e_j} = (\mathbf{g} \times \mathbf{h})_{e_j}$ for these images. From the corresponding properties of the e_j we have

$$P_{e_j} P_{e_k} = \delta_{jk} P_{e_k} \quad \text{and} \quad \sum P_{e_j} = I.$$

By the commutations (4.1) the P_{e_j} are linear operators in both $V_{\mathbf{g}}$ and $V_{\mathbf{h}}$, so that in particular their ranges $V_{e_j} = P_{e_j}(V)$ are subspaces in both vector space structures. Thus we have a decomposition $V = \oplus V_{e_j}$ of the vector group into subgroups invariant under both actions. If $[V_{\mathbf{f}}:F]$ is finite then we have

$$[V_{\mathbf{g}}:K][K:F] = [V_{\mathbf{f}}:F],$$

and the same for $\mathbf{h}$, this being (3.1). These relations remain true if we substitute V_{e_j} for V throughout, for the V_{e_j} also admit the extensions $\mathbf{g}$ and $\mathbf{h}$. It is consistent with this that one or more of the V_{e_j} can be 0. We shall have an example of this in a moment.

When $[K:F] = 2$ there is an easy explicit form for the idempotents. Let $K = F(\mathbf{a})$, $\pm \mathbf{a}$ being the roots of the irreducible polynomial $p(x) = x^2 - c \in F[x]$. (Here we assume the characteristic is not 2.) There are now two idempotents in the decomposition. Denote them by $e_\pm$. One finds by experimentation that

$$(4.4) \qquad e_\pm = (1/2)\left\{(1 \otimes 1) \pm \frac{1}{c}(\mathbf{a} \otimes \mathbf{a})\right\}.$$

The common subspaces $V_\pm = P_\pm(V)$ are, respectively, $\{v \in V: \quad P_\pm(v) = v\}$. By (4.4) we have

$$P_\pm = (1/2)I \pm (1/2c)\mathbf{g_a h_a},$$

so $P_\pm(v) = v$ if and only if

$$(1/2c)\mathbf{g_a h_a}(v) = \mp(1/2)v.$$

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Applying $\mathbf{g_a}$ to both sides of this equation we get

$$v \in V_{\pm} \text{ if and only if } \mathbf{h_a}(v) = \mp \mathbf{g_a}(v).$$

That is to say, on V_- the two actions agree, and on V_+ they are related by composition with the automorphism of K fixing F determined by the map $\mathbf{a} \rightarrow (-\mathbf{a})$. Call this automorphism σ. Then $(\sigma \otimes I)e_+ = e_-$, and if we put $V_- = W$, $V_+ = W^\sigma$, then

$$\mathbf{g}|_W = \mathbf{h}|_W, \quad \mathbf{g}|_{(W^\sigma)} = (\mathbf{h}|_{(W^\sigma)}) \circ \sigma. \tag{4.5}$$

An instructive example arises by the specialization $K = \mathbf{C}$, $F = \mathbf{R}$, $V = \mathbf{C} \oplus \mathbf{C}$ as an additive group. For $\mathbf{f}$ we take the natural multiplication of $\mathbf{R}$ on $\mathbf{C} \oplus \mathbf{C}$, and we consider the extensions $\mathbf{g}$, $\mathbf{h}$ defined as

$$\mathbf{g}_z(u, v) = (zu, zv), \quad \mathbf{h}_z(u, v) = (zu, \bar{z}v)$$

for $z, u, v \in \mathbf{C}$. Tracing through the previous calculation we find

$$V_+ = \mathbf{C} \oplus 0,$$
$$V_- = 0 \oplus \mathbf{C}.$$

The automorphism σ is now the conjugation automorphism of $\mathbf{C}$, and (4.5) is fulfilled. If we modify the example by putting

$$\mathbf{g}_z(u, v) = (zu, zv), \quad \mathbf{h}_z(u, v) = (\bar{z}u, \bar{z}v),$$

which is to say, $\mathbf{g}$ is as before and $\mathbf{h} = \mathbf{g} \circ \sigma$, then we find

$$V_+ = \mathbf{C} \oplus \mathbf{C},$$
$$V_- = 0.$$

This concludes the example.

For separably finite extensions we have a partial analog of (4.5) and for Galois extensions there is a full generalization. To these developments we now turn. They depend upon a comparison of the two actions in the individual invariant subgroups, and upon some information about the primitive idempotents. The main facts are given in Propositions 4.2 to 4.6.

PROPOSITION 4.2. *Let K be separably finite over F, let $\mathbf{g}, \mathbf{h} \in \mathbf{S}(V, K)$ be commuting extensions to K of an action of $F \subset K$ in V, and let $(0) \neq W = (\mathbf{g} \times \mathbf{h})_e(V)$, e a primitive idempotent. Then*

$$(1 \otimes x - x \otimes 1)e = 0 \quad \textit{for all } x \in K \tag{4.6}$$

if and only if $\mathbf{g}$ and $\mathbf{h}$ agree on W,

$$\mathbf{g}|_W = \mathbf{h}|_W. \tag{4.7}$$

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Proof. Assume (4.7). Then

$$\mathbf{g}_x(\mathbf{g} \times \mathbf{h})_e = \mathbf{h}_x(\mathbf{g} \times \mathbf{h})_e \quad \text{for all } x \in K.$$

Putting $\mathbf{g}_x = (\mathbf{g} \times \mathbf{h})_{x\otimes 1}$, $\mathbf{h}_x = (\mathbf{g} \times \mathbf{h})_{1\otimes x}$, so that

$$\mathbf{g}_x(\mathbf{g} \times \mathbf{h})_e = (\mathbf{g} \times \mathbf{h})_{(x\otimes 1)e} \quad \text{and}$$

$$\mathbf{h}_x(\mathbf{g} \times \mathbf{h})_e = (\mathbf{g} \times \mathbf{h})_{(1\otimes x)e},$$

we have

$$0 = (\mathbf{g} \times \mathbf{h})_{(1\otimes x - x\otimes 1)e} \quad \text{for all } x \in K.$$

Therefore

$$(1 \otimes x - x \otimes 1)e \in (K \otimes_F K)e \cap \ker(\mathbf{g} \times \mathbf{h}).$$

Now $\ker(\mathbf{g} \times \mathbf{h})$ is an ideal in $K \otimes_F K$ and therefore either

$$\ker(\mathbf{g} \times \mathbf{h}) \supset (K \otimes_F K)e_j$$

or

$$\ker(\mathbf{g} \times \mathbf{h}) \cap (K \otimes_F K)e_j = (0) \quad \text{for each } j.$$

By the hypothesis $(\mathbf{g} \times \mathbf{h})_e \neq 0$ therefore we have

$$\ker(\mathbf{g} \times \mathbf{h}) \cap (K \otimes_F K)e = (0).$$

Since $(1 \otimes x - x \otimes 1)e$ is in this intersection it must vanish, this for all $x \in K$. Thus (4.7) implies (4.6).

Conversely, if e satisfies (4.6) then

$$0 = (\mathbf{g} \times \mathbf{h})_{(1\otimes x - x\otimes 1)e} = (\mathbf{g}_x - \mathbf{h}_x)(\mathbf{g} \times \mathbf{h})_e \quad \text{for all } x \in K,$$

and the vanishing of this operator is the content of (4.7). This completes the proof of the proposition.

PROPOSITION 4.3. *Let K be separably finite over F. There exists precisely one idempotent $e \in K \otimes_F K$ such that*

$$(1 \otimes x - x \otimes 1)e = 0 \quad \textit{for all } x \in K.$$

This idempotent is necessarily primitive.

Proof. We view K as an F-space, with multiplication by F as the scalar action. This action extends to scalar multiplication **sm** by K acting in K^+ as vector group. Since

$$I = (\mathbf{sm} \times \mathbf{sm})_{1\otimes 1} = \sum(\mathbf{sm} \times \mathbf{sm})_{e_j},$$

summing over the primitive idempotents in $K \otimes_F K$, there exists at least one e_j such that

$$(\mathbf{sm} \times \mathbf{sm})_{e_j} \neq 0.$$

Then on the space $W = (\mathbf{sm} \times \mathbf{sm})_{e_j}(K^+)$ we have (4.7) with $\mathbf{g} = \mathbf{h} = \mathbf{sm}$. Hence by Proposition 4.2 the idempotent e_j satisfies (4.6).

If e is any element of $K \otimes_F K$ satisfying (4.6) then

$$(x \otimes y)e = (x \otimes 1)(1 \otimes y)e = (x \otimes 1)(y \otimes 1)e = (xy \otimes 1)e,$$

whence it follows that $(K \otimes_F K)e = (K \otimes_F 1)e$. If e is any idempotent in $K \otimes_F K$ then $(K \otimes_F 1)e$ is a field (isomorphic to K). Combining these two statements we have that if e is an idempotent satisfying (4.6) then $(K \otimes_F K)e$ is a field, hence a minimal ideal, whence e is primitive.

If there were two idempotents e_1, e_2 satisfying (4.6) they would be primitive, and therefore $e_1e_2 = 0$. This being so, $e_1 + e_2$ would be a decomposable idempotent satisfying (4.6). Since this cannot be, there cannot be two idempotent solutions to (4.6). This completes the proof of the proposition.

If σ is an automorphism of K fixing F then $\sigma \otimes I$ is an automorphism of $K \otimes_F K$. If K is separably finite over F and e_{id} is the primitive idempotent specified in Proposition 4.3 then $(\sigma \otimes I)e_{\mathrm{id}}$ is a primitive idempotent. We denote it by e_σ,

(4.8) $$e_\sigma = (\sigma \otimes I)e_{\mathrm{id}}.$$

The foregoing propositions have e_σ-analogs. We state them both together in

PROPOSITION 4.4. *Let K be separably finite over F. For each automorphism σ of K fixing F there exists a primitive idempotent $e_\sigma \in K \otimes_F K$ which is the unique idempotent solution to the equation*

(4.9) $$(1 \otimes x - \sigma(x) \otimes 1)e = 0 \quad \text{for all } x \in K.$$

For any commuting pair $\mathbf{g}, \mathbf{h} \in \mathbf{S}(V, K)$ of extensions to K of an action of F, if $(\mathbf{g} \times \mathbf{h})_{e_\sigma} \neq 0$ then (4.9) is equivalent to

(4.10) $$\mathbf{g} \circ \sigma|_W = \mathbf{h}|_W$$

where $W = (\mathbf{g} \times \mathbf{h})_{e_\sigma}(V)$.

Proof. The application of $\sigma \otimes I$ to (4.6) yields (4.9). The argument for (4.10) is similar, and is omitted.

The earlier propositions may now be regarded as the description of the special case of the idempotent e_{id} corresponding to the identity automorphism of K.

We say an idempotent $e \in K \otimes_F K$ is *associated with* an automorphism σ of K fixing F if e has the form e_σ of (4.8). We have just shown that each such automorphism σ has precisely one idempotent associated with it, namely e_σ, and now we argue that an idempotent can be associated with only one automorphism. We write $\mathbf{G}(K/F)$ for the group of automorphisms of K fixing F.

PROPOSITION 4.5. *Let K be separably finite over F. For any σ, $\tau \in \mathbf{G}(K/F)$ the associated idempotents e_σ and e_τ are equal if and only if $\sigma = \tau$.*

Proof. If $e_\sigma = e_\tau = e$ then

$$(\sigma(x) \otimes 1)e = (\tau(x) \otimes 1)e$$

by (4.9), whence

$$(\{\sigma(x) - \tau(x)\} \otimes 1)e = 0 \quad \text{for all } x \in K.$$

The map $y \rightarrow (y \otimes 1)e$ of K to $(K \otimes 1)e$ is an isomorphism of fields, so that in particular its kernel is (0). We have just observed that $\sigma(x) - \tau(x)$ is in this kernel, and so vanishes, which is to say $\sigma = \tau$. Thus $e_\sigma = e_\tau$ entails $\sigma = \tau$. Since the converse is trivial the proof is complete.

We have the following criterion for an idempotent to be associated with an automorphism.

PROPOSITION 4.6. *Let K be separably finite over F. An idempotent e in $K \otimes_F K$ is associated with an element of $\mathbf{G}(K/F)$ if and only if $(K \otimes 1)e = (1 \otimes K)e$.*

Proof. If $e = e_\sigma$ with $\sigma \in \mathbf{G}(K/F)$ then $(1 \otimes K)e = (K \otimes 1)e$ by (4.9). Conversely, if $(1 \otimes K)e = (K \otimes 1)e$ then for every $x \in K$ there exists $y \in K$ such that $(x \otimes 1)e = (1 \otimes y)e$. If also $(x \otimes 1)e = (1 \otimes y')e$ then $(1 \otimes y)e = (1 \otimes y')e$, or

$$(1 \otimes (y - y'))e = 0.$$

Then $(y - y')$ is in the kernel of the map $z \rightarrow (1 \otimes z)e$, and so vanishes. We therefore have a map $\phi: K \rightarrow K$ such that

$$(1 \otimes x)e = (\phi(x) \otimes 1)e$$

for all $x \in K$. If $\phi(x) = \phi(x')$ then $(1 \otimes x)e = (1 \otimes x')e$, whence $x = x'$, which is to say ϕ is injective. By symmetry it is also surjective. We have

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$$\begin{aligned}(\phi(x_1+x_2) \otimes 1)e &= (1 \otimes (x_1+x_2))e \\ &= (1 \otimes x_1)e + (1 \otimes x_2)e \\ &= (\phi(x_1) \otimes 1)e + (\phi(x_2) \otimes 1)e \\ &= (\{\phi(x_1) + \phi(x_2)\} \otimes 1)e,\end{aligned}$$

so ϕ preserves addition. By a similar argument it preserves multiplication. If $x \in F$ then

$$(\phi(x) \otimes 1)e = (1 \otimes x)e = (x \otimes 1)e,$$

so $\phi(x) = x$. Thus $\phi \in \mathbf{G}(K/F)$, and we have verified (4.9) for e with respect to ϕ. This completes the proof.

COROLLARY. *If the idempotent $e \in K \otimes_F K$ is associated with an automorphism then $(K \otimes_F K)e \simeq K$, and in particular*

$$[(K \otimes_F K)e:F] = [K:F].$$

Proof. If $(1 \otimes K)e = (K \otimes 1)e$ then

$$\begin{aligned}(x \otimes y)e &= (x \otimes 1)(1 \otimes y)e = (x \otimes 1)(\phi(y) \otimes 1)e \\ &= (x\phi(y) \otimes 1)e \in (K \otimes 1)e,\end{aligned}$$

so $(K \otimes_F K)e \subset (K \otimes 1)e$, whence $(K \otimes_F K)e = (K \otimes 1)e$, and the conclusions follow.

Still under the assumption that K is separably finite over F, we note that there may exist idempotents in $K \otimes_F K$ not associated with automorphisms of K. For example, let $\mathbf{a}$ be a cube root of 2, and take $K = \mathbf{Q}(\mathbf{a})$. We have

$$K \otimes_{\mathbf{Q}} K = (K \otimes_{\mathbf{Q}} K)e_{\mathrm{id}} \oplus (\text{complementary ideals}).$$

There must be at least one complementary ideal because

$$[(K \otimes_{\mathbf{Q}} K)e_{\mathrm{id}}:\mathbf{Q}] = 3$$

by the Corollary, and

$$[K \otimes_{\mathbf{Q}} K:\mathbf{Q}] = 9.$$

But the complementary ideals cannot be generated by idempotents associated with automorphisms because there are none except **id**. However, for commuting pairs of extension actions there is a form of the general result (4.2) which holds for the exceptional idempotents, replacing

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the analysis of Proposition 4.4. In stating this result we shall also review the whole position and include the known case for the sake of clarity.

THEOREM 2. *Let K be separably finite over F. Every pair* $\mathbf{g}, \mathbf{h} \in \mathbf{S}(V, K)$ *of commuting extensions to K of an action in V of F gives rise to a direct sum decomposition $V = \oplus V_j$ of V into subgroups invariant under both actions. In each of these subgroups the restrictions thereto of* $\mathbf{g}$ *and* $\mathbf{h}$ *either*

(i) *are related by composition with an automorphism of K, or*

(ii) *have equivalent forms which are the restrictions to a pair of isomorphic subfields of an action of an extension field of K.*

$$\begin{tikzcd} K \arrow[d, "\sigma"'] \arrow[dr, "\mathbf{h}|_{V_j}"] & \\ K \arrow[r, "\mathbf{g}|_{V_j}"'] & \mathrm{End}(V) \end{tikzcd}$$

Before taking up the proof we point out that, according to the assertion, in the first case $\mathbf{g}|_{V_j}$ and $\mathbf{h}|_{V_j}$ are equivalent to each other by an automorphism of K (see figure), but that no equivalence of $\mathbf{g}|_{V_j}$ and $\mathbf{h}|_{V_j}$ is asserted in the second case. Indeed, in the second case an equivalence of these actions would violate the injectivity of the action of which they are restrictions (within equivalence). (This is an action of a field, and so is injective. See Section 2.)

Proof of theorem. In the notation of (4.3) et seq. V_j is V_{e_j}, the range of $(\mathbf{g} \times \mathbf{h})_{e_j}$. The first case of the theorem is that in which e_j is associated with an automorphism of K, and this has already been treated (see (4.9)). If e_j is not associated with an automorphism then $(K \otimes_F K)e_j$ is nevertheless a field, and this field acts in V_{e_j} by the action

$$a^{(e_j)} = (\mathbf{g} \times \mathbf{h})\,|_{V_{e_j}}^{(K \otimes_F K)e_j}.$$

Hereafter we omit the subscript j. We define a map

$$\sigma_e:(1 \otimes K)e \to (K \otimes 1)e$$

by

$$\sigma_e\{(1 \otimes x)e\} = (x \otimes 1)e.$$

σ_e may not be extendable to an automorphism of $(K \otimes_F K)e$, but it is an isomorphism of the subfield $(1 \otimes K)e$ onto the subfield $(K \otimes 1)e$, each isomorphic to K. We have

$$a^{e}_{(1 \otimes x)e}(v) = \mathbf{h}_x(v)$$

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and

$$a^e_{(x\otimes 1)e}(v) = g_x(v) \quad \text{for all } v \in V_e.$$

Therefore, by using the isomorphisms of $(K \otimes 1)e$ and $(1 \otimes K)e$ with K, we have that $\mathbf{g}|_{V_e}$ and $\mathbf{h}|_{V_e}$ are equivalent to the restrictions to $(K \otimes 1)e$ and $(1 \otimes K)e$ respectively of a^e. By Proposition 4.6 these subfields are distinct. This completes the proof.

By Proposition 4.5 we have a bijection between $\mathbf{G}(K/F)$ and the set of primitive idempotents associated with automorphisms. If K is Galois over F, so that $\#\mathbf{G}(K/F) = [K:F]$, then that set exhausts the primitive idempotents (because its cardinality is the maximum possible number $[K:F]$ of primitive idempotents). Case (ii) of Theorem 2 does not then arise, and the result assumes a rather prettier form. Despite some redundancy we state it in full.

THEOREM 3. *Let K be a Galois extension of a field F. Every pair $\mathbf{g}$, $\mathbf{h} \in \mathbf{S}(V, K)$ of commuting extensions to K of a scalar action of F in V determines a decomposition $V = \oplus V_\sigma$ of V into subgroups V_σ indexed by the Galois group $\mathbf{G}(K/F)$. Some of the V_σ may be (0). $\mathbf{G}(K/F)$ acts transitively on them, V_σ being the image under the action of σ on V_{id}. All the subgroups V_σ are invariant under both actions $\mathbf{g}$ and $\mathbf{h}$, and these actions restricted to V_σ are related there by composition with the automorphism σ,*

$$\mathbf{h}|_{V_\sigma} = (\mathbf{g}|_{V_\sigma}) \circ \sigma. \tag{4.11}$$

Proof. The primitive idempotents in $K \otimes_F K$ are labelled by $\mathbf{G}(K/F)$ by the formula

$$e_\sigma = (\sigma \otimes I)e_{\mathrm{id}}.$$

We carry this over to the projections and invariant subgroups by putting

$$P_\sigma = (\mathbf{g} \times \mathbf{h})_{e_\sigma}, \quad V_\sigma = P_\sigma(V).$$

Then the map $(\sigma, P_\alpha) \rightarrow P_{\sigma\alpha}$ defines an action of $\mathbf{G}(K/F)$ on the projections, and thereby on the subgroups, which is transitive and takes V_{id} to V_σ. The subgroups V_σ are invariant under $\mathbf{g}$ and $\mathbf{h}$ by (4.1), as before. We have

$$(1 \otimes x - \sigma(x) \otimes 1)e = 0,$$

and therefore (4.11) by (4.10) on each $V_\sigma \neq (0)$. This completes the proof.

5. Common lines. The comparison of two actions on a common invariant subgroup (Section 4) suggests the comparison of two actions on a common line. Given an action $\mathbf{g} \in \mathbf{S}(V, K)$, a line L in $V_{\mathbf{g}}$ is the orbit $L = \{\mathbf{g}_c(v): \quad c \in K\}$ of a point v under the $\mathbf{g}$-action of K. We write $\mathbf{P}V_{\mathbf{g}}$ for the set of lines in $V_{\mathbf{g}}$, or, in an alternate locution, the set of $\mathbf{g}$-lines in V.

In the circumstances of Theorem 3 the set of $\mathbf{g}$-lines in each invariant subgroup V_σ is identical with the set of $\mathbf{h}$-lines there,

$$\mathbf{P}(V_\sigma)_{\mathbf{g}} = \mathbf{P}(V_\sigma)_{\mathbf{h}} \quad \text{for all } \sigma \in \mathbf{G}(K/F),$$

because the orbit of $v \in V_\sigma$ under $\mathbf{g}$ is identical with its orbit under $\mathbf{g} \circ \sigma$, and now we invoke (4.10). This may not be the case when K is not Galois over F. For example, let $\mathbf{a}$ be a cube root of 2, let ω be a cube root of 1, put $V = \mathbf{Q}(\mathbf{a}, \omega)^+$, and take $K = \mathbf{Q}(\mathbf{a})$, $F = \mathbf{Q}$. For $\mathbf{g} \in \mathbf{S}(V, K)$ we take the multiplication of $\mathbf{Q}(\mathbf{a})$ on $\mathbf{Q}(\mathbf{a}, \omega)$. Let

$$\phi:\mathbf{Q}(\mathbf{a}\omega) \rightarrow \mathbf{Q}(\mathbf{a})$$

be the conjugacy isomorphism, and put $\mathbf{h} = \mathbf{g} \circ \phi$. Then

$$\mathbf{h} \in \mathbf{S}(V, \mathbf{Q}(\mathbf{a}\omega)) \quad \text{and} \quad [\mathbf{g}, \mathbf{h}] = 0.$$

Nonetheless $\mathbf{P}V_{\mathbf{g}} \neq \mathbf{P}V_{\mathbf{h}}$. For instance, the $\mathbf{h}$-line through $1 \in V$ contains $\mathbf{a}\omega$ but the $\mathbf{g}$-line does not.

The point of these instances is that equivalent scalar actions determine identical sets of lines, whereas inequivalent actions need not, even if they commute. The relation between equivalence, commutativity, and line structure is fully clarified by the following result.

THEOREM 4. *Let V be an additive abelian group admitting scalar action by fields K and E. The following conditions on a pair of actions $\mathbf{g} \in \mathbf{S}(V, K)$ and $\mathbf{h} \in \mathbf{S}(V, E)$ are equivalent*:

(i) $\mathbf{g} \simeq \mathbf{h}$.

(ii) $[\mathbf{g}, \mathbf{h}] = 0$ *and* $\mathbf{P}V_{\mathbf{g}} = \mathbf{P}V_{\mathbf{h}}$.

Proof. We recall from Section 2 that (i) means there exists an isomorphism ϕ of E onto K such that $\mathbf{h} = \mathbf{g} \circ \phi$. Evidently if this is so then (ii) holds. For the converse we begin with the observation that, given $L \in \mathbf{P}V_g$, each $v \in L$, $v \neq 0$, determines $\mathbf{v}$-coordinates on L relative to $\mathbf{g}$ as follows. Each $w \in L$ determines a unique $c \in K$ such that $w = g_c(v)$, and we take c as the v-coordinate of w relative to $\mathbf{g}$. Since $L \in PV_{\mathbf{h}}$ also, by hypothesis, we have also v-coordinates on L relative to $\mathbf{h}$. We claim there is

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a bijection $\phi^v \in K^E$ determined by $v \in L$ mapping the **h**-coordinates to the **g**-coordinates, which is to say

(5.1) $\quad \mathbf{h}_c(v) = (\mathbf{g} \circ \phi^v)_c(v)$

for all $c \in E$. Indeed, since $L = \mathbf{g}_K(v) = \mathbf{h}_E(v)$ in an obvious notation, for any $c \in E$ there exists a unique $d \in K$ such that $\mathbf{h}_c(v) = \mathbf{g}_d(v)$, and we define the required bijection by $\phi^v(c) = d$.

We claim now that $\phi^v = \phi^w$ for all $v, w \in L$, $0 \neq v$, $0 \neq w$. To see this we use the commutativity of **g** and **h** as follows.

$$\mathbf{h}_c\mathbf{g}_d(v) = \mathbf{g}_d\mathbf{h}_c(v) = \mathbf{g}_d\mathbf{g}_{(\phi^v(c))}(v) = \mathbf{g}_{(\phi^v(c))}\mathbf{g}_d(v).$$

Put $w = \mathbf{g}_d(v)$. Then

$$\mathbf{h}_c(w) = \mathbf{g}_{(\phi^v(c))}(w).$$

By (5.1) applied to $w \in L$ we have

$$\mathbf{h}_c(w) = \mathbf{g}_{(\phi^w(c))}(w).$$

Hence

$$\mathbf{g}_{(\phi^v(c))}(w) = \mathbf{g}_{(\phi^w(c))}(w),$$

whence $\phi^v(c) = \phi^w(c)$ for all $c \in E$, by the injectivity of **g**. Since $w = \mathbf{g}_d(v)$ is the generic point of L we have reached the desired conclusion.

We may therefore put $\phi^v = \phi$ for all $\mathbf{v} \in L$, $\phi \in K^E$ a bijection. We have $\mathbf{h} = \mathbf{g} \circ \phi$ on L. That is to say, $\mathbf{g} \circ \phi$ is a scalar action (of E in L). Therefore $\phi \in \mathrm{Iso}(E, K)$, as we have shown in Section 2.

We now have that for each $L \in \mathbf{P}V_{\mathbf{g}} = \mathbf{P}V_{\mathbf{h}}$ there exists $\phi \in \mathrm{Iso}(E, K)$ such that

$$\mathbf{h}_c(v) = (\mathbf{g} \circ \phi)_c(v) \quad \text{for all } v \in L,\ c \in E.$$

Let $v_1, v_2 \in V$ be **g**-linearly independent, and let $\phi_1, \phi_2, \phi_{12}$ be the isomorphisms determined by the **g**-lines through v_1, v_2, and $v_1 + v_2$ respectively. For any $c \in E$ we then have

$$\begin{aligned}\mathbf{g}_{\phi_{12}(c)}(v_1) + \mathbf{g}_{\phi_{12}(c)}(v_2) &= \mathbf{g}_{\phi_{12}(c)}(v_1 + v_2) = \mathbf{h}_c(v_1 + v_2)\\ &= \mathbf{h}_c(v_1) + \mathbf{h}_c(v_2) = \mathbf{g}_{\phi_1(c)}(v_1) + \mathbf{g}_{\phi_2(c)}(v_2).\end{aligned}$$

We now compare the first and last terms, collecting together terms in v_1 and terms in v_2. The result is the equation

$$\mathbf{g}_{\{\phi_{12}(c) - \phi_1(c)\}}(v_1) + \mathbf{g}_{\{\phi_{12}(c) - \phi_2(c)\}}(v_2) = 0.$$

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But v_1 and v_2 are $\mathbf{g}$-linearly independent. Therefore both bracketed expressions vanish, which is to say

$$\phi_1(c) = \phi_{12}(c) = \phi_2(c),$$

this for all $c \in E$. This means that all $L \in \mathbf{P}V_g = \mathbf{P}V_h$ determine one and the same $\phi \in \mathrm{Iso}(E, K)$, and $\mathbf{h} = \mathbf{g} \circ \phi$. This completes the proof of the theorem.

In the special case $E = K$ the conditions (ii) produce an automorphism of the field. The result thus resembles (but differs from) the fundamental theorem of projective geometry as formulated by Artin ([**2**], page 88). We hope to discuss this in a later note.

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The Rockefeller University,
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