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AN OPEN MAPPING THEOREM ON HOMOGENEOUS SPACES

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Abstract

We shall prove an open mapping theorem concerning a Polish group acting transitively on a complete metric space.

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Let X be a topological space and G be a topological transformation group on X which is a Polish (separable, complete metric) group acting transitively on X. This action is denoted by the map ψ from $G \times X$ to X with $\psi(g, x) = g \cdot x$ for $g \in G$ and $x \in X$.

We shall consider an open mapping theorem on the map $G \ni g \to \psi(g, x) = g \cdot x \in X$ from G to X for each fixed $x \in X$.

Even if there is a famous open mapping theorem on Banach spaces or topological linear spaces, the proof of this case is different from that of Banach space cases.

We shall call ψ *continuous* if ψ is continuous as the map from $G \times X$ to X. We shall call ψ *separately continuous* if

(1) for each fixed $g \in G$, the map $x \to g \cdot x$ from X to X is continuous,

(2) for each fixed $x \in X$, the map $g \to g \cdot x$ from G to X is continuous.

Now, we have the following theorem.

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THEOREM A. *Let G be a complete metric group acting on a metric space X. If the map $\Psi: G \times X \to X$ defined by $\Psi(g, x) = g \cdot x$ is separately continuous, then Ψ is continuous.*

PROOF. Due to the inequality:

$$d(gx, g_0x_0) \leq d(gx, gx_0) + d(gx_0, g_0x_0),$$

the continuity of ψ at $(g_0, x_0) \in G \times X$ follows from the claim that for any $\varepsilon > 0$ there exist $\delta > 0$ and a neighborhood U of g_0 such that $d(gx, gx_0) < \varepsilon$ whenever $d(x, x_0) < \delta$ and $g \in U$. Fix $x_0 \in X$, and set

$$A_{m,n} = \left\{ g \in G: d(gx, gx_0) \leq \frac{1}{m} \text{ if } d(x, x_0) < \frac{1}{n} \right\}.$$

The separate continuity of ψ yields the closedness of each $A_{m,n}$ and $G = \bigcup_{n=1}^{\infty} A_{m,n}$. Furthermore, $A_{m,n} \subset A_{m,n+1}$ by construction. Baire's category property of G implies that every non-empty open subset U of G contains a non-empty open subset $V \subset U$ such that $V \subset A_{m,n(m)}$ for some $n(m) \in \mathbb{N}$. Thus, we can find a sequence of open subsets O_m of G such that

$$O_m \subset A_{m,n(m)}, \text{ diameter } (O_m) < 1/m,$$
$$O_m \supset \overline{O}_{m+1}.$$

By the completeness of G, there exists a point $g_0 \in \bigcap_{m=1}^{\infty} O_m$. It then follows that $d(gx, gx_0) \leq 1/m$ if $d(x, x_0) < 1/n(m)$ and $g \in O_m$. Hence ψ is continuous at (g_0, x_0) by the first observation. Therefore, for any $\varepsilon_1 > 0$ there exists $\delta > 0$ and a neighborhood U of g_0 such that $d(gx, g_0x_0) < \varepsilon_1$ whenever $g \in U$ and $d(x, x_0) < \delta$. for an arbitrary $g_1 \in G$ and $\varepsilon > 0$, choose $\varepsilon_1 > 0$ so that $d(g_1g_0^{-1}y, g_1x_0) < \varepsilon$ whenever $d(y, g_0x_0) < \varepsilon_1$. We then set $V = g_1g_0^{-1}U$ as a neighborhood of g_1 and conclude that if $g \in V$ and $d(x, x_0) < \delta$ then $g_0g_1^{-1}g \in U$ so that $d(g_0g_1^{-1}gx, g_0x_0) < \varepsilon_1$ and therefore

$$d(gx, g_1x_0) < \varepsilon \quad \text{if } g \in V \text{ and } d(x, x_0) < \delta.$$

Thus ψ is continuous at $(g_1, x_0) \in G \times X$.

Next, we shall state an open mapping theorem in this case. In this note, a group G acting transitively on a metric space X means that for each $x, y \in X$ there exists an element $g \in G$ with $gx = y$.

THEOREM B. *Let G be a Polish group acting transitively on a complete metric space X. For each $x \in X$ the map $G \ni g \to \psi(g, x) = g \cdot x \in X$ is open.*

PROOF. Let $B(x, \delta) = \{y ; d(x, y) < \delta\}$, that is, the open ball with center x and radius $\delta > 0$. Let $x_0 \in X$ be fixed. We shall prove that for any neighbourhood (nbd in short) U of the unit e of G, there exists an open ball with

$$\overline{Ux_0} \supset B(x_0, \delta)$$

for some $\delta > 0$.

If this is proved, then we can prove the theorem as follows: for any $x_0 \in X$ and for any nbd U_0 of the unit e of G, there exist $\varepsilon_0 > 0$ and $B(x_0, \varepsilon_0) \subset \overline{U_0 x_0}$. Now, by induction we will construct two types of decreasing sequences of nbds of e: U_n and V_n for a given U_0. We want to prove that any point y_0 of $B(x_0, \varepsilon_0)$ is contained in $\overline{U_0^4 x_0}$.

There exists $V_0 \subset U_0$ such that $V_0 = V_0^{-1}$ (symmetric), $V_0^2 \subset U_0$ and $\overline{V_0 y_0} \subset B(x_0, \varepsilon_0)$. Choose $\delta_0 > 0$ such that

$$\overline{V_0 y_0} \supset B(y_0, \delta_0) \quad \text{and} \quad \delta_0 < (1/2)\varepsilon_0.$$

Then there exists $g_0 \in U_0$ with $g_0 \cdot x_0 = x_1 \in B(y_0, \delta_0)$. Choose a nbd U_1 of e such that

$$\overline{U_1 x_1} \subset B(y_0, \delta_0) \quad \text{with } U_1 = U_1^{-1}, \ U_1^2 \subset U_0$$

and

$$d(U_1 g_0) \ (= \text{diameter of } U_1 g_0) \leq (1/2)\varepsilon_0.$$

Choose further $0 < \varepsilon_1 < (1/2)\varepsilon_0$ so that

$$\overline{U_1 x_1} \supset B(x_1, \varepsilon_1).$$

Then, there exists $h_0 \in V_0$ such that

$$y_1 = h_0 y_0 \in B(x_1, \varepsilon_1).$$

Next, choose V_1 such that $V_1 = V_1^{-1}$, $V_1^2 \subset V_0$, with

$$\overline{V_1 y_1} \subset B(x_1, \varepsilon_1) \quad \text{and} \quad d(V_1 h_0) < (1/2)\delta_0.$$

Then there exists $0 < \delta_1 < (1/2)\delta_0$ such that

$$\overline{V_1 y_1} \supset B(y_1, \delta_1).$$

Continue this process to get $\{U_n\}$, $\{V_n\}$, $\{\varepsilon_n\}$, $\{\delta_n\}$, $\{g_n\}$, $\{h_n\}$, $\{x_n\}$, and $\{y_n\}$ such that

$$\begin{aligned}
&x_{n+1} = g_n \cdot x_n = \cdots = g_n \cdot g_{n-1} \cdots g_0 \cdot x_0, \\
&y_{n+1} = h_n \cdot y_n = \cdots = h_n \cdot h_{n-1} \cdots h_0 \cdot y_0, \\
&d(U_n g_{n-1} \cdots g_0) < (1/2)\varepsilon_{n-1}, \qquad g_n \in U_n, \\
&d(V_n h_{n-1} \cdots h_0) < (1/2)\delta_{n-1}, \qquad h_n \in V_n,
\end{aligned}$$

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with $0 < \varepsilon_n < (1/2)\varepsilon_{n-1}$ and $0 < \delta_n < (1/2)\delta_{n-1}$ and

$$\overline{U_n x_n} \supset B(x_n, \varepsilon_n) \supset \overline{V_n y_n}, \qquad x_{n+1} = g_n \cdot x_n \in B(y_n, \delta_n).$$

Let $\lim_{n\to\infty} g_n \cdot g_{n-1} \cdots g_n = g$ and $\lim_{n\to\infty} h_n h_{n-1} \cdots h_0 = h$. Then $g \in \overline{U_0^2}$, $h \in \overline{U_0^2}$ and

$$d(x_n, y_n) < \varepsilon_n \quad \text{implies} \quad g \cdot x_0 = h \cdot y_0.$$

Therefore, we have $y_0 = h^{-1} g \cdot x_0 \in \overline{U^4} x_0$.

From this result, we have that for arbitrary $x_0 \in X$ and for arbitrary nbd U, there exists $\varepsilon > 0$ with $U x_0 \supset B(x_0, \varepsilon)$. Hence the proof is complete, if the following lemma is proved.

LEMMA. *For any nbd U of e in G, $\overline{U x_0}$ contains an open ball $B(x_0, \delta)$ form some $\delta > 0$.*

PROOF. Let $\{g_n\}$ be a sequence of elements in G such that $\bigcup_n g_n U = G$, since G is a Polish group. Hence $\bigcup_n g_n U x_0 = X$ so that there exists an n_0 such that $g_{n_0} \overline{U x_0}$ contains an open ball; hence $\overline{U x_0}$ contains an open ball $B(y, \delta)$ because $\overline{U x_0} = g_{n_0}^{-1} g_{n_0} \overline{U x_0}$. Choose a nbd V of e with $V = V^{-1}$ and $V^2 \subset U$. Then $\overline{V x_0} \supset B(y, \delta)$ means that there exists $g \in V$ such that $g \cdot x_0 \in B(y, \delta)$; hence there exists $\varepsilon > 0$ such that $B(g \cdot x_0, \varepsilon) \subset B(y, \delta) \subset \overline{V x_0}$. Thus $g^{-1} B(g \cdot x_0, \varepsilon) \subset g^{-1} B(y, \delta) \subset g^{-1} \overline{V x_0} \subset \overline{V^2 x_0} \subset \overline{U x_0}$. But $g^{-1} B(g \cdot x_0, \varepsilon)$ is an open set containing x_0, so that there exists $\delta_1 > 0$ such that $B(x_0, \delta_1) \subset g^{-1} B(g \cdot x_0, \varepsilon) \subset \overline{U x_0}$.

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