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A CHARACTERISATION OF COMPACT MINIMAL HYPERSURFACES IN A UNIT SPHERE*

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Dedicated to Professor Hisao Nakagawa on his 60th birthday

In this note, we show that the totally geodesic sphere, Clifford torus and Cartan hypersurface are the only compact minimal hypersurfaces in $S^4(1)$ with constant scalar curvature if the Ricci curvature is not less than -1.

Let M be a compact minimal hypersurface in a unit sphere $S^{n+1}(1)$. It is well-known that if $S \leqslant n$, M is a totally geodesic hypersurface or Clifford torus, where S is the squared norm of the second fundamental form (see Chern, do Carmo and Kobayashi [3]). Yau [8] also characterised the totally geodesic hypersurface and Clifford torus by the sectional curvature which is not less than 0. When $n = 3$, the author and Jiang [2] proved that if the Ricci curvature is not less than $1/2$, then M is totally goedesic. Doi [5] proved that if the Ricci curvature $\mathrm{Ric}(M) \geqslant 0$ and the scalar curvature is constant, then M is a totally geodesic hypersurface or Clifford torus. On the other hand, we also know that the following Cartan hypersurface has constant scalar curvature and its Ricci curvature is not less than -1:

$$2x_5^3 + 3(x_1^2 + x_2^2)x_5 - 6(x_3^2 + x_4^2)x_5 + 3\sqrt{3}(x_1^2 - x_2^2)x_4 + 6\sqrt{3}x_1x_2x_3 = 2,$$

where $(x_1, \ldots, x_5)$ is the natural coordinate system in $\mathbb{R}^5$. Hence, the known examples of 3-dimensional compact minimal hypersurfaces in the unit sphere $S^4(1)$ with constant scalar curvature are only the totally geodesic sphere $S^3(1)$, Clifford torus and Cartan hypersurface above. Their Ricci curvatures are not less than -1. A natural problem is that if M is a compact minimal hypersurface in $S^4(1)$ with constant scalar curvature and its Ricci curvature is not less than -1, then is it one of the above minimal hypersurfaces?

In this note, we give an affirmative answer for the above problem, that is, we prove the following:

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Theorem. *Let M be a 3-dimensional compact minimal hypersurface in a unit sphere $S^4(1)$ with constant scalar curvature. If the Ricci curvature of M is not less than -1, then M is totally geodesic, or a Clifford torus or a Cartan hypersurface.*

Proof: For any point p in M, we choose a local orthonormal frame field such that $h_{ij} = \lambda_i \delta_{ij}$, where h_{ij} is the component of the second fundamental form. We consider function $f = \sum \lambda_i^3$ which does not depend on the choice of the local orthonormal frame field. If f is constant, then M has constant principal curvatures since M is minimal and has constant scalar curvature. Hence M is an isoparametric hypersurface. According to the classification of isoparametric hypersurfaces due to Cartan when $n = 3$, we know that Theorem is true. Next we will prove that f must be constant.

(a) If there exists a point p in M such that $f(p) = 0$, we have, at point p,

$$\lambda_1 + \lambda_2 + \lambda_3 = 0 \tag{1}$$

$$\lambda_1^3 + \lambda_2^2 + \lambda_3^2 = S, \tag{2}$$

$$\lambda_1^3 + \lambda_2^3 + \lambda_3^3 = 0 \tag{3}$$

where S is the squared norm of the second fundamental form. According to the Gaussian equation, we get that S is constant since M has constant scalar curvature. From (1), (2) and (3), we have $\lambda_1 = -\sqrt{S/2}$, $\lambda_2 = 0$ and $\lambda_3 = \sqrt{S/2}$ at point p. From the Gaussian equation and the assumptions of the Theorem, we obtain

$$R_{ii} = 2 - \lambda_i^2 \geqslant -1,$$

where R_{ii} is the component of the Ricci curvature tensor. Hence, at point p, $R_{11} = 2 - S/2 \geqslant -1$. Thus $S \leqslant 6$. Since S is constant, we know that $S = 0$, 3 or 6 on M from the result in [1]. Then [4, Corollary 1] due to de Almeida and Brito implies that M is isoparametric. Therefore the Theorem is valid in this case.

(b) If $f(p) \neq 0$ for any point p in M, without loss of generality, we assume $f < 0$. Because M is compact, we know that there exists a point p in M such that $f(p) = \max f < 0$. Therefore, at point p,

$$\lambda_1 + \lambda_2 + \lambda_3 = 0$$
$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = S,$$
$$\lambda_1^3 + \lambda_2^3 + \lambda_3^3 = \max f.$$

From the following Lemma, we know that f is constnat or λ_1, λ_2 and λ_3 are distinct at point p.

Lemma. (See Okumura [6]) *Let a_1, a_2 and a_3 be real numbers satisfying $\sum a_i = 0$ and $\sum a_i^2 = k^2$ for $k > 0$. Then*

$$\left|\sum a_i^3\right| \leqslant k^3/\sqrt{6}.$$

The equality holds if and only if two of them are equal with each other.

If f is constant, then the Theorem is valid. If λ_1, λ_2 and λ_3 are distinct at point p, from the assumptions of the Theorem and $f(p) = \max f$, we have

$$\sum h_{iik} = 0 \qquad \text{for any } k, \tag{4}$$

$$\sum \lambda_i h_{iik} = 0 \qquad \text{for any } k, \tag{5}$$

$$\sum \lambda_i^2 h_{iik} = 0 \qquad \text{for any } k, \tag{6}$$

where h_{ijk} is the component of the covariant derivative of the second fundamental form. Therefore from (4), (5) and (6), we conclude that $h_{iik} = 0$ for any i and k since λ_1, λ_2 and λ_3 are distinct. A direct and simple computation yields

$$\Delta f = 3(3 - S)f + 6 \sum \lambda_i h_{ijk}^2,$$

where Δ is the Laplacian on M. Since f obtains its maximum at point p, we get

$$(3 - S)f + 2 \sum \lambda_i h_{ijk}^2 \leqslant 0.$$

Because $\lambda_1 + \lambda_2 + \lambda_3 = 0$ and $h_{iik} = 0$ for any i and k, we have

$$\begin{aligned} \sum \lambda_i h_{ijk}^2 &= (1/3) \sum (\lambda_i + \lambda_j + \lambda_k) h_{ijk}^2 \\ &= (1/3) \sum_{i \neq j \neq k} (\lambda_i + \lambda_j + \lambda_k) h_{ijk}^2 + \sum_{i \neq k} (2\lambda_i + \lambda_k) h_{ijk}^2 + \sum_i \lambda_i h_{iii}^2 \\ &= 0. \end{aligned}$$

Hence $(3 - S)f \leqslant 0$. Thus $S \leqslant 3$ from $f(p) < 0$. From the result due to Chern, do Carmo and Kobayashi [3], we obtain that M is totally geodesic or a Clifford torus.

From (a) and (b), we complete the proof of the Theorem. $\square$

Making use of the generalised maximum principle due to Omori [7] and Yau [9] and the similar proof to the one of the Theorem, we can prove the following statement.

Theorem 1. *Let M be a 3-dimensional complete minimal hypersurface in $S^4(1)$ with constant scalar curvature. If the Ricci curvature of M is not less than -1, then $S = 0$, 3 or 6, where S is the squared norm of the second fundamental form.*

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