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RESEARCH ARTICLE

Tail risk driven by investment losses and exogenous shocks

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Abstract
Consider a company whose business carries the potential for investment losses and is additionally vulnerable to exogenous shocks. The unpredictability of the shocks makes it challenging for both the company and the regulator to accurately assess their impact, potentially leading to an underestimation of solvency capital when employing traditional approaches. In this paper, we utilize a stylized model to conduct an extreme value analysis of the tail risk of the company under a Fréchet-type and a Gumbel-type shock. Our main results explicitly demonstrate the different roles of investment risk and shock risk in driving large losses. Furthermore, we derive asymptotic estimates for the value at risk and expected shortfall of the total loss. Numerical studies are conducted to examine the accuracy of the obtained estimates.

1. Introduction

1.1. Motivations

Consider a company—such as an investment bank, an insurer, a mortgage provider, or a pension fund—whose business involves the potential for investment losses. In accordance with a certain regulatory framework, the company is required to hold an adequate solvency capital, typically determined based on past experience. In addition to investment losses, the company may also be vulnerable to exogenous shocks. The unpredictability of such shocks poses challenges to both the company and the regulator in accurately assessing their impact, potentially leading to an underestimation of the solvency capital when employing traditional approaches.

Here, we use the general term "shock" to refer to an unexpected event that may cause significant financial consequences. Shocks can originate from various sources, including financial distress, economic instability, policy changes, natural disasters, and technological failures.

Our consideration of shocks aligns with various contemporary regulatory frameworks in the banking, financial services, and insurance industries. In banking, the Basel III reforms set a primary objective "to improve the banking sector's ability to absorb shocks arising from financial and economic stress, whatever the source, thus reducing the risk of spillover from the financial sector to the real economy."[1] See also Borio *et al*. (2020), who review post-crisis financial regulatory reforms and emphasize that at the heart of these reforms lies the notion of shock-absorbing capacity. In the insurance industry, several authoritative regulatory frameworks run in parallel. For instance, the Solvency II Directive 2009/138/EC, Article 105, outlines provisions for calculating the basic solvency capital

[1]See the 2011 Basel Committee on Banking Supervision document "Basel III: A global regulatory framework for more resilient banks and banking systems" available at https://www.bis.org/publ/bcbs189.pdf.

requirement, taking into account extreme or exceptional events.[2] Notably, the European Insurance and Occupational Pensions Authority has organized a series of stress tests to assess the European insurance sector's resilience against a range of shocks, such as the impact of the COVID-19 pandemic in its 2021 stress test, as well as adverse market fluctuations that may occur during a financial crisis.[3]

Climate change acts as a primary driver of climate-related natural disaster shocks and financial shocks, which have profound impacts on the environment, economy, and society. According to Swiss Re, insured losses have exhibited a consistent annual growth rate of 5–7% since 1992, with the trend anticipated to persist. This upward trajectory can be attributed to a confluence of economic and natural factors, including the effects of climate change.[4] An example of climate-change shocks as a determinant for bankruptcies is the 2018 Camp Fire, which triggered the bankruptcy of Pacific Gas and Electric Company (PG&E) in January 2019 and the liquidation of Merced Property and Casualty Company in December 2018. See Appendix A.1 for a snapshot. Among the already vast and still fast-growing literature examining the economic consequences of climate change, we refer to the following two most recent papers, which particularly align with the motivation of our current study. Cantelmo *et al.* (2023) employ a dynamic stochastic general equilibrium model to assess the long-term macroeconomic and welfare effects of natural disaster shocks, modeled as exogenous. Pankratz *et al.* (2023) present empirical evidence that increased heat exposure negatively impacts firm financial performance, but capital market participants do not fully anticipate the economic consequences of heat as a first-order physical climate risk.

The COVID-19 pandemic underscores the unpredictability and potentially devastating consequences of exogenous shocks. Shortly after its outbreak, it spurred a substantial body of research focusing on its impacts on financial markets. In contrast to events like the global financial crisis, political shifts, or regulatory changes, which are at least partially endogenous, COVID-19 represents a truly exogenous shock to firms. This perspective is highlighted by Ramelli and Wagner (2020), who investigate the initial market reactions to COVID-19 and emphasize the importance of precautionary cash holdings for firms confronting a crisis like COVID-19. Bartik *et al.* (2020) conduct a survey of small businesses between March 28 and April 4, 2020, a critical period when both the progression of COVID-19 and the government's response were uncertain, shedding light on the financial fragility of many small businesses facing a major exogenous economic shock.[5]

Arguably, most, if not all, major bankruptcies are attributed to unexpected shocks. When a company faces unexpected shocks, it may experience severe financial distress, operational disruptions, and a loss of market share. The impact of such shocks can be amplified when the company is unprepared. Failure to anticipate or effectively respond to these shocks can put the company in financial instability and ultimately result in bankruptcy. A stark example is the recent collapse of Silicon Valley Bank (SVB). See Appendix A.1 for a snapshot.

1.2. Overview of the present work

We envision a company that faces both investment losses and exogenous shocks. Our goal is to gain a quantitative understanding of the roles of the two risk factors in driving large losses. To ensure the clear delivery of our message, we utilize a highly stylized model for ease of presentation. Nevertheless, we note that it is straightforward to extend this model in various ways to incorporate more realistic features, depending on the context. See Remark 5.1 for related discussions.

[2] See the latest 2021 version available at http://data.europa.eu/eli/dir/2009/138/2021-10-19.

[3] Refer to https://www.eiopa.europa.eu/browse/financial-stability/insurance-stress-test_en.

[4] Refer to the Sigma 1/2023 report titled "Natural Catastrophes and Inflation in 2022: A Perfect Storm", available at https://www.swissre.com/institute/research/sigma-research/sigma-2023-01.html.

[5] See also the BIS July 2021 report "Early lessons from the COVID-19 pandemic on the Basel reforms" available at https://www.bis.org/bcbs/publ/d521.pdf.

Specifically, suppose the company holds an initial capital $u > 0$ and incurs an investment loss of amount uX, where X denotes the overall negative return rate. Besides the investment loss, the company is vulnerable to an exogenous, hence unpredictable, shock, which can incur an additional loss of amount Y. Thus, the total loss is

$$L = uX + Y. \tag{1.1}$$

As we are concerned with the tail risk of the company, we focus on the tail probability of L; that is, we study the asymptotic behavior of $P(L > \ell u)$ as u becomes large for $\ell > 0$.

We make the following standing assumptions:

Assumption 1.1. *The negative return rate X and the shock variable Y satisfy the following:*

- *X follows a distribution function F supported on $(-\infty, \hat{x}]$, where $0 < \hat{x} < \infty$ denotes the upper endpoint of F (i.e., the essential upper bound of X);*
- *Y follows a distribution function G supported on $\mathbb{R}$;*
- *X and Y are independent.*

Both the negative return rate X and the shock variable Y are modeled as real-valued random variables. For X, a positive value indicates a loss, while a negative value indicates a gain in the investment portfolio. Likewise, for Y, a positive value signifies a bad shock, whereas a negative value signifies a good shock. The upper bound assumption on X is practically relevant and indeed commonly assumed in mathematical finance. We allow $\hat{x}$ to exceed 1 to account for the possibility of a short position. The independence assumption between X and Y greatly simplifies the problem, but it is not too unrealistic in view of the exogeneity of the shock. Indeed, it is a fundamental approach in modeling to introduce shocks that are exogenous to, and thus independent of, underlying factors. Outstanding early works in finance, economics, and insurance that exemplify this include Merton (1976), Bernanke (1983), Black and Litterman (1992), Cochrane (1991), Frees *et al.* (1996), Cox *et al.* (2000), Froot (2001), and Lindskog and McNeil (2003). Recent reviews of exogenous shocks in broader contexts include Atanasov and Black (2016), who survey shock-based methods in corporate finance and accounting research with a focus on exogenous shocks, Miklian and Hoelscher (2022), who outline an analytical lens suggesting how small- and medium-sized enterprises experience exogenous shocks, and Röglinger *et al.* (2022), who further conceptualize the interplay of exogenous shocks and business process management.

Remark 1.1 Arguably, the financial consequence of a shock should be linked to the business size. To reflect this linkage, we can replace Y with $v(u)Z$ for some positive and monotonically increasing function $v(u)$ and a real-valued random variable Z. Then the problem becomes

$$P(uX + v(u)Z > \ell u), \qquad u \to \infty, \tag{1.2}$$

where $\ell > 0$. There are three cases: $u = o(v(u))$, $v(u) \asymp u$, and $v(u) = o(u)$ (refer to Subsection 2.1 for such notational conventions). For the first case, (1.2) roughly reduces to $P(Z > 0)$. For the second case, as $v(u)$ is of the same order as u, (1.2) roughly reduces to $P(cX + Z > \ell c)$ for c varying over a certain closed subinterval of $(0, \infty)$. For the third case, (1.2) becomes

$$P\left(\frac{u}{v(u)}X + Z > \ell\frac{u}{v(u)}\right) = P(\tilde{u}X + Z > \ell\tilde{u}),$$

where $\tilde{u} = \frac{u}{v(u)} \to \infty$. This reduces to our original problem. Only the third case involves tail risk, making it the most interesting to us. Thus, it does not incur much loss of generality to restrict the study to the tail probability of L in (1.1).

While we have formulated our study within the general context of a company facing both investment risk and shock risk, we provide a concrete example below to illustrate its immediate applicability in the insurance context.

Example 1.1 Consider an insurer who holds an initial capital $x > 0$ at the beginning of a year. Suppose that the policies it underwrites incur a collection of claims with a total amount of S to be paid at the end of the year. In return, the insurer receives a total premium income equal to $(1 + \theta) E [S]$ at the beginning of the year, where $\theta > 0$ is the safety loading coefficient. Under a certain regulatory framework, the insurer is required to hold a solvency capital of amount $C > 0$; that is, $x + (1 + \theta) E [S] \geq C$. The insurer invests the remaining capital, $x + (1 + \theta) E [S] - C$, in risky assets to earn higher returns. Then the insurer's terminal wealth is

$$C (1 + r) + (x + (1 + \theta) E [S] - C) (1 + R) - S,$$

where $r > 0$ is a risk-free rate and $R \in \mathbb{R}$ is the overall return rate of its investment portfolio. Note that conservative insurance regulatory frameworks essentially prohibit insurers from engaging in short sales; see, for example, Molk and Partnoy (2019). Thus, we may assume a lower bound for R.

Consider the situation that the terminal wealth runs low, say, lower than a critical level y, in which case the insurer may be forced to initiate a rehabilitation plan. This has the probability

$$\begin{aligned} & P (C (1 + r) + (x + (1 + \theta) E [S] - C) (1 + R) - S < y) \\ = & P ((x + (1 + \theta) E [S] - C) (-R) + S > x + (1 + \theta) E [S] + Cr - y) \\ = & P (uX + Y > \ell u) , \end{aligned}$$

with $X = -R$, $Y = S$, $u = x + (1 + \theta) E [S] - C$, and

$$\ell = \frac{x + (1 + \theta) E [S] + Cr - y}{x + (1 + \theta) E [S] - C}.$$

Note that the three conditions in Assumption 1.1 become natural in this example. Remarkably, $\ell > 0$ necessarily varies within a certain range, which motivates us to derive *uniform* asymptotic formulas with respect to ℓ. In conclusion, our study of the tail probability of L has implications for insurance regulation.

We carry out our asymptotic study of $P (L > \ell u)$ for $\hat{x} \leq \ell < \infty$, which describes an ultimate tail area corresponding to a catastrophic loss. We have excluded $0 < \ell < \hat{x}$ from the current study because in this case, the tail probability $P (L > \ell u)$ roughly reduces to $P (X > \ell)$ and does not involve tail risk. We make the assumption that the distribution of Y is of the Fréchet or Gumbel type. In the Fréchet case, Y exhibits a power-like tail, and our analysis shows that $P (L > \ell u)$ is primarily determined by the tail of Y; see Theorem 3.1. In the Gumbel case, Y may have a heavy tail (but less heavy than a power-like tail) or a light tail, and our analysis shows that $P (L > \ell u)$ is jointly determined by the tails of X and Y; see Theorem 4.1. In summary, concerning the causation of large losses, if Y is of the Fréchet-type then Y plays a first-order role, but if Y is of the Gumbel-type then X and Y play a joint role.

A remarkable feature of our asymptotic formulas is that they hold *uniformly* for ℓ over a certain range. This uniformity not only enhances the theoretical value of the results but also amplifies their practical applicability. To attain uniformity in our asymptotic formulas, substantial effort is required. As applications of our main results, we further derive asymptotic estimates for both the value at risk (VaR) and expected shortfall (ES) of the loss L. In this pursuit, *the uniformity of our main results becomes pivotal*.

The rest of the paper is organized as follows: Section 2 collects necessary preliminaries; Section 3 considers a Fréchet-type shock; Section 4 considers a Gumbel-type shock; Section 5 conducts numerical studies to examine the accuracy of the obtained estimates; Section 6 concludes the work with a few remarks; finally, Appendix A collects additional information, prepares necessary lemmas, and compiles the proofs of the main results.

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2. Preliminaries

2.1. Notational conventions

Throughout this paper, all limit relationships are for $u \to \infty$ unless stated otherwise. For two positive functions $h_1(\cdot)$ and $h_2(\cdot)$ satisfying

$$m_* = \liminf_{u\to\infty} \frac{h_1(u)}{h_2(u)} \leq \limsup_{u\to\infty} \frac{h_1(u)}{h_2(u)} = m^*$$

for some $0 \leq m_* \leq m^* \leq \infty$, we write $h_1(u) = o(h_2(u))$ if $m^* = 0$, write $h_1(u) \sim h_2(u)$ if $m_* = m^* = 1$, write $h_1(u) \lesssim h_2(u)$ or $h_2(u) \gtrsim h_1(u)$ if $m^* = 1$, and write $h_1(u) \asymp h_2(u)$ if $0 < m_* \leq m^* < \infty$. Furthermore, when the two functions involve another argument $\ell \in \mathbb{R}$ for which case we rewrite them as $h_1(u, \ell)$ and $h_2(u, \ell)$, we often equip asymptotic relations with certain uniformity with respect to ℓ. For example, we say that $h_1(u, \ell) \sim h_2(u, \ell)$ holds uniformly for $\ell \in D \neq \emptyset$ if

$$\lim_{u\to\infty} \sup_{\ell \in D} \left| \frac{h_1(u, \ell)}{h_2(u, \ell)} - 1 \right| = 0.$$

For $x \in \mathbb{R}$, denote its positive part by $x_+ = x1_{(x>0)} = \max\{x, 0\} = x \vee 0$, where $1_{(\cdot)}$ is the indicator function of $(\cdot)$. For a general risk variable Z distributed by U and for $0 < q < 1$, its VaR at level q is $\mathrm{VaR}_q[Z] = U^{\leftarrow}(q) = \inf\{x \in \mathbb{R} : U(x) \geq q\}$, and its ES at level q, assuming the integrability of Z_+, is

$$\mathrm{ES}_q[Z] = \frac{1}{1-q}\int_q^1 \mathrm{VaR}_p[Z]dp = \mathrm{VaR}_q[Z] + \frac{1}{1-q}\int_{\mathrm{VaR}_q[Z]}^{\infty} \overline{U}(x)dx, \tag{2.1}$$

where $\overline{U} = 1 - U$ denotes the right tail of U. For the last step in (2.1), refer to Proposition 8.13 of McNeil *et al.* (2015).

2.2. Highlights of extreme value theory

To present our main results, we need to collect some basics from extreme value theory.

A positive measurable function h is said to be regularly varying at ∞ with index $\alpha \in \mathbb{R}$, written as $h \in \mathrm{RV}_\alpha$, if

$$\lim_{u\to\infty} \frac{h(su)}{h(u)} = s^\alpha \qquad \text{for every } s > 0.$$

When $\alpha = 0$, this defines a slowly varying function. In a natural way, the concept of regular variation can be extended to encompass rapid variation as its extreme. Precisely, a positive measurable function h is said to be rapidly varying with index ∞, written as $h \in \mathrm{RV}_\infty$, if

$$\lim_{u\to\infty} \frac{h(su)}{h(u)} = \begin{cases} \infty & \text{for every } s > 1, \\ 0 & \text{for every } 0 < s < 1. \end{cases}$$

The class $\mathrm{RV}_{-\infty}$ is defined in a symmetric way.

A distribution function U is said to belong to the maximum domain of attraction (MDA) of a non-degenerate distribution function V, denoted by $U \in \mathrm{MDA}(V)$, if there exist some constants $c_n > 0$ and $d_n \in \mathbb{R}$ for $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} U^n(c_n x + d_n) = V(x), \qquad x \in \mathbb{R}. \tag{2.2}$$

According to the classical Fisher–Tippett–Gnedenko theorem, as summarized in Theorem 3.2.3 of Embrechts *et al.* (1997), V must have the type of one of the following three distributions:

- Fréchet: $\Phi_\alpha(x) = \exp\{-x^{-\alpha}\}$ for $x > 0$ and $\alpha > 0$;
- Weibull: $\Psi_\alpha(x) = \exp\{-(-x)^\alpha\}$ for $x \leq 0$ and $\alpha > 0$;
- Gumbel: $\Lambda(x) = \exp\{-e^{-x}\}$ for $x \in \mathbb{R}$.

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For latter use, we highlight the characterization theorems for the three MDAs. The reader is referred to Chapter 1 of Resnick (1987), Chapter 3 of Embrechts *et al.* (1997), and Chapter 2 of Beirlant *et al.* (2006) for comprehensive treatments. For a distribution function U, denote by $\hat{z} \le \infty$ the upper endpoint of its support. If $\hat{z} < \infty$, U necessarily assigns no mass to $\hat{z}$.

The Fréchet and Weibull MDAs have characterizations in terms of regular variation. Actually, $U \in \text{MDA}\,(\Phi_\alpha)$, $\alpha > 0$, if and only if $\hat{z} = \infty$ and $\overline{U} \in \text{RV}_{-\alpha}$; see, for example, Theorem 3.3.7 of Embrechts *et al.* (1997). Moreover, $U \in \text{MDA}\,(\Psi_\alpha)$, $\alpha > 0$, if and only if $\hat{z} < \infty$ and $\overline{U}\left(\hat{z} - (\cdot)^{-1}\right) \in \text{RV}_{-\alpha}$; see, for example, Theorem 3.3.12 of Embrechts *et al.* (1997).

The Gumbel MDA is more complicated. It has a very broad coverage, ranging from moderately heavy-tailed distributions (such as lognormal) to light-tailed distributions (such as normal), and further to distributions with finite upper endpoints (namely, $\hat{z} < \infty$). According to Proposition 1.4 of Resnick (1987), $U \in \text{MDA}\,(\Lambda)$ if and only if $\overline{U}(x)$ is equivalent to the tail of a Von Mises distribution function; that is, there is some $z_0 < \hat{z}$ such that

$$\overline{U}(x) = b(x) \exp\left\{ -\int_{z_0}^{x} \frac{1}{a(y)} dy \right\}, \qquad z_0 < x < \hat{z}, \tag{2.3}$$

where $a(\cdot)$, called auxiliary function, is positive and differentiable with $\lim_{x\uparrow\hat{z}} a'(x) = 0$, and $b(\cdot)$ is positive with $\lim_{x\uparrow\hat{z}} b(x) = b_0 > 0$.

3. Under a Fréchet-type shock

Theorem 3.1 *In addition to Assumption 1.1, assume that* $G \in \text{MDA}(\Phi_\alpha)$ *for some* $\alpha > 0$.

(a) *For some arbitrarily fixed* $\ell_* > \hat{x}$, *it holds uniformly for* $\ell \in [\ell_*, \infty)$ *that*

$$P\left(L > \ell u\right) \sim E\left[(1 - \ell^{-1}X)^{-\alpha}\right] \overline{G}(\ell u). \tag{3.1}$$

(b) *If in addition* $E\left[(\hat{x} - X)^{-\kappa}\right] < \infty$ *for some* $\kappa > \alpha$, *then* (3.1) *holds uniformly for* $\ell \in [\hat{x}, \infty)$.

Theorem 3.1 shows that Y plays a first-order role, while X plays a second-order role in driving the tail risk. In item (b), the additional assumption implies that Y has a heavier tail than that of $(\hat{x} - X)^{-1}$. We remark that it is both nontrivial and meaningful to extend the uniformity region of ℓ to $[\hat{x}, \infty)$ to cover the critical point $\hat{x}$.

Now, we apply Theorem 3.1 to derive asymptotic estimates for the VaR and ES of L. In doing so, the uniformity established in Theorem 3.1 becomes crucial.

Theorem 3.2 *Define* $\hat{c} = E\left[(\hat{x} - X)^{-\alpha}\right] \le \infty$. *Consider a high confidence level* $q_u \in (0, 1)$ *satisfying*

$$1 - q_u \sim c\overline{G}(u) \tag{3.2}$$

for some constant $c \in (0, \hat{c}] \cap (0, \infty)$. *Denote by* $\hat{\ell}$ *the unique solution to the equation*

$$E\left[(\ell - X)^{-\alpha}\right] = c. \tag{3.3}$$

(a) *Under the conditions of Theorem 3.1(a), we have*

$$\text{VaR}_{q_u}[L] \sim \hat{\ell} u. \tag{3.4}$$

(b) *If further* $\alpha > 1$ *and* $c \in (0, \hat{c})$, *then*

$$\text{ES}_{q_u}[L] \sim \left(\hat{\ell} + c^{-1} \int_{\hat{\ell}}^{\infty} E\left[(y - X)^{-\alpha}\right] dy \right) u. \tag{3.5}$$

As explained before, the independence between X and Y in Assumption 1.1 reflects the exogeneity of the shock. For a general shock that is not entirely exogenous to the market, this independence assumption becomes irrelevant, and we need to assume a certain dependence structure to capture the endogeneity of the shock. Following a reviewer's request, we illustrate that under a dependence structure called bivariate regular variation (BRV), we can derive an explicit uniform asymptotic formula for the tail probability of L. We choose BRV for this illustration because it nicely couples heavy-tailed marginals with tail dependence. There are many other dependence structures that enable us to derive explicit uniform asymptotic formulas for both the current case of Fréchet-type shocks and the case of Gumbel-type shocks in the next subsection, but we will not expand on such discussions in this paper.

For a nonnegative random variable ξ distributed by U, note that $\overline{U} \in \mathrm{RV}_{-\alpha}$ for $\alpha > 0$ can be equivalently written as

$$\lim_{u\to\infty} \frac{1}{\overline{U}(u)} P\left(\frac{\xi}{u} > s\right) = s^{-\alpha} \qquad s > 0.$$

This naturally extends to BRV. Precisely, a nonnegative random vector (ξ, η) is said to follow BRV if there exist a reference distribution U and a non-degenerate limit measure ν such that

$$\lim_{u\to\infty} \frac{1}{\overline{U}(u)} P\left(\frac{(\xi, \eta)}{u} \in B\right) = \nu(B)$$

holds for every relatively compact and ν-continuous (i.e., $\nu(\partial B) = 0$) Borel set $B \subset [0, \infty]^2 \backslash \{(0, 0)\}$. Necessarily, the limit measure ν is homogeneous in the sense that there exists some index $\alpha > 0$ such that $\nu\,(tB) = t^{-\alpha}\nu\,(B)$ for every $t > 0$ and every Borel set $B \in [0, \infty]^2 \backslash \{(0, 0)\}$. For this case, we write $(\xi, \eta) \in \mathrm{BRV}_{-\alpha}(\nu, \overline{U})$. We refer the reader to Resnick (1987, 2007) for textbook treatments of multivariate regular variation.

Back to our stylized model (1.1). Define

$$\xi = \frac{1}{\hat{x} - X} \qquad \text{and} \qquad \eta = Y_+. \tag{3.6}$$

Proposition 3.1 *Assume that* (ξ, η) *defined by* (3.6) *follows* $\mathrm{BRV}_{-\alpha}(\nu, \overline{U})$ *for some* $\alpha > 0$. *For* $\ell_b = \hat{x} + bu^{-1/2}$ *for arbitrarily fixed* $b \geq 0$, *we have*

$$\lim_{u\to\infty} \frac{P\,(L > \ell_b u)}{\overline{U}\left(\sqrt{u}\right)} = \nu\,(A_b)\,, \tag{3.7}$$

where $A_b = \left\{(x, y) \in \mathbb{R}_+^2 : y - \frac{1}{x} > b\right\}$.

It is easy to see that $\nu\,(A_b)$ is non-increasing and continuous in b, with $0 = \nu\,(A_\infty) \leq \nu\,(A_0) < \infty$. Thus, the convergence (3.7) is automatically uniform for $0 \leq b < \infty$, meaning that

$$\lim_{u\to\infty} \sup_{0\leq b<\infty} \left| \frac{P\,(L > \ell_b u)}{\overline{U}\left(\sqrt{u}\right)} - \nu\,(A_b) \right| = 0.$$

Moreover, if ν shows tail dependence, that is, $\nu\left((1, \infty)^2\right) > 0$, we have $0 < \nu\,(A_b) < \infty$ for all $0 \leq b < \infty$. Then it holds locally uniformly that

$$P\,(L > \ell_b u) \sim \nu\,(A_b)\,\overline{U}\left(\sqrt{u}\right).$$

Remarkably, Proposition 3.1 shows that the tail behavior of L is jointly determined by the tails of X and Y, contrasting with the conclusion of Theorem 3.1.

4. Under a Gumbel-type shock

We assume that $G \in \mathrm{MDA}(\Lambda)$ fulfills the representation (2.3) with an infinite upper endpoint and an auxiliary function $a(\cdot)$. In the theorem below, we need a function $\phi(\cdot)$ specified as follows: (1) if $a(\cdot)$ is bounded, we choose $\phi(u) = \infty$; (2) If $a(\cdot)$ is unbounded, then by the argument after (A5), we can always find a positive and increasing function $\phi(\cdot)$ such that $\phi(u) \to \infty$ and $a(\phi(u)u) = o(u)$.

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As usual, define the gamma function

$$\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx, \qquad r > 0. \tag{4.1}$$

Theorem 4.1 *In addition to Assumption 1.1, assume that $F \in \mathrm{MDA}(\Psi_\beta)$ for some $\beta > 0$ and that $G \in \mathrm{MDA}(\Lambda)$ fulfills the representation (2.3) with an infinite upper endpoint. If the auxiliary function $a(\cdot)$ is unbounded, further assume that it is non-decreasing. Then for some arbitrarily fixed $\ell_* > \hat{x}$ and for the function $\phi(\cdot)$ specified above, it holds uniformly for $\ell \in [\ell_*, \phi(u))$ that*

$$P(L > \ell u) \sim \Gamma(\beta+1) \overline{F}\left(\hat{x} - u^{-1} a(\ell u - \hat{x} u)\right) \overline{G}(\ell u - \hat{x} u). \tag{4.2}$$

Recall that the assumption $F \in \mathrm{MDA}(\Psi_\beta)$ implies $\overline{F}\left(\hat{x} - (\cdot)^{-1}\right) \in \mathrm{RV}_{-\beta}$. Also recall that the assumption $G \in \mathrm{MDA}(\Lambda)$ implies $\overline{G} \in \mathrm{RV}_{-\infty}$ and consequently Y_+ has a finite moment of any positive order; see, for example, Corollary 3.3.32 of Embrechts *et al.* (1997). Therefore, Y has a lighter tail than that of $(\hat{x} - X)^{-1}$. In contrast to the conclusion of Theorem 3.1, Theorem 4.1 shows that X and Y play a joint role in driving the tail risk.

Now we apply Theorem 4.1 to derive asymptotic estimates for the VaR and ES of L. In doing so, the uniformity established in Theorem 4.1 becomes crucial.

Theorem 4.2 *Consider a high confidence level $q_u \in (0, 1)$ satisfying*

$$1 - q_u \sim \Gamma(\beta+1) \overline{F}\left(\hat{x} - u^{-1} a(cu)\right) \overline{G}(cu) \qquad \textit{for some } c > 0. \tag{4.3}$$

(a) Under the conditions of Theorem 4.1, we have

$$\mathrm{VaR}_{q_u}[L] \sim \left(\hat{x} + c\right) u. \tag{4.4}$$

(b) If further $u^{-\frac{1}{\delta}} \lesssim a(u) \lesssim u^{1-\delta}$ for some $0 < \delta < 1$, then

$$\mathrm{ES}_{q_u}[L] - \mathrm{VaR}_{q_u}[L] \sim a\left(\mathrm{VaR}_{q_u}[L] - \hat{x} u\right). \tag{4.5}$$

Since $a(u) = o(u)$ from (A5), a combination of (4.4)–(4.5) implies that $\mathrm{VaR}_{q_u}[L]$ and $\mathrm{ES}_{q_u}[L]$ share the same tail behavior, both asymptotic to $\left(\hat{x} + c\right) u$. Nevertheless, (4.5) contains an additional merit by offering a precise asymptotic estimate for the difference between $\mathrm{ES}_{q_u}[L]$ and $\mathrm{VaR}_{q_u}[L]$. This becomes meaningful in some circumstances (e.g., when we possess the true value of $\mathrm{VaR}_{q_u}[L]$ or a highly accurate estimate for it).

Our study still misses the scenario that both the negative return rate X and the shock variable Y are of Gumbel-type. For the sake of completeness, as requested by a reviewer, we present a result under a special case of such a Gumbel–Gumbel scenario. Note that in our setting, X always has a finite upper endpoint, while Y is unbounded. We assume that both the tail of X as it approaches its upper endpoint $\hat{x}$ and the ultimate tail of Y are of the exponential power form. These distributions, although not entirely general, arguably encompass most useful examples in the Gumbel MDA.

Proposition 4.1 *In addition to Assumption 1.1, assume that*

$$\begin{cases} \overline{F}(x) \sim K_1 e^{-c_1 \left(\hat{x} - x\right)^{-\tau_1}}, & x \uparrow \hat{x}, \\ \overline{G}(y) \sim K_2 e^{-c_2 y^{\tau_2}}, & y \to \infty, \end{cases} \tag{4.6}$$

where $K_1, c_1, \tau_1, K_2, c_2, \tau_2$ are all positive constants. Then

$$P\left(L > \hat{x} u\right) \sim K_1 K_2 c_2 \left(\frac{2\pi s_*^{\rho+2}}{c_1 \rho(\rho+1)}\right)^{\frac{1}{2}} u^{\frac{\tau_1 \tau_2}{2(\tau_1+\tau_2)}} \exp\left\{-t_* u^{\frac{\tau_1 \tau_2}{\tau_1+\tau_2}}\right\}, \tag{4.7}$$

where $\rho = \frac{\tau_1}{\tau_2}$, $s_* = \left(\frac{c_1 \rho}{c_2}\right)^{\frac{1}{\rho+1}}$, and $t_* = c_1 s_*^{-\rho} + c_2 s_*$.

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As a sanity check, letting all parameters in (4.6) be 1, the asymptotic formula (4.7) reduces to

$$P\left(L > \hat{x}u\right) \sim \sqrt{\pi}u^{\frac{1}{4}}e^{-2\sqrt{u}}.$$

Note that $P\left(L > \hat{x}u\right) = P\left(\xi Y > u\right)$, where $\xi = \frac{1}{\hat{x}-X}$. This result is consistent with Example 2.1 of Tang (2008).

As the proof of Proposition 4.1 shows, it is usually quite troublesome to derive precise asymptotic formulas for the Gumbel–Gumbel scenario, and the obtained formulas are typically quite involved. Challenges arise mainly because most powerful techniques dealing with regular variation are not applicable anymore. For this reason, we refrain from pursuing general results for the Gumbel–Gumbel scenario, and we do not provide uniformity with respect to ℓ for Proposition 4.1. Asymptotic formulas for the Gumbel–Gumbel scenario may offer certain theoretical insights, but admittedly, they often exhibit poor numerical performance.

5. Numerical studies

5.1. Models for the negative return rate and the shock variable

In this section, we conduct numerical studies to examine the accuracy of the asymptotic estimates for $P(L > \ell u)$ as well as for the VaR and ES of L obtained in Sections 3–4. First introduce three distributions:

(a) We always assume that the negative return rate X follows a scaled beta (α, β) distribution over $[-0.75, 0.5]$ for $\alpha, \beta > 0$. Precisely, F has the probability density function

$$f(x) = 0.8\frac{\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}(0.8x+0.6)^{\alpha-1}(0.4-0.8x)^{\beta-1}, \qquad -0.75 \le x \le 0.5. \tag{5.1}$$

(b) In Subsection 5.2, we assume that the shock variable Y follows a mixed Lomax distribution, which assigns a probability 0.4 over $(-\infty, 0)$ according to the Lomax (α, θ_1) distribution and assigns a probability 0.6 over $(0, \infty)$ according to the Lomax (α, θ_2) distribution, where the shape parameter α and the two scale parameters θ_1 and θ_2 are all positive. Precisely, G has the probability density function

$$g(y) = 0.4 \times \frac{\alpha}{\theta_1}\left(1-\frac{y}{\theta_1}\right)^{-\alpha-1}1_{(y<0)} + 0.6 \times \frac{\alpha}{\theta_2}\left(1+\frac{y}{\theta_2}\right)^{-\alpha-1}1_{(y\ge 0)}. \tag{5.2}$$

(c) In Subsection 5.3, we assume that the shock variable Y follows a mixed Weibull distribution, which assigns a probability 0.4 over $(-\infty, 0)$ according to the Weibull (τ, λ_1) distribution and assigns a probability 0.6 over $(0, \infty)$ according to the Weibull (τ, λ_2) distribution, where the shape parameter τ and the two scale parameters λ_1 and λ_2 are all positive. Precisely, G has the probability density function

$$g(y) = 0.4 \times \frac{\tau}{\lambda_1}\left(-\frac{y}{\lambda_1}\right)^{\tau-1}e^{-(-y/\lambda_1)^{\tau}}1_{(y<0)} + 0.6 \times \frac{\tau}{\lambda_2}\left(\frac{y}{\lambda_2}\right)^{\tau-1}e^{-(y/\lambda_2)^{\tau}}1_{(y\ge 0)}. \tag{5.3}$$

The specification in (5.1) accounts for both investment gains (corresponding to negative values of X) and investment losses (corresponding to positive values of X). In both (5.2) and (5.3), the specifications of the probabilities 0.4 and 0.6 signify the perception that bad shocks (corresponding to positive values of Y) are more likely than good shocks (corresponding to negative values of Y). Furthermore, we allow Y to be distributed differently over $(-\infty, 0)$ and $(0, \infty)$ to reflect the reality that good shocks and bad shocks may exhibit different patterns.

All numerical results in this paper are realized in `Python`. We adopt the `quad` function in the `scipy.integrate` module to obtain the true value of $P(L > \ell u)$. Given $u > 0$ and $0 < q < 1$, by letting $P(L > v_{q_L}) = 1 - q$, we adopt the `fsolve` function in the `scipy.optimize` module to obtain v_{q_L} as the true value of $\mathrm{VaR}_q[L]$. Then we follow (2.1) to compute the true value of $\mathrm{ES}_q[L]$.

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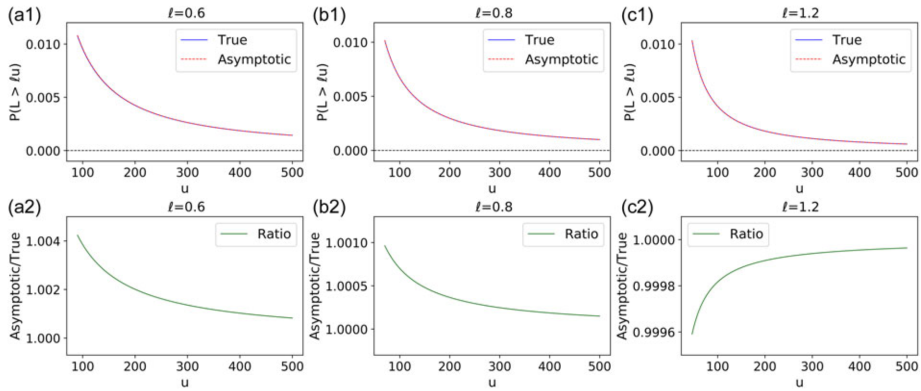

Figure 1. *Compare the estimate (3.1) for $P(L > \ell u)$ with its true value.*

Remark 5.1 We have selected these standard distributions (5.1)–(5.3) mainly to facilitate our numerical studies. It is important to note that, for general cases, directly computing $P(L > \ell u)$, $\mathrm{VaR}_q[L]$, and $\mathrm{ES}_q[L]$ may become challenging or even impossible. In practice, X as the negative return rate of an investment portfolio may be modeled as a randomly weighted sum of dependent negative return rates of individual assets, while Y may represent the accumulating result of a sequence of shocks and hence be modeled as the sum of a random number of dependent shock variables. In such a situation, it is generally impossible to get the exact distributions of X and Y. However, the theorems in Sections 3–4 only require conditions on the tails X and Y rather than their exact distributions. There are established procedures in asymptotic analysis to deal with such intricate stochastic structures and determine conditions under which X and Y are amenable to the theorems in Sections 3–4, enabling us to still easily derive asymptotic estimates for $P(L > \ell u)$, $\mathrm{VaR}_q[L]$, and $\mathrm{ES}_q[L]$. To save space, we will not expand on discussions along this line, but refer the interested reader to Ng *et al.* (2002) and Tang and Yuan (2014)) for some results readily applicable here.

5.2. For Theorems 3.1 and 3.2

Example 5.1 Let X follow a scaled beta (6,5) distribution described by (5.1). Then $E[X] \approx -0.0682$, which means that the expected return rate is approximately 6.82%, and $F(0) \approx 0.6331$, which is the probability that the risky investment results in an overall profitable outcome. Let Y follow a mixed Lomax distribution described by (5.2) with $\alpha = 1.2$, $\theta_1 = 1$, and $\theta_2 = 2$, so that Y has a finite mean.

Figure 1 plots the estimate (3.1) for $P(L > \ell u)$ against its true value in subfigures (a1)–(c1) and their ratio in subfigures (a2)–(c2), for $\ell = 0.6$, 0.8, and 1.2. We allow u to vary within a certain interval to ensure that $P(L > \ell u)$ roughly varies from 0.001 to 0.01, a range corresponding to tail risk in most regulatory frameworks. All subfigures confirm that the estimate closely matches the true value. Noticeably, as u increases, the ratio approaches 1 from above in subfigures (a2)–(b2) but from below in subfigure (c2), with relative errors ranging from 0.5% to 0.05% across all cases.

Tables 1–2 tabulate the estimates (3.4)–(3.5) for $\mathrm{VaR}_q[L]$ and $\mathrm{ES}_q[L]$, as well as the ratios of these estimates to the corresponding true values, for $u = 25$, 50, 100, and 150, and $q = 95\%$, 97.5%, 99%, 99.5%, 99.75%, and 99.9%, respectively. To apply (3.4)–(3.5), given u and q, by (3.2) we calculate $c = \frac{1-q}{\overline{G}(u)}$ and then adopt the `fsolve` function to solve (3.3) to obtain $\hat{\ell}$. The tables show that the ratios are always close to 1, especially for larger values of u.

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Table 1. *Compare the estimate (3.4) for* $\mathrm{VaR}_q[L]$ *with its true value.*

		$\mathrm{VaR}_q[L]$					
q		95.00%	97.50%	99.00%	99.50%	99.75%	99.90%
	Asymptotic	14.53	25.31	54.85	98.58	176.70	380.94
u=25	True	13.58	25.34	57.31	104.56	188.96	409.55
	Ratio	1.0700	0.9988	0.9570	0.9427	0.9352	0.9302
	Asymptotic		27.15	56.43	101.35	182.21	394.12
u=50	True		26.09	56.70	103.47	187.60	408.00
	Ratio		1.0406	0.9952	0.9795	0.9713	0.9660
	Asymptotic			58.77	102.48	183.83	399.15
u=100	True			57.83	102.53	185.57	405.23
	Ratio			1.0163	0.9995	0.9906	0.9850
	Asymptotic				104.00	183.98	399.48
u=150	True				103.31	184.46	402.89
	Ratio				1.0066	0.9974	0.9915

Table 2. *Compare the estimate (3.5) for* $\mathrm{ES}_q[L]$ *with its true value.*

		$\mathrm{ES}_q[L]$					
q		95.00%	97.50%	99.00%	99.50%	99.75%	99.90%
	Asymptotic	87.25	155.77	335.45	598.77	1068.08	2293.86
u=25	True	92.23	166.30	360.39	644.79	1151.65	2475.50
	Ratio	0.9459	0.9367	0.9308	0.9286	0.9274	0.9266
	Asymptotic		161.44	347.32	620.50	1107.71	2380.54
u=50	True		165.89	359.28	643.42	1150.13	2473.88
	Ratio		0.9732	0.9667	0.9644	0.9631	0.9623
	Asymptotic			353.22	0.8314	1126.89	2424.32
u=100	True			358.29	1.1285	1147.47	2470.83
	Ratio			0.9859	0.9834	0.9821	0.9812
	Asymptotic				633.72	1132.21	2437.25
u=150	True				640.19	1145.32	2468.00
	Ratio				0.9899	0.9886	0.9875

5.3. For Theorems 4.1 and 4.2

Example 5.2 *Let X follow a scaled beta* $(0.6, 0.5)$ *distribution described by* (5.1). *Then* $F \in \mathrm{MDA}(\Psi_{0.5})$ *with upper endpoint* $\hat{x} = 0.5$. *We still have the same mean* $E[X] \approx -0.0682$ *but* $F(0) \approx 0.5144$. *Let Y follow a mixed Weibull distribution described by* (5.3) *with* $\tau = 0.9$, $\lambda_1 = 1.5$, *and* $\lambda_2 = 2$. *Then* $G \in \mathrm{MDA}(\Lambda)$ *with an auxiliary function* $a(y) = \lambda_2^{\tau}\tau^{-1}y^{1-\tau}$ *for* $y > 0$. *Notably,* $g(0) = \infty$, *which is not problematic but may be interpreted as reflecting the reality that there are significantly more small shocks than large shocks.*

Figure 2 plots the estimate (4.2) for $P(L > \ell u)$ against its true value in subfigures (a1)–(c1) and their ratio in subfigures (a2)–(c2), for $\ell = 0.6$, 0.8, and 1, where we allow u to vary within a certain interval to ensure that $P(L > \ell u)$ roughly varies from 0.001 to 0.01. All subfigures show that the estimate closely matches the true value and the ratio approaches 1 from below as u increases, indicating a slight underestimation.

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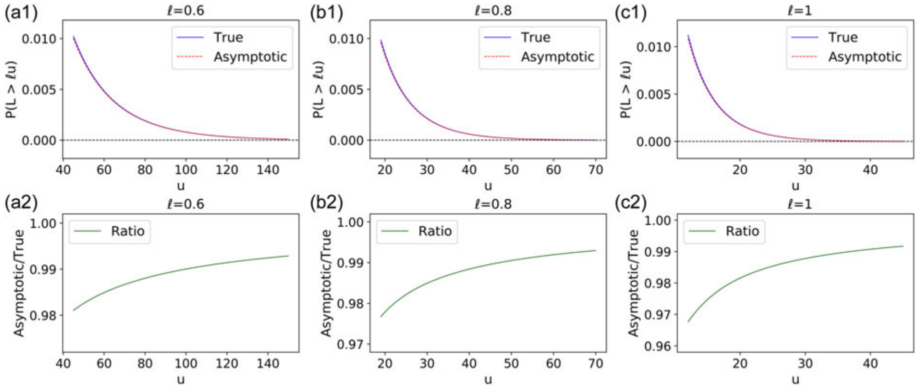

Figure 2. *Compare the estimate (4.2) for $P(L > \ell u)$ with its true value.*

Table 3. *Compare the estimate (4.4) for* $\mathrm{VaR}_q[L]$ *with its true value.*

		$\mathrm{VaR}_q[L]$					
q		95.00%	97.50%	99%	99.50%	99.75%	99.90%
	Asymptotic	7.471	9.153	11.453	13.240	15.061	17.512
u=10	True	7.586	9.256	11.549	13.334	15.153	17.603
	Ratio	0.9849	0.9889	0.9917	0.9930	0.9939	0.9948
	Asymptotic	11.622	13.272	15.537	17.303	19.106	21.536
u=20	True	11.714	13.341	15.595	17.356	19.156	21.584
	Ratio	0.9921	0.9948	0.9963	0.9969	0.9974	0.9978
	Asymptotic	16.141	17.773	20.017	21.770	23.562	25.981
u=30	True	16.239	17.838	20.067	21.815	23.603	26.018
	Ratio	0.9940	0.9963	0.9975	0.9980	0.9983	0.9986

Tables 3–4 tabulate the estimates for $\mathrm{VaR}_q[L]$ and $\mathrm{ES}_q[L]$ based on (4.4)–(4.5), as well as the ratios of these estimates to the corresponding true values, for $u = 10$, 20, and 30, and $q = 95\%$, 97.5%, 99%, 99.5%, 99.75%, and 99.9%, respectively. To apply (4.4)–(4.5), given u and q, we understand (4.3) as an equality and then adopt the `fsolve` function to solve it to obtain c. The tables show that the ratios approach 1 and the estimates improve as u increases.

5.4. Empirical discussions

To elaborate on our thoughts regarding empirical studies concerning the obtained theoretical results, we interpret our stylized model in the insurance context, where X represents the annual negative return rate of an insurance company and Y represents its annual loss due to climate shocks. In this context, our study of the tail probability of L in (1.1) can help us gain a quantitative understanding of the roles of the two risk factors in driving potentially large losses. For a representative insurance company, the overall negative return rate X of its investment portfolio can be approximated as the industry average. Then useful datasets for modeling X include: (1) The Dow Jones U.S. Select Insurance Index, which, as part of the Dow Jones U.S. Broad Stock Market Index, tracks the performance of U.S. insurance companies since 1991, including 53 constituents; (2) The S&P Insurance Select Industry Index, which is a subset of the S&P Total Market Index, focusing specifically on U.S. insurance companies since 2003, including 49 constituents. Datasets useful for modeling the climate shock loss Y include: (1) The Spatial Hazard Events and Losses Database for the U.S. (SHELDUS), which records and analyzes hazard events and

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Table 4. *Compare the estimate for* $\mathrm{ES}_q[L]$ *based on (4.5) with its true value.*

q		$\mathrm{ES}_q[L]$					
		95.00%	97.50%	99%	99.50%	99.75%	99.90%
u=10	Asymptotic	9.741	11.544	13.952	15.800	17.672	20.181
	True	10.060	11.800	14.166	15.995	17.854	20.350
	Ratio	0.9683	0.9783	0.9849	0.9878	0.9898	0.9917
u=20	Asymptotic	13.798	15.606	17.998	19.833	21.692	24.184
	True	14.137	15.847	18.182	19.992	21.834	24.310
	Ratio	0.9760	0.9848	0.9899	0.9920	0.9935	0.9948
u=30	Asymptotic	18.242	20.068	22.453	24.281	26.132	28.616
	True	18.629	20.320	22.636	24.435	26.268	28.733
	Ratio	0.9792	0.9876	0.9919	0.9937	0.9948	0.9959

their associated losses in the U.S. since 1960; (2) The Emergency Events Database (EM-DAT), which provides comprehensive information on natural and technological disasters globally since 1900.

A potential issue in utilizing these datasets for our study is that a major climate event may concurrently drive up both the negative return rate X and the climate shock loss Y, which violates our independence assumption. We propose two methods to mitigate this issue. First, we can eliminate the effects of climate shocks on the investment portfolio by employing intervention analysis (Box and Tiao, 1975) to pinpoint the occurrence of such shocks. Then we utilize the resulting negative return rate to inform the modeling of X to ensure its independence of Y. Second, we can instead use the data for Y during a period subsequent to that of X so that the unpredictability of climate events well justifies the independence between X and Y.

Note that the three types for the limit distribution in (2.2) can be unified through a shape parameter (often called the extreme value index) $\gamma \in \mathbb{R}$ to

$$V_\gamma = \exp\left\{-(1+\gamma x)^{-1/\gamma}\right\},$$

for $x \in \mathbb{R}$ such that $1+\gamma x > 0$, where in case $\gamma = 0$ the right-hand side is understood as $\exp\{-e^{-x}\}$, namely, the limit as $\gamma \to 0$. Precisely, when $\gamma > 0$, we have the Fréchet case with $\alpha = \frac{1}{\gamma}$, when $\gamma < 0$, we have the Weibull case with $\alpha = -\frac{1}{\gamma}$, while when $\gamma = 0$, we have the Gumbel case. A three-parameter family can be constructed by introducing a real-valued location parameter and a positive scale parameter.

To implement empirical studies, the first step is to estimate the shape parameter $\gamma \in \mathbb{R}$, which is a classical topic in univariate extreme value theory. This estimation will help us determine which MDA highlighted in Subsection 2.2 is relevant. Traditional methods include: (1) the block-maxima method, which divides data into non-overlapping blocks of equal size and fits the block maxima to a three-parameter limit distribution; (2) the peaks-over-threshold method, which restricts attention to data over a given large threshold and fits the exceedances to a corresponding generalized Pareto distribution; (3) the Pickands estimation, which relies on the fact that, for $U \in \mathrm{MDA}\left(V_\gamma\right)$,

$$\lim_{t\to\infty} \frac{\tilde{U}(2t)-\tilde{U}(t)}{\tilde{U}(t)-\tilde{U}(t/2)} = 2^\gamma,$$

where $\tilde{U}(t) = U^{\leftarrow}(1-1/t)$, and then constructs an empirical version of the expression for γ. These three methods work for $\gamma \in \mathbb{R}$, and, in particular, the peaks-over-threshold method can estimate the upper endpoint of U if the estimated value for γ is negative. When we are certain that the shape parameter γ is positive, indicating heavy-tailed losses, as is often the case in climate losses, we can resort to: (4) the Hill estimation, which relies on the fact that, for $U \in \mathrm{MDA}\,(\Phi_\alpha)$ for $\alpha = \frac{1}{\gamma} > 0$,

$$\lim_{t\to\infty} \frac{1}{\overline{U}(t)} \int_t^\infty (\ln x - \ln t)\, dU(x) = \gamma,$$

and then constructs an empirical version of this expression for $\gamma > 0$.

Each of these methods has its advantages and limitations. Many improved estimators for the general case of $\gamma \in \mathbb{R}$ have been developed since Dekkers *et al.* (1989). To keep the paper short, we will not expand on this discussion here, but we refer the reader to Sections 6.4–6.5 of Embrechts *et al.* (1997), Section 3 of de Haan and Ferreina (2006), Chapters 4–6 of Beirlant *et al.* (2006), and Section 5.1 of McNeil *et al.* (2015), among other monographs, for comprehensive reviews of these methods, case studies, and further discussions for data exhibiting serial features. The latest developments can be found in Beirlant *et al.* (2005), Fraga Alves *et al.* (2009), and Buitendag *et al.* (2019); see also the recent reviews by Gomes and Guillou (2015) and Fedotenkov (2020).

6. Concluding remarks

In this paper, we envision a company facing both investment risk (quantified as a real-valued, upper-bounded random variable denoting the negative return rate) and shock risk (quantified as an independent random variable denoting the shock loss). We utilize a stylized model to conduct an extreme value analysis of its tail risk across various extreme scenarios. Our main results explicitly demonstrate the different roles of the two risk factors in driving large losses.

As an initial step in exploring this topic, our work gives rise to a series of new research problems. First, the standing assumption in our current work is the exogeneity of shocks. It is desirable to also consider shocks that are partially or entirely endogenous to the financial market, which would allow us to delineate the interplay between the two risk factors. Proposition 3.1 represents a preliminary exploration in the current static setting. Second, it is meaningful to extend the study to a dynamic setting, be it continuous-time or discrete-time (i.e., multi-period). In a dynamic setting, we can incorporate additional practical features of investment risk and shock risk. This extension may yield new insights into their roles in driving tail risk. Third, our entire work relies on certain tail assumptions rooted in extreme value theory. However, it is typically infeasible to know the tails of these risk factors. In addition to their intricate stochastic structures, as highlighted in Remark 5.1, these risk factors are surrounded by multiple layers of uncertainty, hindering the precise estimation of their distributions. Consequently, it becomes important to robustify the estimation against model uncertainty. This consideration is particularly relevant when analyzing tail risk.

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A. Appendix

A.1. Snapshots

Two bankruptcies triggered by the Camp Fire

The 2018 Camp Fire—the deadliest and most destructive wildfire in California's history—serves as a perfect example of climate-change shocks as a determinant for bankruptcies. This disaster was triggered by the faulty power line of PG&E, ultimately leading to PG&E filing for Chapter 11 protection in January 2019, citing expected wildfire liabilities of $30 billion. This event is often regarded as the first major climate-change-induced bankruptcy. Prior to the bankruptcy, PG&E, as one of the largest U.S. energy

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utilities, was rated by S&P Global, Moody's, and Fitch as investment grade, and by Sustainalytics in the top 10% of its peers in the environmental category.

Less media attention has been given to the fact that the 2018 Camp Fire also directly caused the liquidation of Merced Property and Casualty Company, a regional insurer established in March 1906. Merced's liquidation occurred in December 2018, with an estimated USD 87 million in liabilities, far exceeding its USD 23 million of capital. Like PG&E, as of year-end 2017, Merced appeared healthy from many aspects, notably holding an A- rating from A.M. Best.[6]

The Silicon Valley Bank collapse

Due to its rapid growth in deposits during the COVID-19 pandemic, SVB made an outsized bet on government debt with durations ranging from 10 to 30 years. According to the SVB 2022 annual report filed on February 24, 2023[7], SVB put at least 75% of its debt as held to maturity. When interest rates began to rise in 2022, depositors started demanding their funds back, triggering a subsequent bank run that ultimately led to the closure of SVB on March 10, 2023. Apparently, when taking this gamble, SVB significantly underestimated the likelihood of interest rate hikes. Moreover, neither rating agencies nor regulators did a better job than SVB during this process. Indeed, it was only on the evening of March 8, 2023, after SVB disclosed a $1.8 billion loss on the sale of bonds, that Moody's downgraded SVB Financial by just one notch, from A3 to Baa1. The same indolent was S&P Global Ratings, who followed suit one day later, downgrading SVB Financial from BBB to BBB-, also by just one notch.

In its self-review of the Federal Reserve's Supervision and Regulation of Silicon Valley Bank conducted in April 2023, the Board of Governors of the Federal Reserve System pointed out in hindsight that "More than a decade of banking system stability and strong performance by banks of all sizes may have led bankers to be overconfident and supervisors to be too accepting. Supervisors should be encouraged to evaluate risks with rigor and consider a range of potential shocks and vulnerabilities, so that they think through the implications of tail events with severe consequences."[8]

A.2. Proofs for Section 3

Potter's bounds

The following Potter's bounds are a restatement of Theorem 1.5.6 of Bingham *et al.* (1987):

Lemma A.1 *Let $h \in \mathrm{RV}_\alpha$ for $\alpha \in \mathbb{R}$. It holds for every $0 < \varepsilon < 1$, all large u, and all $s > 0$ that*

$$(1-\varepsilon)\left(s^{\alpha+\varepsilon} \wedge s^{\alpha-\varepsilon}\right) \le \frac{h(su)}{h(u)} \le (1+\varepsilon)\left(s^{\alpha+\varepsilon} \vee s^{\alpha-\varepsilon}\right).$$

Lemma A.1 forms a foundation for the proofs of our main results.

Proof of Theorem 3.1

(a) It is easy to see that the expectation $E\left[(1-\ell^{-1}X)^{-\alpha}\right]$ appearing in (3.1) is uniformly away from both 0 and ∞ for $\ell \in [\ell_*, \infty)$. Actually, it holds for all $\ell \in [\ell_*, \infty)$ that

[6] For further discussions regarding Merced's liquidation, refer to the Property and Casualty Insurance Compensation Corporation report titled "Why Insurers Fail 2022: Mapping the road to ruin: Lessons learned from four recent insurer failures", available at available at https://www.pacicc.ca/publication/research/why-insurers-fail/.

[7] Available at https://ir.svb.com/financials/annual-reports-and-proxies/default.aspx.

[8] Available at https://www.federalreserve.gov/publications/files/svb-review-20230428.pdf.

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$$\begin{aligned}0 < P(X>0) &\le E\left[(1-\ell^{-1}X)^{-\alpha}1_{(X>0)}\right]\\ &\le E\left[(1-\ell^{-1}X)^{-\alpha}\right]\\ &\le \left(1-\ell^{-1}\hat{x}\right)^{-\alpha} \le \left(1-\ell_*^{-1}\hat{x}\right)^{-\alpha} < \infty. \end{aligned} \tag{A1}$$

Now we turn to prove the uniform asymptotic formula (3.1). By the last assertion of Theorem 1.5.2 of Bingham *et al.* (1987), it is easy to see that the convergence

$$\frac{P\left(Y>(1-\ell^{-1}x)\ell u\right)}{\overline{G}(\ell u)} \to (1-\ell^{-1}x)^{-\alpha}$$

holds uniformly for $\ell \in [\ell_*, \infty)$ and $x \le \hat{x}$. Thus, by (A1), it holds uniformly for $\ell \in [\ell_*, \infty)$ that

$$\frac{P(L>\ell u)}{\overline{G}(\ell u)} = \int_{-\infty}^{\hat{x}} \frac{P\left(Y>(1-\ell^{-1}x)\ell u\right)}{\overline{G}(\ell u)} dF(x) \to E[(1-\ell^{-1}X)^{-\alpha}].$$

(b) Under the strengthened moment condition, the expectation $E\left[(1-\ell^{-1}X)^{-\alpha}\right]$ in (3.1) for $\ell \in [\hat{x}, \infty)$ is uniformly away from both 0 and ∞. Actually, the proof for the lower bound in (A1) is still valid. For the upper bound, we slightly modify the proof as

$$\begin{aligned}E\left[(1-\ell^{-1}X)^{-\alpha}\right] &= E\left[(1-\ell^{-1}X)^{-\alpha}\left(1_{(X>0)}+1_{(X\le 0)}\right)\right]\\ &\le E\left[(1-\hat{x}^{-1}X)^{-\alpha}1_{(X>0)}\right] + E\left[1_{(X\le 0)}\right]\\ &\le \hat{x}^{\alpha}E\left[(\hat{x}-X)^{-\alpha}\right] + P(X\le 0)\\ &< \infty.\end{aligned}$$

To prove the uniformity of (3.1) for $\ell \in [\hat{x}, \infty)$, introduce $h(u)$ to be a positive, non-decreasing, and slowly varying function diverging to ∞ as $u \to \infty$. According to whether $\ell u - uX \le h(u)$, split

$$\begin{aligned}&P(Y>\ell u-uX)\\ &= P(Y>\ell u-uX, \ell u-uX\le h(u)) + P(Y>\ell u-uX, \ell u-uX>h(u))\\ &= I_1+I_2.\end{aligned}$$

Clearly, $I_1 \le P(\ell u - uX \le h(u))$. When $\ell > \hat{x}$, the probability $P(\ell u - uX \le h(u))$ is exactly 0 for all large u. When $\ell = \hat{x}$, this probability is bounded by

$$P(\hat{x}u-uX\le h(u)) = P\left((\hat{x}-X)^{-1} \ge \frac{u}{h(u)}\right) = o(1)\left(\frac{u}{h(u)}\right)^{-\kappa} = o(\overline{G}(\hat{x}u)),$$

where the last step follows from the conditions $G \in \mathrm{MDA}(\Phi_\alpha)$ and $\kappa > \alpha$. Thus, in any case, uniformly for $\ell \in [\hat{x}, \infty)$,

$$I_1 = o(\overline{G}(\ell u)).$$

For I_2, by conditioning on X and applying Lemma A.1, it holds for arbitrarily fixed $0 < \varepsilon < (\kappa - \alpha) \wedge \alpha$, all large u, and uniformly for $\ell \in [\hat{x}, \infty)$ that

$$\frac{I_2}{\overline{G}(\ell u)} \le (1+\varepsilon)E\left[(1-\ell^{-1}X)^{-\alpha+\varepsilon} \vee (1-\ell^{-1}X)^{-\alpha-\varepsilon}\right]. \tag{A2}$$

If we can get rid of $\varepsilon > 0$ on the right-hand side of (A2), then an asymptotic upper bound for (3.1) will be established. Moreover, a corresponding asymptotic lower bound for (3.1) can be established similarly and we will conclude the proof.

Now we show how to get rid of $\varepsilon > 0$ on the right-hand side of (A2). For arbitrarily fixed large $M > 0$ and small $0 < \delta < 1$, according to the value of X belonging to $(-\infty, -M)$, $[-M, 0]$, $(0, (1-\delta)\hat{x}]$, or $((1-\delta)\hat{x}, \hat{x}]$, we split the expectation in (A2) into four parts as

$$\sum_{i=1}^{4} J_i = E\left[\left((1-\ell^{-1}X)^{-\alpha+\varepsilon} \vee (1-\ell^{-1}X)^{-\alpha-\varepsilon}\right) 1_{(-\infty,-M)\cup[-M,0]\cup(0,(1-\delta)\hat{x}]\cup((1-\delta)\hat{x},\hat{x}]}\right].$$

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Clearly, we have

$$J_1 \le E[1_{(X<-M)}].$$

We also have

$$\begin{aligned} J_2 &= E\left[(1-\ell^{-1}X)^{-\alpha+\varepsilon}1_{(-M\le X\le 0)}\right] \\ &\le (1+\hat{x}^{-1}M)^{\varepsilon}E\left[(1-\ell^{-1}X)^{-\alpha}1_{(-M\le X\le 0)}\right] \\ &\le (1+\hat{x}^{-1}M)^{\varepsilon}\left(E\left[(1-\ell^{-1}X)^{-\alpha}1_{(X\le 0)}\right]-\hat{x}^{\alpha}E\left[(\hat{x}-X)^{-\alpha}1_{(X<-M)}\right]\right). \end{aligned}$$

Moreover,

$$\begin{aligned} J_3 &= E\left[(1-\ell^{-1}X)^{-\alpha-\varepsilon}1_{\left(0<X\le(1-\delta)\hat{x}\right)}\right] \\ &\le \delta^{-\varepsilon}E\left[(1-\ell^{-1}X)^{-\alpha}1_{\left(0<X\le(1-\delta)\hat{x}\right)}\right] \\ &\le \delta^{-\varepsilon}\left(E\left[(1-\ell^{-1}X)^{-\alpha}1_{(X>0)}\right]-E\left[1_{\left(X>(1-\delta)\hat{x}\right)}\right]\right). \end{aligned}$$

Finally,

$$\begin{aligned} J_4 &= E\left[(1-\ell^{-1}X)^{-\alpha-\varepsilon}1_{\left((1-\delta)\hat{x}<X\le\hat{x}\right)}\right] \\ &\le E\left[(1-\hat{x}^{-1}X)^{-\alpha-\varepsilon}1_{\left((1-\delta)\hat{x}<X\le\hat{x}\right)}\right] \\ &\le E\left[\left(1-\hat{x}^{-1}X\right)^{-\kappa}1_{\left((1-\delta)\hat{x}<X\le\hat{x}\right)}\right]. \end{aligned}$$

Plug these bounds into (A2), first let $\varepsilon\downarrow 0$, then let both $M\uparrow\infty$ and $\delta\downarrow 0$. Also keep in mind that $E\left[(\hat{x}-X)^{-\alpha}\right]<\infty$ and $P(X=\hat{x})=0$. We eventually arrive at

$$\frac{I_2}{\overline{G}(\ell u)}\lesssim E\left[(1-\ell^{-1}X)^{-\alpha}1_{(X\le 0)}\right]+E\left[(1-\ell^{-1}X)^{-\alpha}1_{(X>0)}\right]=E[(1-\ell^{-1}X)^{-\alpha}].$$

Proof of Theorem 3.2

(a) We start with collecting two facts. First, for arbitrarily fixed $\ell>\hat{x}$, applying Theorem 3.1(a) but relaxing the uniformity requirement, we can rewrite (3.1) as

$$P\left(L>\ell u\right)\sim E\left[(\ell-X)^{-\alpha}\right]\overline{G}(u). \tag{A3}$$

Second, due to the strict monotonicity of $E\left[(\ell-X)^{-\alpha}\right]$ in $\ell\in[\hat{x},\infty)$ and the range of c, Equation (3.3) has a unique solution $\hat{\ell}\in[\hat{x},\infty)$, where $\hat{\ell}=\hat{x}$ occurs only if $\hat{c}<\infty$ and $c=\hat{c}$.

Now we turn to the proof of (3.4). If $\hat{\ell}>\hat{x}$, we arbitrarily choose ℓ_1 and ℓ_2 such that $\hat{x}<\ell_1<\hat{\ell}<\ell_2<\infty$. By applying (A3) to both $P\left(L>\ell_1 u\right)$ and $P\left(L>\ell_2 u\right)$ and keeping in mind equation (3.3), it is easy to see the following two-sided inequality:

$$\lim_{u\to\infty}\frac{P(L>\ell_1 u)}{\overline{G}(u)}>c>\lim_{u\to\infty}\frac{P(L>\ell_2 u)}{\overline{G}(u)}. \tag{A4}$$

If $\hat{\ell}=\hat{x}$, we arbitrarily choose $\ell_1\in\mathcal{C}(F)\cap(0,\hat{x})$ and $\hat{x}<\ell_2<\infty$, where $\mathcal{C}(F)$ denotes the set of continuity points of F. The right-hand inequality in (A4) still holds based on the same reasoning. Moreover, it is obvious that $P(L>\ell_1 u)\to\overline{F}(\ell_1)>0$. Then the first limit in (A4) becomes ∞ and thus the left-hand inequality in (A4) remains valid. For both cases, it follows from (A4) that, for all large u,

$$P(L>\ell_1 u)>1-q_u>P(L>\ell_2 u),$$

which gives $\ell_1 u\le \mathrm{VaR}_{q_u}[L]\le\ell_2 u$. By letting $\ell_1\uparrow\hat{\ell}$ and $\ell_2\downarrow\hat{\ell}$, we obtain (3.4).

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(b) The condition $\alpha > 1$ ensures the integrability of Y_+ and hence of L_+. Thus, the ES of L is finite. Starting from (2.1) and applying a change of variables $x = yu$, we obtain

$$\mathrm{ES}_{q_u}[L] = \mathrm{VaR}_{q_u}[L] + \frac{u}{1-q_u}\int_{u^{-1}\mathrm{VaR}_{q_u}[L]}^{\infty} P\left(L > yu\right) dy.$$

Since $c \in (0, \hat{c})$, following the discussion at the beginning of the proof of item (a), we have $\hat{\ell} \in (\hat{x}, \infty)$. By (3.4), the uniform asymptotic formula (3.1) can be applied to the integrand above. We continue to derive

$$\begin{aligned}\mathrm{ES}_{q_u}[L] &\sim \hat{\ell}u + \frac{u}{1-q_u}\int_{u^{-1}\mathrm{VaR}_{q_u}[L]}^{\infty} E\left[(1-y^{-1}X)^{-\alpha}\right]\overline{G}(yu)dy \\ &\sim \hat{\ell}u + c^{-1}u\int_{u^{-1}\mathrm{VaR}_{q_u}[L]}^{\infty} E\left[(1-y^{-1}X)^{-\alpha}\right]\frac{\overline{G}(yu)}{\overline{G}(u)}dy,\end{aligned}$$

where the last step is due to (3.2). For the last integral above, subject to a discussion on the upper bound for $\frac{\overline{G}(yu)}{\overline{G}(u)}$ by using Lemma A.1, we apply the dominated convergence theorem to obtain

$$\lim_{u\to\infty}\int_{u^{-1}\mathrm{VaR}_{q_u}[L]}^{\infty} E\left[(1-y^{-1}X)^{-\alpha}\right]\frac{\overline{G}(yu)}{\overline{G}(u)}dy = \int_{\hat{\ell}}^{\infty} E\left[(y-X)^{-\alpha}\right]dy.$$

This proves (3.5).

Proof of Proposition 3.1

According to whether $Y > 0$, we do the split

$$\begin{aligned}& P\left(L > \ell_b u\right) \\ &= P\left(uX + Y > \ell_b u, Y > 0\right) + P\left(uX + Y > \ell_b u, Y \le 0\right) \\ &= P\left(uX + Y_+ > \ell_b u\right) - P\left(X > \ell_b, Y \le 0\right) + P\left(uX + Y > \ell_b u, Y \le 0\right).\end{aligned}$$

The second term in the right-hand side above is 0 because $\ell_b = \hat{x} + bu^{-1/2} \ge \hat{x}$. Similarly, the third term is 0 too. Then by (3.6), we rewrite

$$\begin{aligned}P\left(L > \ell_b u\right) &= P(uX + Y_+ > \ell_b u) \\ &= P\left(\eta - \frac{u}{\xi} > (\ell_b - \hat{x})u\right) \\ &= P\left(\eta - \frac{u}{\xi} > b\sqrt{u}\right) \\ &= P\left(\frac{(\xi,\eta)}{\sqrt{u}} \in A_b\right).\end{aligned}$$

Thus, the desired result follows from the assumption that $(\xi, \eta) \in \mathrm{BRV}_{-\alpha}(\nu, \overline{U})$.

A.3. Proofs for Section 4

Preliminaries about the Gumbel MDA

To prepare the proofs for Section 4, we collect some well-known results about the Gumbel MDA.

Let $U \in \mathrm{MDA}(\Lambda)$ with the representation (2.3) and upper endpoint $\hat{z} \le \infty$. At the core of analyzing the Gumbel MDA is its auxiliary function $a(\cdot)$ in (2.3). It possesses the following nice properties. First, by Lemma 1.2 of Resnick (1987),

$$\begin{cases}\lim\limits_{x\to\infty}\frac{a(x)}{x} = 0, & \text{if } \hat{z} = \infty, \\ \lim\limits_{x\uparrow\hat{z}}\frac{a(x)}{\hat{z}-x} = 0, & \text{if } \hat{z} < \infty.\end{cases} \tag{A5}$$

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Therefore, for the first case of $\hat{z}=\infty$, it is easy to see that there is always a positive and increasing function $\phi(\cdot)$ such that $\phi(x)\to\infty$ and $a(\phi(x)x)=o(x)$ as $x\to\infty$.

As an example, we check the Weibull-like distribution with tail

$$\overline{U}(x)\sim Ke^{-cx^{\tau}}, \qquad x\geq 0,\ K,c,\tau>0,$$

which has been employed in Proposition 4.1 and Example 5.2. Its auxiliary function is $a(x)=c^{-1}\tau^{-1}x^{1-\tau}$, $x>0$. Then it is easy to see that the function $\phi(\cdot)$ can be chosen as $\phi(x)=x^{r}$ for $0<r<\frac{\tau}{1-\tau}$ for $0<\tau<1$ and as $\phi(\cdot)=\infty$ for $\tau\geq 1$.

Second, by Lemma 1.3 of Resnick (1987), the convergence

$$\frac{a\left(x+sa(x)\right)}{a(x)}\to 1, \qquad x\uparrow\hat{z}, \tag{A6}$$

holds locally uniformly in $s\in\mathbb{R}$. This means that $a(\cdot)$ is self-neglecting.

Third, it follows immediately from (2.3) and (A6) that the convergence

$$\frac{\overline{U}\left(x+sa(x)\right)}{\overline{U}(x)}\to e^{-s}, \qquad x\uparrow\hat{z}, \tag{A7}$$

holds locally uniformly in $s\in\mathbb{R}$. This means that $a(\cdot)$ serves as a scale of the tail $\overline{U}$ as $\overline{U}$ decays to 0. Theorem 3.3.27 of Embrechts *et al.* (1997) states that (A7) actually provides another characterization for the Gumbel MDA.

The following lemma is a restatement of Lemma 3.4 of Tang and Yang (2012):

Lemma A.2 *Let* $U\in\mathrm{MDA}(\Lambda)$ *with the representation* (2.3) *and upper endpoint* $\hat{z}\leq\infty$. *Then, for arbitrary* $0<\varepsilon<1$, *there is some* $z_0<\hat{z}$ *such that, for all* $z_0<x<\hat{z}$ *and all* $s\geq 0$,

$$\frac{\overline{U}\left(x+sa(x)\right)}{\overline{U}(x)}\leq(1+\varepsilon)\left(1+\varepsilon s\right)^{-1/\varepsilon}.$$

Proof of Theorem 4.1

We follow the proof of Theorem 3.1(a) of Hashorva *et al.* (2010), but we need to address some technical issues arising from the uniformity requirement. By conditioning on Y and applying the change of variables $y=\ell u-\hat{x}u+wa(\ell u-\hat{x}u)$, we derive

$$\begin{aligned}P\left(L>\ell u\right)&=\int_{\ell u-\hat{x}u}^{\infty}\overline{F}\left(\frac{\ell u-y}{u}\right)dG(y)\\&=-\int_{w=0}^{\infty}\overline{F}\left(\hat{x}-u^{-1}wa(\ell u-\hat{x}u)\right)d\overline{G}\left(\ell u-\hat{x}u+wa(\ell u-\hat{x}u)\right).\end{aligned}\tag{A8}$$

For every $u>0$, introduce a nonnegative random variable W_u with tail satisfying

$$P(W_u>w)=\frac{\overline{G}\left(\ell u-\hat{x}u+wa(\ell u-\hat{x}u)\right)}{\overline{G}(\ell u-\hat{x}u)}, \qquad 0\leq w<\infty. \tag{A9}$$

By the scaling property (A7) of the Gumbel MDA, the condition $Y\in\mathrm{MDA}(\Lambda)$ with the representation (2.3) implies that

$$\lim_{u\to\infty}P(W_u>w)=e^{-w}, \qquad 0\leq w<\infty;$$

that is, W_u converges in distribution to an exponential random variable W with mean 1. We can rewrite the right-hand side of (A8) in terms of the expectation with respect to W_u, as

$$P\left(L>\ell u\right)=\overline{G}(\ell u-\hat{x}u)E\left[\overline{F}\left(\hat{x}-u^{-1}W_u a(\ell u-\hat{x}u)\right)\right]. \tag{A10}$$

For arbitrarily fixed $0<\delta<1$, split the expectation above into two parts as

$$E\left[\overline{F}\left(\hat{x}-u^{-1}W_u a(\ell u-\hat{x}u)\right)\left(1_{\left(W_u\leq\frac{\delta u}{a((\ell-\hat{x})u)}\right)}+1_{\left(W_u>\frac{\delta u}{a((\ell-\hat{x})u)}\right)}\right)\right]=E\left[I_1+I_2\right].$$

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We notice that $\frac{u}{a((\ell-\hat{x})u)} \to \infty$ holds uniformly for $\ell \in [\ell_*, \phi(u))$. Actually, this is trivial if $a(\cdot)$ is bounded. If $a(\cdot)$ is unbounded, by the conditions on $a(\cdot)$, we have $\frac{u}{a((\ell-\hat{x})u)} \geq \frac{u}{a(\phi(u)u)} \to \infty$.

For I_1, by the condition $\overline{F}\left(\hat{x}-(\,\cdot\,)^{-1}\right) \in \mathrm{RV}_{-\beta}$ and Lemma A.1, for arbitrarily fixed $0<\varepsilon_1<\beta$, it holds for δ small enough and for all large u that

$$\frac{I_1}{\overline{F}\left(\hat{x}-u^{-1}a(\ell u-\hat{x}u)\right)} \leq (1+\varepsilon_1)\left(W_u^{\beta+\varepsilon_1} \vee W_u^{\beta-\varepsilon_1}\right).$$

We claim that $W_u^{\beta+\varepsilon_1} \vee W_u^{\beta-\varepsilon_1}$ on the right-hand side above are uniformly integrable for all large u. Actually, applying Lemma A.2 to (A9), for arbitrarily fixed $\varepsilon_2>0$, it holds for all large $u>0$ and all $w \geq 0$ that

$$P(W_u > w) \leq (1+\varepsilon_2)(1+\varepsilon_2 w)^{-1/\varepsilon_2}. \tag{A11}$$

By specifying $\varepsilon_2 < 1/(\beta+\varepsilon_1)$, one sees that the collection of random variables W_u, indexed by all large u, are stochastically bounded by a positive random variable with tail of order w^{-1/ε_2} as $w \to \infty$, and thus the claim. Then we apply the dominated convergence theorem to obtain

$$\begin{aligned}
\lim_{u\to\infty} \frac{E\left[I_1\right]}{\overline{F}\left(\hat{x}-u^{-1}a(\ell u-\hat{x}u)\right)} &= E\left[\lim_{u\to\infty} \frac{\overline{F}\left(\hat{x}-u^{-1}W_u a(\ell u-\hat{x}u)\right)}{\overline{F}\left(\hat{x}-u^{-1}a(\ell u-\hat{x}u)\right)} 1_{\left(W_u \leq \frac{\delta u}{a((\ell-\hat{x})u)}\right)}\right] \\
&= E[W^\beta] \\
&= \Gamma(\beta+1),
\end{aligned} \tag{A12}$$

where the second step is due to $\overline{F}\left(\hat{x}-(\,\cdot\,)^{-1}\right) \in \mathrm{RV}_{-\beta}$.

For I_2, by (A11) with $\varepsilon_2 < 1/(\beta+\varepsilon_1)$,

$$\begin{aligned}
E\left[I_2\right] &\leq P\left(W_u > \frac{\delta u}{a((\ell-\hat{x})u)}\right) \\
&\leq (1+\varepsilon_2)\left(1+\frac{\varepsilon_2 \delta u}{a((\ell-\hat{x})u)}\right)^{-1/\varepsilon_2} \\
&= o(1)\overline{F}\left(\hat{x}-u^{-1}a(\ell u-\hat{x}u)\right),
\end{aligned} \tag{A13}$$

where the last step is due to $\overline{F}\left(\hat{x}-(\,\cdot\,)^{-1}\right) \in \mathrm{RV}_{-\beta}$ and $1/\varepsilon_2 > \beta$.

Finally, a combination of the two estimates (A12)–(A13) for $E\left[I_1\right]$ and $E\left[I_2\right]$ gives

$$E\left[\overline{F}\left(\hat{x}-u^{-1}W_u a(\ell u-\hat{x}u)\right)\right] \sim \Gamma(\beta+1)\overline{F}\left(\hat{x}-u^{-1}a(\ell u-\hat{x}u)\right).$$

Plugging this into (A10) yields (4.2).

Proof of Theorem 4.2

We need to prepare two elementary lemmas. The following first lemma, to be used in the proof of Theorem 4.2, is a corollary of Lemma A.2:

Lemma A.3 *Let $U \in \mathrm{MDA}(\Lambda)$ with the representation (2.3) and upper endpoint $\hat{z}=\infty$. If $a(u) \lesssim u^{1-\delta}$ for some $0<\delta<1$, then it holds for every $d_1>1$ and $d_2 \in \mathbb{R}$ that*

$$\lim_{u\to\infty} u^{d_2} \frac{\overline{U}\left(d_1 u\right)}{\overline{U}(u)} = 0.$$

Proof. For arbitrary $0<\varepsilon<1$, by Lemma A.2, it holds for all large u that

$$u^{d_2} \frac{\overline{U}\left(d_1 u\right)}{\overline{U}(u)} \leq (1+\varepsilon)\, u^{d_2}\left(1+\varepsilon \frac{(d_1-1)u}{a(u)}\right)^{-\frac{1}{\varepsilon}} \asymp u^{d_2}\left(\frac{u}{a(u)}\right)^{-\frac{1}{\varepsilon}} \lesssim u^{d_2-\frac{\delta}{\varepsilon}}.$$

Thus, we can always find $\varepsilon>0$ such that $d_2-\frac{\delta}{\varepsilon}<0$. □

The following second lemma is to be used in the proof of Theorem 4.2 too:

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Lemma A.4 *Under the conditions of Theorem 4.1, let* $h: (0, \infty) \to (0, \infty)$ *be a function satisfying* $h(u) \sim \ell u$ *for some* $\ell > \hat{x}$*. Then it holds for arbitrarily fixed* $s \in \mathbb{R}$ *that*

$$P\left(L > h(u) + sa\left(h(u) - \hat{x}u\right)\right) \sim e^{-s} P\left(L > h(u)\right). \tag{A14}$$

Proof. Thanks to the uniformity established in Theorem 4.1, we apply (4.2) with ℓu replaced by $h(u) + sa\left(h(u) - \hat{x}u\right)$ to obtain

$$\begin{aligned} P\left(L > h(u) + sa\left(h(u) - \hat{x}u\right)\right) \sim \Gamma(\beta+1)\overline{F}\left(\hat{x} - u^{-1}a\left(\left(h(u) - \hat{x}u\right) + sa\left(h(u) - \hat{x}u\right)\right)\right) \\ \times \overline{G}\left(\left(h(u) - \hat{x}u\right) + sa\left(h(u) - \hat{x}u\right)\right). \end{aligned} \tag{A15}$$

Observe the right-hand side. We have

$$\overline{F}\left(\hat{x} - u^{-1}a\left(\left(h(u) - \hat{x}u\right) + sa\left(h(u) - \hat{x}u\right)\right)\right) \sim \overline{F}\left(\hat{x} - u^{-1}a(h(u) - \hat{x}u)\right)$$

due to the self-neglecting property (A6) of the auxiliary function $a(\cdot)$ and the regular variation of $\overline{F}\left(\hat{x} - (\cdot)^{-1}\right)$. Moreover,

$$\overline{G}\left(\left(h(u) - \hat{x}u\right) + sa\left(h(u) - \hat{x}u\right)\right) \sim e^{-s}\overline{G}\left(h(u) - \hat{x}u\right)$$

due to the scaling property (A7) of the Gumbel MDA. It follows from (A15) that

$$\begin{aligned} P\left(L > h(u) + sa\left(h(u) - \hat{x}u\right)\right) &\sim e^{-s}\Gamma(\beta+1)\overline{F}\left(\hat{x} - u^{-1}a(h(u) - \hat{x}u)\right)\overline{G}\left(h(u) - \hat{x}u\right) \\ &\sim e^{-s}P\left(L > h(u)\right), \end{aligned}$$

where the last step applies (4.2) again. This proves (A14). □

Now we are ready to show the Proof of Theorem 4.2:

(a) By the very definition of $\mathrm{VaR}_{q_u}[L]$, we have

$$P\left(L > \mathrm{VaR}_{q_u}[L]\right) \le 1 - q_u \le P\left(L \ge \mathrm{VaR}_{q_u}[L]\right). \tag{A16}$$

Applying Lemma A.4 with $h(u) = (\hat{x} + c)u$, then applying Theorem 4.1, we derive

$$\begin{aligned} P\left(L > (\hat{x}+c)u + sa(cu)\right) &\sim e^{-s}P\left(L > (\hat{x}+c)u\right) \\ &\sim e^{-s}\Gamma(\beta+1)\overline{F}\left(\hat{x} - u^{-1}a(cu)\right)\overline{G}(cu) \\ &\sim e^{-s}(1-q_u), \end{aligned}$$

where the last step is due to (4.3). By specifying $s = -1$ and 1 and comparing the resulting relations with the two sides of (A16) accordingly, we see that the following strict inequalities hold for all large u:

$$\begin{cases} P\left(L > \mathrm{VaR}_{q_u}[L]\right) < P\left(L > (\hat{x}+c)u - a(cu)\right), \\ P\left(L \ge \mathrm{VaR}_{q_u}[L]\right) < P\left(L > (\hat{x}+c)u + a(cu)\right). \end{cases}$$

These strict inequalities jointly imply that, for all large u,

$$\begin{aligned} \mathrm{VaR}_{q_u}[L] &> (\hat{x}+c)u - a(cu) \\ &\sim (\hat{x}+c)u \\ &\sim (\hat{x}+c)u + a(cu) > \mathrm{VaR}_{q_u}[L], \end{aligned}$$

where we have applied $a(u) = o(u)$ by (A5). This proves (4.4).

(b) As in the proof of Theorem 3.2(b), we start with

$$\mathrm{ES}_{q_u}[L] = \mathrm{VaR}_{q_u}[L] + \frac{1}{1-q_u}\int_{\mathrm{VaR}_{q_u}[L]}^{\infty} P(L > x)dx.$$

Thus, it suffices to prove that

$$I = \int_{\mathrm{VaR}_{q_u}[L]}^{\infty} P(L > x)dx \sim (1-q_u)a\left(\mathrm{VaR}_{q_u}[L] - \hat{x}u\right). \tag{A17}$$

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In terms of $\phi(u)$ specified in Theorem 4.1, we split I into two parts as

$$I = \int_{u\phi(u)}^{\infty} + \int_{\mathrm{VaR}_{q_u}[L]}^{u\phi(u)} = I_1 + I_2.$$

First deal with I_1. Recall the expression for L in (1.1). Since $\phi(u) \to \infty$, by Lemma A.3 and a dominated convergence argument, it holds for any $M > 0$ that

$$I_1 \le \int_{u\phi(u)}^{\infty} P(Y > x - \hat{x}u)dx = o(1)u^{-M}\overline{G}(cu). \tag{A18}$$

On the other hand, by (4.3),

$$(1 - q_u)a\left(\mathrm{VaR}_{q_u}[L] - \hat{x}u\right) \sim \Gamma(\beta+1)\overline{F}\left(\hat{x} - u^{-1}a(cu)\right)\overline{G}(cu)a\left(\mathrm{VaR}_{q_u}[L] - \hat{x}u\right).$$

Observe the right-hand side. The regular variation of $\overline{F}\left(\hat{x} - (\cdot)^{-1}\right)$ implies that, for any $0 < \varepsilon < 1$,

$$\overline{F}\left(\hat{x} - u^{-1}a(cu)\right) \gtrsim (1-\varepsilon)\left(u^{-1}a(cu)\right)^{\beta+\varepsilon}.$$

By the condition on $a(\cdot)$ and the result (4.4),

$$a\left(\mathrm{VaR}_{q_u}[L] - \hat{x}u\right) \gtrsim \left(\mathrm{VaR}_{q_u}[L] - \hat{x}u\right)^{-\frac{1}{\delta}} \sim (cu)^{-\frac{1}{\delta}}.$$

Put together, it follows that

$$\begin{aligned}(1 - q_u)a\left(\mathrm{VaR}_{q_u}[L] - \hat{x}u\right) &\gtrsim \Gamma(\beta+1)(1-\varepsilon)\left(u^{-1}a(cu)\right)^{\beta+\varepsilon}\overline{G}(cu)\,(cu)^{-\frac{1}{\delta}} \\ &\gtrsim \Gamma(\beta+1)(1-\varepsilon)\left(u^{-1}(cu)^{-\frac{1}{\delta}}\right)^{\beta+\varepsilon}\overline{G}(cu)\,(cu)^{-\frac{1}{\delta}} \\ &\asymp u^{-\left(1+\frac{1}{\delta}\right)(\beta+\varepsilon)-\frac{1}{\delta}}\overline{G}(cu).\end{aligned} \tag{A19}$$

Comparing (A19) with (A18) in which M is chosen to be large enough, it follows that

$$I_1 = o(1)(1-q_u)a\left(\mathrm{VaR}_{q_u}[L] - \hat{x}u\right). \tag{A20}$$

Next deal with I_2. By (4.4), the uniform asymptotic formula (4.2) is applicable. We derive

$$\begin{aligned}I_2 &\sim \Gamma(\beta+1)\int_{\mathrm{VaR}_{q_u}[L]}^{u\phi(u)} \overline{F}\left(\hat{x} - u^{-1}a(x-\hat{x}u)\right)\overline{G}(x-\hat{x}u)dx \\ &= \Gamma(\beta+1)\left(\int_{\mathrm{VaR}_{q_u}[L]}^{\infty} - \int_{u\phi(u)}^{\infty}\right)\overline{F}\left(\hat{x} - u^{-1}a(x-\hat{x}u)\right)\overline{G}(x-\hat{x}u)dx \\ &= I_{21} - I_{22}.\end{aligned}$$

Following the proof for (A20), we have

$$I_{22} = o(1)(1-q_u)a\left(\mathrm{VaR}_{q_u}[L] - \hat{x}u\right). \tag{A21}$$

To deal with I_{21}, for notational convenience, introduce $\Delta_u = \mathrm{VaR}_{q_u}[L] - \hat{x}u$, which is asymptotic to cu due to (4.2). By the change of variables $x = \mathrm{VaR}_{q_u}[L] + za(\Delta_u)$, we rewrite I_{21} as

$$\begin{aligned}I_{21} &\sim \Gamma(\beta+1)a(\Delta_u)\int_0^{\infty}\overline{F}\left(\hat{x} - u^{-1}a\left(\Delta_u + za(\Delta_u)\right)\right)\overline{G}\left(\Delta_u + za(\Delta_u)\right)dz \\ &= \Gamma(\beta+1)a(\Delta_u)J(u).\end{aligned} \tag{A22}$$

Observe that

$$\frac{J(u)}{\overline{F}\left(\hat{x} - u^{-1}a(\Delta_u)\right)\overline{G}(\Delta_u)} = \int_0^{\infty}\frac{\overline{F}\left(\hat{x} - u^{-1}a\left(\Delta_u + za(\Delta_u)\right)\right)}{\overline{F}\left(\hat{x} - u^{-1}a(\Delta_u)\right)} \times \frac{\overline{G}\left(\Delta_u + za(\Delta_u)\right)}{\overline{G}(\Delta_u)}dz. \tag{A23}$$

For arbitrarily fixed $0 < \epsilon < 1$, by $\lim_{x\to\infty} a'(x) = 0$, it holds for all large u and all $z \ge 0$ that $a\left(\Delta_u + za(\Delta_u)\right) \le (1+\epsilon z)\,a(\Delta_u)$. Then by the regular variation of $\overline{F}\left(\hat{x} - (\cdot)^{-1}\right)$ and Lemma A.1, it holds for all $z \geq 0$ and all large u that

$$\frac{\overline{F}\left(\hat{x} - u^{-1}a\left(\Delta_u + za(\Delta_u)\right)\right)}{\overline{F}\left(\hat{x} - u^{-1}a(\Delta_u)\right)} \leq \frac{\overline{F}\left(\hat{x} - u^{-1}\left(1 + \epsilon z\right)a(\Delta_u)\right)}{\overline{F}\left(\hat{x} - u^{-1}a(\Delta_u)\right)} \leq (1+\epsilon)\left(1+\epsilon z\right)^{\beta+\epsilon}.$$

By Lemma A.2, the inequality

$$\frac{\overline{G}\left(\Delta_u + za(\Delta_u)\right)}{\overline{G}(\Delta_u)} \leq (1+\epsilon)\left(1+\epsilon z\right)^{-1/\epsilon}$$

holds for all $z \geq 0$ and all large u. By choosing ϵ small enough such that $1/\epsilon - \epsilon > 1 + \beta$, these bounds indicate that, for all large u, the integrand in (A23) is bounded by an integrable function. Therefore, applying the dominated convergence theorem to (A23) gives

$$\lim_{u\to\infty} \frac{J(u)}{\overline{F}\left(\hat{x} - u^{-1}a(\Delta_u)\right)\overline{G}(\Delta_u)} = \int_0^\infty 1 \times e^{-z}dz = 1, \tag{A24}$$

where the convergence of the first ratio to 1 is due to the self-neglecting property (A6) of $a(\cdot)$ and the regular variation of $\overline{F}\left(\hat{x} - (\cdot)^{-1}\right)$, while the convergence of the second ratio to e^{-z} is due to the scaling property (A7) of $G \in \mathrm{MDA}(\Lambda)$. It follows from (A22) and (A24) that

$$\begin{aligned} I_{21} &\sim \Gamma(\beta+1)a\left(\mathrm{VaR}_{q_u}[L] - \hat{x}u\right)\overline{F}\left(\hat{x} - u^{-1}a\left(\mathrm{VaR}_{q_u}[L] - \hat{x}u\right)\right)\overline{G}\left(\mathrm{VaR}_{q_u}[L] - \hat{x}u\right) \\ &\sim a\left(\mathrm{VaR}_{q_u}[L] - \hat{x}u\right)P\left(L > \mathrm{VaR}_{q_u}[L]\right), \end{aligned} \tag{A25}$$

where the last step applies Theorem 4.1 with ℓu replaced by $\mathrm{VaR}_{q_u}[L]$.

Comparing (A25) with (A17), it remains to show that

$$P\left(L > \mathrm{VaR}_{q_u}[L]\right) \sim 1 - q_u. \tag{A26}$$

Actually, for arbitrarily fixed $s < 0$, it holds for all large u that

$$\begin{aligned} 1 - q_u &\leq P\left(L \geq \mathrm{VaR}_{q_u}[L]\right) \\ &\leq P\left(L > \mathrm{VaR}_{q_u}[L] + sa\left(\mathrm{VaR}_{q_u}[L] - \hat{x}u\right)\right) \\ &\sim e^{-s}P\left(L > \mathrm{VaR}_{q_u}[L]\right), \end{aligned}$$

where the last step is due to Lemma A.4 with $h(u) = \mathrm{VaR}_{q_u}[L]$. By the arbitrariness of $s < 0$, we obtain $1 - q_u \lesssim P\left(L > \mathrm{VaR}_{q_u}[L]\right)$. This, together with the obvious inequality $P\left(L > \mathrm{VaR}_{q_u}[L]\right) \leq (1 - q_u)$, gives (A26).

Proof of Proposition 4.1

For the proof of Proposition 4.1, we need to derive asymptotics, as $x \to \infty$, for the integral

$$I = \int_0^\infty \exp\left\{-\left(c_1 s^{-\rho} + c_2 s\right)x\right\}ds, \qquad c_1, c_2, \rho > 0. \tag{A27}$$

Observe the function $t = c_1 s^{-\rho} + c_2 s$ for $s > 0$. By analyzing the derivative

$$\frac{dt}{ds} = -c_1\rho s^{-\rho-1} + c_2,$$

we see that the function attains its global minimum t_* at the root $s = s_*$. For later reference, we list here

$$\begin{cases} -c_1\rho s_*^{-\rho-1} + c_2 = 0, \\ s_* = \left(\frac{c_1\rho}{c_2}\right)^{\frac{1}{\rho+1}}, \\ t_* = c_1 s_*^{-\rho} + c_2 s_*. \end{cases} \tag{A28}$$

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Lemma A.5 *The integral I in (A27) satisfies*

$$I \sim \left(\frac{2\pi s_*^{\rho+2}}{c_1 \rho(\rho+1)} \right)^{\frac{1}{2}} x^{-\frac{1}{2}} e^{-t_* x}, \qquad x \to \infty. \tag{A29}$$

Proof. The working limit procedure in this proof is as $x \to \infty$. We split the integral I into two parts as

$$I = \left(\int_{|s-s_*| \le \frac{\ln x}{\sqrt{x}}} + \int_{s>0: |s-s_*| > \frac{\ln x}{\sqrt{x}}} \right) = I_1 + I_2. \tag{A30}$$

We are going to derive asymptotics for I_1 and show that $I_2 = o(I_1)$.

By the third equation in (A28), we rewrite I_1 as

$$I_1 = e^{-t_* x} \int_{|s-s_*| \le \frac{\ln x}{\sqrt{x}}} \exp\left\{ -\left(c_1 \left(s^{-\rho} - s_*^{-\rho} \right) + c_2 \left(s - s_* \right) \right) x \right\} ds.$$

Over the range $|s - s_*| \le \frac{\ln x}{\sqrt{x}}$, we do Taylor's expansion

$$s^{-\rho} - s_*^{-\rho} = -\rho s_*^{-\rho-1} (s - s_*) + \frac{1}{2} \rho(\rho+1) s_*^{-\rho-2} (s - s_*)^2 + O\left((s - s_*)^3 \right),$$

where the remainder is $o(x^{-1})$. By the change of variables $h = s - s_*$, it follows that

$$I_1 \sim e^{-t_* x} \int_{|h| \le \frac{\ln x}{\sqrt{x}}} \exp\left\{ -\left(-c_1 \rho s_*^{-\rho-1} h + \frac{1}{2} c_1 \rho(\rho+1) s_*^{-\rho-2} h^2 + c_2 h \right) x \right\} dh.$$

Due to the first equation in (A28), the h terms in the exponent above cancel out. The remaining part is an even function of h. Therefore,

$$I_1 \sim 2e^{-t_* x} \int_{0<h \le \frac{\ln x}{\sqrt{x}}} \exp\left\{ -\frac{1}{2} c_1 \rho(\rho+1) s_*^{-\rho-2} h^2 x \right\} dh.$$

By the change of variables $v = h^2 x$ again,

$$\begin{aligned} I_1 &\sim x^{-\frac{1}{2}} e^{-t_* x} \int_{0<v \le \ln^2 x} \exp\left\{ -\frac{1}{2} c_1 \rho(\rho+1) s_*^{-\rho-2} v \right\} v^{-\frac{1}{2}} dv \\ &\sim x^{-\frac{1}{2}} e^{-t_* x} \int_0^{\infty} \exp\left\{ -\frac{1}{2} c_1 \rho(\rho+1) s_*^{-\rho-2} v \right\} v^{-\frac{1}{2}} dv \\ &= \left(\frac{2\pi s_*^{\rho+2}}{c_1 \rho(\rho+1)} \right)^{\frac{1}{2}} x^{-\frac{1}{2}} e^{-t_* x}, \end{aligned}$$

where the last step applies the gamma integral (4.1) subject to a change of variables.

Next, we need to show that $I_2 = o\left(x^{-\frac{1}{2}} e^{-t_* x} \right)$. We only consider $\int_{s>s_*+\frac{\ln x}{\sqrt{x}}}$ as the consideration of the other part $\int_{0<s<s_*-\frac{\ln x}{\sqrt{x}}}$ is similar. For some large constant $M > s_*$, we further split $\int_{s>s_*+\frac{\ln x}{\sqrt{x}}}$ into two parts as

$$\left(\int_{s_*+\frac{\ln x}{\sqrt{x}} < s \le M} + \int_{s>M} \right) \exp\left\{ -\left(c_1 s^{-\rho} + c_2 s \right) x \right\} ds = I_{21} + I_{22}.$$

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In I_{21}, the function $c_1 s^{-\rho} + c_2 s$ is increasing since $s > s_*$. We have

$$\begin{aligned}
c_1 s^{-\rho} + c_2 s &\geq c_1 \left(s_* + \frac{\ln x}{\sqrt{x}} \right)^{-\rho} + c_2 \left(s_* + \frac{\ln x}{\sqrt{x}} \right) \\
&= c_1 s_*^{-\rho} \left(1 - \rho \frac{\ln x}{s_* \sqrt{x}} + \frac{1}{2} \rho(\rho+1) \left(\frac{\ln x}{s_* \sqrt{x}} \right)^2 + O(1) \left(\frac{\ln x}{s_* \sqrt{x}} \right)^3 \right) + c_2 \left(s_* + \frac{\ln x}{\sqrt{x}} \right) \\
&= t_* + \frac{1}{2} c_1 s_*^{-\rho} \rho(\rho+1) \left(\frac{\ln x}{\sqrt{x} s_*} \right)^2 + o(x^{-1}),
\end{aligned}$$

where the second step applies Taylor's expansion and the last step is due to the first and third equations in (A28). Thus, for arbitrarily fixed $M > 0$,

$$I_{21} \leq M \exp \left\{ - \left(t_* + c_1 s_*^{-\rho} \frac{\rho(\rho+1)}{2} \left(\frac{\ln x}{\sqrt{x} s_*} \right)^2 + o(x^{-1}) \right) x \right\} = o \left(x^{-\frac{1}{2}} e^{-t_* x} \right).$$

For I_{22}, by the change of variables $v = s - M$, we have

$$\begin{aligned}
I_{22} &= \int_0^\infty \exp \left\{ - \left(c_1 (v+M)^{-\rho} + c_2 (v+M) \right) x \right\} dv \\
&\leq e^{-c_2 M x} \int_0^\infty e^{-c_2 v x} dv,
\end{aligned}$$

which is $o\left(x^{-\frac{1}{2}} e^{-t_* x} \right)$ for M large enough. This ends the proof of Lemma A.5. □

Now we are ready to show the Proof of Proposition 4.1:

This proof builds on Lemma A.5. Recall the specifications in (4.6). Define $\xi = \frac{1}{\hat{x} - X}$, which is nonnegative with an ultimate tail

$$P(\xi > z) = P\left(X > \hat{x} - \frac{1}{z} \right) \sim K_1 e^{-c_1 z^{\tau_1}}, \qquad z \to \infty.$$

We have

$$P\left(L > \hat{x} u \right) = P(\xi Y > u) = \int_0^\infty P\left(\xi > \frac{u}{y} \right) dP(Y \leq y).$$

Note that the tails of ξ and Y are both rapidly varying. Thus, by Lemma A.5 of Tang and Tsitsiashvili (2004), we can replace them with their asymptotics and obtain

$$P\left(L > \hat{x} u \right) \sim -K_1 K_2 \int_0^\infty \exp \left\{ -c_1 \left(\frac{u}{y} \right)^{\tau_1} \right\} d e^{-c_2 y^{\tau_2}}.$$

Applying the change of variables $r = y^{\tau_2}$,

$$P\left(L > \hat{x} u \right) \sim K_1 K_2 c_2 \int_0^\infty \exp \left\{ -c_1 u^{\tau_1} r^{-\frac{\tau_1}{\tau_2}} - c_2 r \right\} dr.$$

Applying the change of variables $r = s u^{\frac{\tau_1 \tau_2}{\tau_1 + \tau_2}}$ again, it follows that

$$P\left(L > \hat{x} u \right) \sim K_1 K_2 c_2 u^{\frac{\tau_1 \tau_2}{\tau_1 + \tau_2}} \int_0^\infty \exp \left\{ - \left(c_1 s^{-\frac{\tau_1}{\tau_2}} + c_2 s \right) u^{\frac{\tau_1 \tau_2}{\tau_1 + \tau_2}} \right\} ds.$$

The integral above is reduced to I in (A27) with $\rho = \frac{\tau_1}{\tau_2}$ and $x = u^{\frac{\tau_1 \tau_2}{\tau_1 + \tau_2}}$. Then by Lemma A.5, we obtain (4.7) and conclude the proof of Proposition 4.1.

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