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FIXED POINT THEOREMS FOR LIPSPCHITZIAN SEMIGROUPS

BY

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ABSTRACT. Let U be a nonempty subset of a Banach space, S a left reversible semitopological semigroup, $\mathcal{S} = \{T_t : t \in \mathcal{S}\}$ a continuous representation of $\mathcal{S}$ as lipschitzian mappings on U into itself, that is for each $s \in \mathcal{S}$, there exists $k_S > 0$ such that $\|T_S(x) - T_S(y)\| \leqq k_S\|x-y\|$ for $x, y \in U$. We first show that if there exists a closed subset C of U such that $\cap_s \overline{co}\{T_t x : t \geqq s\} \subseteq C$ for all $x \in U$ then $\mathcal{S}$ with $\limsup_s k_S < \sqrt{2}$ has a common fixed point in a Hilbert space. Next, we prove that the theorem is valid in a Banach space E if $\limsup_s k_S < \tilde{N}(E)^{-1/2}$.

1. **Introduction.** Let S be a semitopoligical semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \longrightarrow a \cdot s$ and $s \longrightarrow s \cdot a$ from S to S are continuous. Let U be a nonempty subset of a Banach space E. Then a family $\mathcal{S} = \{T_t : t \in S\}$ of mappings from U into itself is said to be a *lipschitzian semigroup* on U if $\mathcal{S}$ satisfies the following:

(1) $T_{ts}(x) = T_t T_s(x)$ *for* $t, s \in S$ *and* $x \in U$;

(2) the mapping $(s, x) \longrightarrow T_s(x)$ *from* $S \times U$ *into* U *is continuous when* $S \times U$ *has the product topology;*

(3) for each $s \in S$, *there exists* $k_s > 0$ *such that* $\|T_s(x) - T_s(y)\| \leqq k_s\|x - y\|$ *for* $x, y \in U$.

A semitopological semigroup S is *left reversible* if any two closed right ideals of S have nonvoid intersection. In this case, $(S, \leqq)$ is a directed system when the binary relation "$\leqq$" on S is defined by $a \leqq b$ if and only if $\{a\} \cup \overline{aS} \supset \{b\} \cup \overline{bS}$. A lipschitzian semigroup on U is said to be a *uniformly k- lipschitzian* if $k_s = k$ for all $s \in S$. Fixed point theorems for uniformly k-lipschitzian semigroups were first studied by Goebel and Kirk [6] and Goebel, Kirk and Thele [7]. Lifschitz [10], Downing and Ray [4] and Ishihara and Takahashi [8] proved that in a Hilbert space a uniformly k- lipschitzian semi-group with $k < \sqrt{2}$ has a common fixed point. Also Casini and Maluta [3] and Ishihara and Takahashi [9] proved that a uniformly k-lipschitzian semigroup in a Banach space E has a common fixed point if $k < \tilde{N}(E)^{-1/2}$, where $\tilde{N}(E)$ is the constant of uniformity of normal structure. In these results, except [7], domain U of semigroups were assumed to be closed and convex.

In this paper, we first show that if S is left reversible and if there exists a closed subset C of U such that $\cap_s \overline{co}\{T_t x : t \geqq s\} \subseteq C$ for all $x \in U$ then a lipschitzian

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semigroup on nonconvex domain in a Hilbert space with $\limsup_s k_s < \sqrt{2}$ has a common fixed point. Next, we prove that the theorem is valid in a Banach space E if $\limsup_s k_s < \tilde{N}(E)^{-1/2}$. These results are the generalization of [5], [8], [9].

2. **Fixed point theorems.** Let $\{B_\alpha : \alpha \in \Lambda\}$ be a decreasing net of bounded subsets of a Banach space E. For a nonempty subset C of E define,

$$r(\{B_\alpha\}, x) = \inf_\alpha \sup\{\|x - y\| : y \in B_\alpha\} :$$

$$r(\{B_\alpha\}, C) = \inf\{r(\{B_\alpha\}, x) : x \in C\};$$

$$\mathcal{A}(\{B_\alpha\}, C) = \{x \in C : r(\{B_\alpha\}, x) = r(\{B_\alpha\}, C)\}.$$

We know that $r(\{B_\alpha\}, \cdot)$ is a continuous convex function on E which satisfies the following:

$$|r(\{B_\alpha\}, x) - r(\{B_\alpha\}, y)| \leqq \|x - y\| \leqq r(\{B_\alpha\}, x) + r(\{B_\alpha\}, y)$$

for each $x, y \in E$. It is easy to see that if E is reflexive and if C is closed convex then $\mathcal{A}(\{B_\alpha\}, C)$ is nonempty and moreover, if E is uniformly convex then it consists of a single point, cf. [11]. For a subset C, we denote by $\bar{co}C$ the closure of the convex hull of C, by $d(C)$ the diameter of C and by $R(C)$ the Chebyshev radius of C, i.e. $R(C) = \inf \sup_{x \in C y \in C} \|x - y\|$. The uniformity $\tilde{N}(E)$ of normal structure of E is defined by

$$\tilde{N}(E) = \sup\left\{\frac{R(C)}{d(C)} : C \textit{ is a nonempty bounded convex subset of } E \textit{ with } d(C) > 0\right\}.$$

It is known that if $\tilde{N}(E) < 1$ then E is reflexive, cf. [1], [9]. [12]. We start with proving a fixed point theorem in a Hilbert space. The following lemma which was proved in [8] plays a crucial role in the proof of the theorem.

LEMMA 1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{B_\alpha : \alpha \in \Lambda\}$ be a decreasing net of nonempty bounded subsets of H and let $\{a\} = \mathcal{A}(\{B_\alpha\}, C)$. Then*

$$r(\{B_\alpha\}, C)^2 + \|a - x\|^2 \leqq r(\{B_\alpha\}, x)^2$$

for every $x \in C$.

We also know the following:

LEMMA 2. *Let C be a nonempty closed convex subset of a Hilbert space H and let $\{B_\alpha : \alpha \in \Lambda\}$ be a decreasing net of nonempty bounded subset of C. Then the asymptotic center a of $\{B_\alpha : \alpha \in \Lambda\}$ in C is an element of $\cap_\alpha \bar{co} B_\alpha$.*

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PROOF. Let z_β be the nearest point to a in $\bar{co}B_\beta$. Then we have $\|y-z_\beta\| \leqq \|y-a\|$ for all $y \in \bar{co}B_\beta$. So we have

$$r(\{B_\alpha\}, z_\beta) \leqq \sup\{\|y-z_\beta\| : y \in B_\beta\} \leqq \sup\{\|y-a\| : y \in B_\beta\}.$$

Let $\{z_{\beta_\gamma}\}$ be a subnet of $\{z_\beta\}$ which converges weakly to z_0. Then we obtain

$$\begin{aligned} r(\{B_\alpha\}, z_0) &\leqq \liminf_\gamma r(\{B_\alpha\}, z_{\beta_\gamma}) \\ &\leqq \liminf_\gamma \sup\{\|y-a\| : y \in B_{\beta_\gamma}\} = r(\{B_\alpha\}, a). \end{aligned}$$

Hence we have $z_0 = a$. Since $\{z_{\beta_\gamma}\}$ is arbitrary, we obtain $\{z_\beta\}$ converges weakly to a. Therefore $a \in \cap_\alpha \bar{co}B_\alpha$.

THEOREM 1. *Let U be a nonempty subset of a Hilbert space H and let S be a left reversible semitopological semigroup. Let $\mathcal{S} = \{T_t : t \in S\}$ be a lipschitzian semigroup on U with $\limsup_s k_s < \sqrt{2}$. Suppose that $\{T_t y : t \in S\}$ is bounded for some $y \in U$ and there exists a closed subset C of U such that $\cap_s \bar{co}\{T_t x : t \geqq s\} \subseteq C$ for all $x \in U$. Then there exists a $z \in C$ such that $T_s z = z$ for all $s \in S$.*

PROOF. Let $B_s(x) = \{T_t x : t \geqq s\}$ for $s \in S$ and $x \in U$. Define $\{x_n : n \geqq 0\}$ by induction as follows:

$$x_0 = y;$$

$$\{x_n\} = \mathcal{A}(\{B_s(x_{n-1})\}, \bar{co}U) \text{ for } n \geqq 1.$$

By Lemma 2, we have $x_n \in \cap_s \bar{co}\{T_t x : t \geqq s\} \subseteq C \subseteq U$ and hence $\{x_n\}$ is well defined. Let $r_n(x) = r(\{B_s(x_{n-1})\}, x)$ and $r_n = r(\{B_s(x_{n-1})\}, \bar{co}U)$ for $n \geqq 1$. Then by Lemma 1, we have $\|x_n - u\|^2 \leqq r_n(x)^2 - r_n^2$ for all $u \in \bar{co}U$ and $n \geqq 1$. Putting $u = T_s x_n$, we have

$$\begin{aligned} \|x_n - T_s x_n\|^2 &= r_n(T_s x_n)^2 - r_n^2 \\ &= (\limsup_t \|T_t x_{n-1} - T_s x_n\|)^2 - r_n^2 \\ &= (\limsup_t \|T_s T_t x_{n-1} - T_s x_n\|)^2 - r_n^2 \\ &\leqq k_s^2 \limsup_t \|T_t x_{n-1} - x_n\|)^2 - r_n^2 \\ &= (k_s^2 - 1) r_n^2. \end{aligned}$$

Let $\eta = \limsup_s k_s^2 - 1$. Then we obtain

$$\begin{aligned} r_{n+1}^2 &\leqq r_{n+1}(x_n)^2 = \limsup_t \|T_t x_n - x_n\|^2 \\ &\leqq (\limsup_s k_s^2 - 1) r_n^2 = \eta r_n^2 \leqq \eta^n r_1^2 \end{aligned}$$

for all $n \geqq 1$. Since

$$\|x_{n+1} - x_n\|^2 \leqq 2\|x_{n+1} - T_t x_n\|^2 + 2\|T_t x_n - x_n\|^2$$

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for all $t \in S$ and $n \geqq 0$, we have

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &\leqq 2 \limsup_t \|T_t x_n - x_{n+1}\|^2 + 2 \limsup_t \|T_t x_n - x_n\|^2 \\ &\leqq 2r_{n+1}^2 + 2r_{n+1}(x_n)^2 \leqq 4\eta^n r_1^2.\end{aligned}$$

Therefore since $\eta < 1$, $\{x_n\}$ is a Cauchy sequence of C. Let $x_n \longrightarrow z$. Then for $s \in S$,

$$\begin{aligned}\|z - T_s z\|^2 &\leqq 3\|z - x_n\|^2 + 3\|x_n - T_s x_n\|^2 + 3\|T_s x_n - T_s z\|^2 \\ &\leqq 3(1 + k_s^2)\|z - x_n\|^2 + 3\|x_n - T_s x_n\|^2 \longrightarrow 0\end{aligned}$$

as $n \longrightarrow \infty$. Therefore $T_s z = z$ for all $s \in S$.

As a direct consequence, we have the following:

COROLLARY 1. *Let U be a nonempty subset of a Hilbert space H and let T be a mapping from U into itself such that*

$$\|T^n x - T^n y\| \leqq k_n \|x - y\|$$

for all $x, y \in U$ and $n \geqq 1$, where $\{k_n\}$ is a positive sequence with $\limsup_n k_n < \sqrt{2}$. *Suppose that $\{T^n y : n \geqq 1\}$ is bounded for some $y \in U$ and there exists a closed subset C of U such that $\bigcap_n \bar{co}\{T^k x : k \geqq n\} \subseteq C$ for all $x \in U$. Then there exists a $z \in C$ such that $Tz = z$.*

If we confine ourselves to nonexpansive or asymptotically nonexpansive semigroups, we have the following result.

THEOREM 2. *Let U be a nonempty subset of a Hilbert space H and let S be a left reversible semitopological semigroup. Let $\mathcal{S} = \{T_t : t \in S\}$ be a lipschitzian semigroup on U with* $\limsup_s k_s \leqq 1$. *Suppose that $\{T_t x : t \in S\}$ is bounded and $\bigcap_s \bar{co}\{T_t x : t \geqq s\} \subseteq U$ for some $x \in U$. Then there exists a $z \in U$ such that $T_s z = z$ for all $s \in S$.*

PROOF. Let $B_s = \{T_t x : t \geqq s\}$ for $s \in S$ and let a be the asymptotic center of $\{B_s\}$ in $\bar{co}U$. Then by Lemma 1, we have

$$r(\{B_s\}, \bar{co}U)^2 + \|a - T_t a\|^2 \leqq r(\{B_s\}, T_t a)^2 \leqq k_t^2 r(\{B_s\}, a)^2$$

for all $t \in S$. Hence we have

$$\limsup_t \|a - T_t a\|^2 \leqq (\limsup_t k_t^2) r(\{B_s\}, \bar{co}U)^2 - r(\{B_s\}, \bar{co}U)^2 = 0.$$

Therefore we obtain

$$\begin{aligned}\|a - T_s a\| &\leqq \limsup_t \|a - T_t a\| + \limsup_t \|T_t a - T_s a\| \\ &\leqq \limsup_t \|a - T_t a\| + k_s \limsup_t \|T_t a - a\| = 0\end{aligned}$$

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for all $s \in S$.

Next, by a method similar to that of the proof of Theorem 1, we prove a fixed point theorem in a Banach space. An important lemma is a result proved in [9], which we state here as:

LEMMA 3. *Let C be a closed convex subset of a reflexive Banach space E. Let $\{B_\alpha : \alpha \in \Lambda\}$ be a decreasing net of nonempty bounded closed convex subsets of C and let $B = \bigcap_\alpha B_\alpha$. Then*

$$r(\{B_\alpha\}, B) \leqq \tilde{N}(E) \inf_\alpha d(B_\alpha).$$

THEOREM 3. *Let U be a nonempty subset of a Banach space E with $\tilde{N}(E) < 1$ and let S be a left reversible semitopological semigroup. Let $\mathcal{S} = \{T_t : t \in S\}$ be a lipschitzian semigroup on U with $\limsup_s k_s < \tilde{N}(E)^{-1/2}$. Suppose that $\{T_t : t \in S\}$ is bounded for some $y \in U$ and there exists a closed subset C of U such that $\cap_s \overline{co}\{T_t x : t \geqq s\} \subseteq C$ for all $x \in U$. Then there exists a $z \in C$ such that $T_s z = z$ for all $s \in S$.*

PROOF. Without loss of generality we may assume that $\limsup_s k_s \geqq 1$. Let $B_s(x) = \overline{co}\{T_t x : t \geqq s\}$ and let $B(x) = \cap_s B_s(x)$ for $s \in S$ and $x \in U$. Define $\{x_n : n \geqq 0\}$ by induction as follows:

$$x_0 = y;$$

$$x_n \in \mathcal{A}(\{B_s(x_{n-1})\}, B(x_{n-1})) \text{ for } n \geqq 1.$$

Well-definedness of $\{x_n\}$ follows from that $B(x) \subseteq C \subseteq U$ for all $x \in U$. Let $r_n(x) = r(\{B_s(x_{n-1})\}, x)$ and $r_n = r(\{B_s(x_{n-1})\}, B(x_{n-1}))$ for $n \geqq 1$. Then from $x_n \in B(x_{n-1}) = \cap_t B_t(x_{n-1})$ for $n \geqq 1$, we have

$$\begin{aligned} r_{n+1}(x_n) &= \limsup_s \|T_s x_n - x_n\| \leqq \limsup_s (\limsup_t \|T_t x_{n-1} - T_s x_n\|) \\ &\leqq (\limsup_s k_s) \limsup_t \|T_t x_{n-1} - x_n\| = (\limsup_s k_s) r_n \\ &\leqq (\limsup_s k_s) \tilde{N}(E) \inf_s d(B_s(x_{n-1})) \end{aligned}$$

and

$$\begin{aligned} \inf_s d(B_s(x_{n-1})) &= \inf_s \sup\{\|T_a x_{n-1} - T_b x_{n-1}\| : a, b \geqq s\} \\ &\leqq \limsup_t (\limsup_s \|T_s x_{n-1} - T_t x_{n-1}\|) \\ &= \limsup_t r_n(T_t x_{n-1}) \leqq (\limsup_t k_t) r_n(x_{n-1}). \end{aligned}$$

Let $\eta = (\limsup_s k_s)^2 \tilde{N}(E)$. Then we have

$$\begin{aligned} r_{n+1}(x_n) &\leqq (\limsup_t k_t) r_n \leqq (\limsup_t k_t)^2 \tilde{N}(E) r_n(x_{n-1}) \\ &= \eta r_n(x_{n-1}) \leqq \eta^n r_1(x_0) \end{aligned}$$

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and

$$\begin{aligned}\|x_{n+1}-x_n\| &\leqq r(\{B_s(x_n)\},x_{n+1})+r(\{B_s(x_n)\},x_n)=r_{n+1}+r_{n+1}(x_n)\\ &\leqq (\limsup_t k_t)^{-1}\eta^{n+1}r_1(x_0)+\eta^n r_1(x_0)\\ &\leqq 2\eta^n r_1(x_0)\end{aligned}$$

for all $n \geqq 1$. So, $\{x_n\}$ is a Cauchy sequence of C and hence $\{x_n\}$ converges to a point $z \in C$. Therefore we have

$$\begin{aligned}\|z-T_sz\| \lim_{n\to\infty}\|x_n-T_sx_n\| &\leqq \lim_{n\to\infty}(r_{n+1}(x_n)+r_{n+1}(T_sx_n))\\ &\leqq \lim_{n\to\infty}(1+k_s)\eta^n r_1(x_0)=0\end{aligned}$$

for all $s \in S$.

We know that if E is uniformly convex then $\tilde{N}(E) < 1$, cf. [2]. Hence the following corollary which is a direct consequence of Theorem 2 is generalization of the result of Goebel and Kirk [6].

Corollary 2. *Let U be a nonempty subset of a Banach space E with $\tilde{N}(E) < 1$ and let T be a mapping from U into itself such that*

$$\|T^nx-T^ny\| \leqq k_n\|x-y\|$$

for all $x, y \in U$ and $n \geqq 1$, where $\{k_n\}$ is a positive sequence with $\limsup_n k_n < \tilde{N}(E)^{-1/2}$. Suppose that $\{T^ny : n \geqq 1\}$ is bounded for some $y \in U$ and there exists a closed subset C of U such that $\bigcap_n \overline{co}\{T^kx : k \geqq n\} \subseteq C$ for all $x \in U$. Then there exists a $z \in C$ such that $Tz = z$.

Remark 1. Casini and Maluta [3] showed that the condition $\tilde{N}(E) < 1$ is weaker than $\epsilon_0(E) < 1$ of [7]. Goebel, Kirk and Thele employed the condition that there exist a bounded closed convex subset C of U such that for each $x \in U$ and $\epsilon > 0$, $dist(T_tx, C) < \epsilon (t \geqq s)$ for some $s \in S$. This condition implies that there exists a closed subset C of U such that $\bigcap_s \overline{co}\{T_tx : t \geqq s\} \subseteq C$ for all $x \in U$ and $\{T_ty : t \in S\}$ is bounded for some $y \in U$. In fact, it is easy to see that $\{T_ty : t \in S\}$ is bounded for all $y \in U$. Let $z \in \bigcap_s \overline{co}\{T_tx : t \geqq s\}$. Then for each $\epsilon > 0$, there exist $s \in S$ such that $dist(T_tx, C) < \epsilon/3$ for every $t \geqq s$. Also there exist $0 \leqq \lambda_i \leqq 1 (\sum_{i=1}^n \lambda_i = 1)$ and $t_i \geqq s$ with $\|z - \sum_{i=1}^n \lambda_i T_{t_i}x\| < \epsilon/3$. For each $1 \leqq i \leqq n$, choose $u_i \in C$ so that $\|T_{t_i}x - u_i\| < (2/3)\epsilon$. Then we have

$$\begin{aligned}dist(z,C) &\leqq \|z-\sum_{i=1}^n\lambda_iu_i\| \leqq \|z-\sum_{i=1}^n\lambda_iT_{t_i}x\|+\sum_{i=1}^n\lambda_i\|T_{t_i}x-u_i\|\\ &\leqq \epsilon/3+\sum_{i=1}^n\lambda_i(2/3)\epsilon=\epsilon.\end{aligned}$$

Since ϵ is arbitrary, we have $z \in C$. Therefore $\bigcap_s \overline{co}\{T_tx : t \geqq s\} \subseteq C$ for all $x \in U$.

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The following example, due to Goebel and Kirk [5], shows that there exists a lipschitzian mapping which is not uniformly k-lipschitzian.

EXAMPLE. Let B denote the unit ball in the Hilbert space l^2 and let T be defined as follows:

$$T : (x_1, x_2, x_3, \cdots) \longrightarrow (0, x_1^2, a_2x_2, a_3x_3, \cdots)$$

where a_i is a sequence of numbers such that $0 \leqq a_i \leqq 1$ and $\prod_{i=2}^{\infty} a_i < 1/\sqrt{2}$. Then $\|Tx - Ty\| \leqq 2\|x - y\|$ for $x, y \in B$ and moreover $\|T^nx - T^ny\| \leqq 2\prod_{i=2}^{n} a_i\|x - y\|$ for $n \geqq 2$. Thus

$$\lim_n 2\prod_{1=2}^{2} a_i = 2\lim_n \prod_{i=2}^{n} a_i < \sqrt{2}.$$

Clearly the mapping T is not uniformly k-lipschitzian with $k < \sqrt{2}$.

REMARK 2. Let γ be a positive real number and let $\mathcal{S} = \{T_t : t \in S\}$ be a lipschitzian semigroup with $\limsup_s k_s < \gamma$. Then, putting $k_s' = \sup_{t \geqq s} k_t$, we have

$$\|T_sx - T_sy\| \leqq k_s\|x - y\| \leqq \sup_{t \geqq s} k_t\|x - y\| = k_s'\|x - y\|$$

and $\lim_s k_s' = \limsup_s k_s$. Hence $\mathcal{S}$ is a lipschitzian semigroup with $\lim_s k_s' < \gamma$.

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