SPLITTING AND DECOMPOSITION BY REGRESSIVE SETS, II

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1. Introduction. In **(3)**, Dekker drew attention to an analogy between (a) the relationship of the recursive sets to the recursively enumerable sets, and (b) the relationship of the retraceable sets to the regressive sets. As was to be expected, this analogy limps in some respects. For example, if a number set α is split by a recursive set, then it is decomposed by a pair of recursively enumerable sets; whereas, as we showed in **(6**, Theorem 2**)**, α may be split by a retraceable set and yet not decomposable (in a liberal sense of the latter term) by a pair of regressive sets. The result for recursive and recursively enumerable sets, of course, follows from the trivial fact that the complement of a recursive set is recursive. On the other hand, the complement of a retraceable set is hardly ever retraceable; indeed, by a result of R. Mansfield $\bar{\alpha}$ is retraceable together with α only if α is recursive; see **(1**; **2)** for generalizations. In view of this, the failure of the recursive:recursively enumerable = retraceable:regressive analogy relative to splitting and decomposition is not too surprising. Indeed, Theorem 2 of **(6)**, which already exhibits this failure, would have appeared in much stronger form were it not for the restricted character of **(6**, Lemma 6**)**. This restriction is now removed: K. I. Appel has very recently found an ingenious proof **(1)** that the union of any finite collection of immune retraceable sets is immune; moreover, the present author has noticed a very simple trick for reducing the corresponding result for *regressive* sets to Appel's theorem. (This reduction provides a very simple proof of the "main result" of **(2**, §3**)**, by reducing the latter directly to Mansfield's result; needless to say, this easy argument for the case of a complementary pair was missed by the authors of **(2)** at the time that paper was written.) This enables us to give, in §§2, 4 below, proofs of much-improved forms of Theorems 2 and 3 of **(6)**, with about the same expenditure of effort as in **(6)**. Further, these proofs do not employ the axiom of choice; our use of that principle in **(6)** was due to our overlooking Lemma 1 in §2 below. In §2, we prove a purely set-theoretical assertion (Theorem 1), exhibit the reduction giving the regressive extension of Appel's theorem, and then note that, as a consequence of this extension and Theorem 1, we obtain a considerably stronger version of **(6**, Theorem 2**)** (which, however, can be strengthened still more in one respect, as we shall prove in §4). Finally, in §3 we present proofs of the main assertions of **(7)**; one of these

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assertions **(7**, Proposition A**)** is, in a sense, complementary to the theorem of Appel. The reference for all notation not explained here is **(6)**.

2. Splitting vs. decomposition for subfamilies of 2^N. We need four definitions:

Definition 1. Let α, β be subsets of N. Then α *splits* $\beta \Leftrightarrow \alpha \cap \beta$ and $\bar{\alpha} \cap \beta$ are both infinite.

Definition 2. Let $\alpha_1, \ldots, \alpha_k$ $(k \geqslant 2)$ and β be subsets of N. Then $\alpha_1, \ldots, \alpha_k$ *decompose* $\beta \Leftrightarrow$ (i) each α_j splits β, and (ii)

$$\beta \subset \bigcup_{1 \leqslant j \leqslant k} \alpha_j.$$

Definition 3. Let $F \subset 2^N$, and let α be a subset of N. α is *F-immune* $\Leftrightarrow \alpha$ is infinite and $(\forall \beta)$ (β an infinite member of $F \Rightarrow \beta \not\subset \alpha$).

Definition 4. Let $F \subset 2^N$, and let α be a subset of N. α is *F-cohesive* $\Leftrightarrow \alpha$ is infinite and $(\forall \beta)$ ($\beta \in F \Rightarrow \beta$ does not split α).

(Whenever the terms *immune* and *cohesive* are used in this paper, it is to be understood that the missing F is the class of recursively enumerable sets.) For what follows, let Q be a countable boolean subalgebra of 2^N containing all of the finite sets; and let F be a (not necessarily countable) subfamily of 2^N such that $F - Q \neq \Box$.

THEOREM 1. *Suppose F has the following properties:*

(1) *F is the union of a countable* (i.e., finite or denumerably infinite) *collection $\{F_i\}_{i \in C}$ of sets such that*

$$(\forall i \in C)([\alpha, \beta \in F_i \text{ and } \alpha \neq \beta] \Rightarrow$$

$$[\alpha \cap \beta \text{ is finite and } (\exists \gamma)(\gamma \in Q \text{ and } \alpha \subset \gamma \subset \overline{\beta - (\alpha \cap \beta)})]);$$

(2) $\alpha \in F \Rightarrow [\alpha \in Q$ or α is *Q-immune*]; *and*

(3) $[\alpha_1, \ldots, \alpha_k \in F$ *and* $\alpha_1, \ldots, \alpha_k$ *are Q-immune*] $\Rightarrow \bigcup_{1 \leqslant j \leqslant k} \alpha_j$ *is Q-immune.*

Then, [β *a Q-immune element of* F] $\Rightarrow$ [β *splits a Q-cohesive set γ such that γ is not decomposed by any finite collection of elements of* F].

Proof. We require for our proof a very simple lemma based on Property (1).

LEMMA 1. *Let β be Q-cohesive, and let $i \in C$. Then, for all but at most one α in F_i, we have: α does not split any Q-cohesive superset of β.*

Proof of Lemma 1. Assume the contrary: let τ_1, τ_2 be distinct elements of F_i, let γ_1 and γ_2 be Q-cohesive supersets of β, and suppose that τ_i splits γ_i, $i = 1, 2$. By (1), there is a set $\lambda \in Q$ such that $\tau_1 \cap \tau_2$ is finite and λ separates τ_1 from $\tau_2 - (\tau_1 \cap \tau_2)$. If λ has finite intersection with γ_2, then λ splits γ_1. Hence, since γ_1 is Q-cohesive, $\lambda \cap \gamma_2$ is infinite. Therefore $\gamma_2 - \lambda$ is finite, since γ_2 is Q-cohesive. But $\lambda \cap (\tau_2 - (\tau_1 \cap \tau_2)) = \Box$; hence, λ splits γ_2, which is a contradiction. Lemma 1 follows.

Now suppose $\beta \in F - Q$. Our next step is to obtain a certain descending sequence $\alpha_0 \supset \alpha_1 \supset \alpha_2 \supset \ldots$ of elements of Q. Let $q_0, q_i, q_2, \ldots$ be an enumeration of the infinite elements of Q. Since $\beta \notin Q$, we have that β is Q-immune. Keeping this in mind, suppose that $q \in Q$ and both $q \cap \beta$, $q \cap \bar{\beta}$ are infinite; then, if q^* is any element of Q, we have either $((q^* \cap q) \cap \beta$ and $(q^* \cap q) \cap \bar{\beta}$ are both infinite) or $((\overline{q^*} \cap q) \cap \beta$ and $(\overline{q^*} \cap q) \cap \bar{\beta}$ are both infinite). In view of this, we are able to define a sequence $\{\alpha_0, \alpha_1, \ldots\}$ of *infinite* sets inductively as follows:

$$\alpha_0 = \begin{cases} q_0 & \text{if } q_0 \cap \beta \text{ and } q_0 \cap \bar{\beta} \text{ are both infinite,} \\ \bar{q}_0 & \text{otherwise;} \end{cases}$$

$$\alpha_{n+1} = \begin{cases} q_{n+1} \cap \alpha_n & \text{if } q_{n+1} \cap \alpha_n \cap \beta \text{ and } q_{n+1} \cap \alpha_n \cap \bar{\beta} \text{ are both infinite,} \\ \bar{q}_{n+1} \cap \alpha_n & \text{otherwise.} \end{cases}$$

Since Q is an algebra, it follows that

(a) $(\forall i)(\exists j)(\alpha_i = q_j)$;

(b) $(\forall i)(\alpha_i \cap \beta$ and $\alpha_i \cap \bar{\beta}$ are both infinite);

and

(c) $(\forall i, j)(j \geqslant i \Rightarrow \alpha_j \subset \alpha_i)$.

Now let sets $\gamma_1^{(n)}, \gamma_2^{(n)}$ be defined thus:

$$\gamma_1^{(0)} = \gamma_2^{(0)} = \square;$$

$$\gamma_1^{(n+1)} = \gamma_1^{(n)} \cup \left\{\text{the least elt of } \alpha_{n+1} \cap \beta \text{ not in } \bigcup_{j \leqslant n} (\gamma_1^{(j)} \cup \gamma_2^{(j)})\right\};$$

$$\gamma_2^{(n+1)} = \gamma_2^{(n)} \cup \left\{\text{the least elt of } \alpha_{n+1} \cap \bar{\beta} \text{ not in } \bigcup_{j \leqslant n} (\gamma_1^{(j)} \cup \gamma_2^{(j)})\right\}.$$

Let $\tilde{\gamma} = \bigcup_n (\gamma_1^{(n)} \cup \gamma_2^{(n)})$. Then $\tilde{\gamma}$ is easily seen to be Q-cohesive and is obviously split by β. In the remainder of the proof, we shall make use of the sequence $\{\alpha_0, \alpha_1, \ldots\}$ in order to obtain a Q-cohesive superset γ of $\tilde{\gamma}$ such that γ is not decomposed by any finite set of elements of F.

Let $K_0, K_1, K_2, \ldots$ be an enumeration of the class of all non-empty finite subsets of the family $\{F_i\}_{i \in C}$; in case C is finite, we let each non-empty subset of $\{F_i\}_{i \in C}$ be repeated infinitely often. $\gamma - \tilde{\gamma}$ is defined as the union of certain sets $\tau^{(n)}$, which we define as follows:

To obtain $\tau^{(n)}$, consider K_n. There are two cases.

Case 1. Each F_i in K_n contains a set which splits some Q-cohesive extension of $\tilde{\gamma}$. Then, by Lemma 1, each F_i in K_n contains *exactly one* such set. If $K_n = \{F_{i_0}, \ldots, F_{i_r}\}$, let $\lambda_{i_0}, \ldots, \lambda_{i_r}$ be these $r + 1$ uniquely determined sets. By (2) and the stipulated property of the λ_{i_j}, each of $\lambda_{i_0}, \ldots, \lambda_{i_r}$ must be Q-immune. Hence, by (3), since $\alpha_n \in Q$, we have that

$$\alpha_n \not\subset \bigcup_{0 \leqslant s \leqslant r} \lambda_{i_s}.$$

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Let m be the smallest number in

$$\alpha_n - \bigcup_{0 \leqslant s \leqslant r} \lambda_{i_s}.$$

Set $\tau^{(n)} = \{m\}$.

Case 2. Case 1 does not hold. Here we set $\tau^{(n)} = \square$. Now define γ as $\tilde{\gamma} \cup \bigcup_n \tau^{(n)}$. γ must be Q-cohesive, by reason of the definition of the sequence $\{\alpha_0, \alpha_1, \ldots\}$ and the fact that for every i we have both $\tilde{\gamma} - \alpha_i$ finite and $(\bigcup_n \tau^{(n)}) - \alpha_i$ finite. Suppose that $\lambda_0, \ldots, \lambda_r$ are distinct elements of F each of which splits γ. Let $F_{i_0}, \ldots, F_{i_r}$ be the corresponding F_i; and let $K_n = \{F_{i_0}, \ldots, F_{i_r}\}$. Then, Case 1 of the definition of $\tau^{(n)}$ applies, and we are obliged to place into γ a number which is not a member of any λ_j, $0 \leqslant j \leqslant r$. Hence, $\lambda_1, \ldots, \lambda_r$ do not decompose γ. The proof of Theorem 1 is complete.

Notice that if $Q \subset F$, then any set γ satisfying the conclusion of Theorem 1 *must* be Q-cohesive.

LEMMA 2 (**1**). *If* $\alpha_1, \ldots, \alpha_k$ *are immune retraceable sets, then* $\bigcup_{1 \leqslant i \leqslant k} \alpha_i$ *is immune.*

We shall now describe the simple procedure whereby the stronger form of Lemma 2 ($\alpha_1, \ldots, \alpha_k$ immune and *regressive*) is reduced to Lemma 2 itself; observe that (**2**, Theorem 1) reduces to Mansfield's theorem by precisely this procedure applied to a complementary pair. In fact, a rather minor modification of the argument in (**1**) provides a direct proof of Lemma 2′ below. We give the reduction procedure anyway, on the grounds that it may be a special instance of some useful "metatheorem."

Suppose, then, that $\alpha_1, \ldots, \alpha_k$ are immune regressive sets, and that $\bigcup_{1 \leqslant i \leqslant k} \alpha_i$ has an infinite recursive subset γ. Two elementary facts which we shall use often are: (a) the intersection of a regressive set with a recursive set is regressive, and (b) the intersection of a retraceable set with a recursive set is retraceable. In view of (a), we may as well assume that $\gamma = \bigcup_{1 \leqslant i \leqslant k} \alpha_i$. Let f_1 be a special regressing function for α_1 (for the notion of a *special* regressing function, see (**2**)) such that the domain of f_1 is a subset of γ; this last assumption is clearly permissible. Let β be an infinite recursive subset of domain(f_1); we assume, with no loss of generality, that β contains the fixed point, a_0, of α_1 under f_1. Now it is easy to obtain $\alpha_1 \cap \beta$ as a regressive set *regressed by a function with domain* β: for each $x \in \beta$, set

$$p(x) = f_1^{x^*}(x), \qquad \text{where } x^* = \mu k(f_1^{\,k}(x) \in \beta \;\&\; x \neq a_0 \Rightarrow k \geqslant 1).$$

It is safe to assume $\alpha_1 \cap \beta$ is *infinite*; for if not, then $\alpha_2 \cup \ldots \cup \alpha_k$ is not immune and we would contradict an induction hypothesis which is clearly valid in the case $k = 1$.

If we now look carefully at the proof of (**3**, Proposition 7), we see that it provides the following: if τ is an infinite set regressed by a special regressing function g such that g has recursive domain, then τ has an infinite *retraceable* subset ξ such that ξ and $\tau - \xi$ are *recursively separated.* We apply this observation to $\alpha_1 \cap \beta$, letting ξ_1 be a retraceable subset of $\alpha_1 \cap \beta$ such that ξ_1 and $(\alpha_1 \cap \beta) - \xi_1$ are separated by a recursive set, say η_1; thus $\xi_1 \subset \eta_1$,

$$(\alpha_1 \cap \beta) - \xi_1 \subset \bar{\eta}_1.$$

The remainder of the reduction takes place inside η_1. It is safe to assume $\eta_1 \cap \alpha_j$ is infinite for $2 \leqslant j \leqslant k$; otherwise, the union of $k - 1$ of the α_i would be non-immune and we would obtain a contradiction to a legitimate induction hypothesis. Thus, in particular, $\eta_1 \cap \alpha_2$ is an infinite regressive set. If f_2 was an originally given special regressing function for α_2, then it is clear how to obtain from f_2 a special regressing function h such that (i) h regresses $\eta_1 \cap \alpha_2$, and (ii) domain$(h) \subset \eta_1$. Let ζ be an infinite recursive subset of domain(h). Again, it is safe to assume $\zeta \cap \alpha_i$ infinite for $1 \leqslant i \leqslant k$; in particular, we may assume $\zeta \cap \xi_1$ and $\zeta \cap \alpha_2$ are infinite. We now repeat the above procedure relative to ζ, obtaining an infinite retraceable set $\xi_2 \subset \zeta \cap \alpha_2$ and a recursive set η_2 such that $\xi_2 \subset \eta_2$, $(\zeta \cap \alpha_2) - \xi_2 \subset \eta_2$. But $\eta_2 \cap \xi_1$ *remains retraceable* since η_2 is recursive; thus we have now replaced *two* of the originally given regressive sets by retraceable sets. It should now be clear that by $k - 2$ more applications of the same procedure, we arrive at a recursive set represented as the union of k immune retraceable sets, in contradiction to Appel's result. The reduction procedure is therefore a success, and we have the desired generalization:

LEMMA 2′. *If $\alpha_1, \ldots, \alpha_k$ are immune regressive sets, then $\bigcup_{1 \leqslant i \leqslant k} \alpha_i$ is immune.*

In order to combine this result with Theorem 1, however, we need one additional step of generalization: we need Lemma 2′ relative to functions partial recursive *in a given set.* But this causes no difficulty: examination of the proof given in **(1)** reveals that it "relativizes" in the usual trivial manner; and it is plain that the reduction procedure discussed above also relativizes trivially. Thus, if we call a set α "β-regressive" just in case it is regressed by a function which is *partial recursive in* β, then the following relativized form of Lemma 2′ is seen to be true:

LEMMA 2^R. *Let β be a set of natural numbers. If $\alpha_1, \ldots, \alpha_k$ are β-immune β-regressive sets, then $\bigcup_{1 \leqslant j \leqslant k} \alpha_j$ is β-immune.*

(Here "β-immune" means, of course, *devoid of infinite subsets which are recursively enumerable in* β.)

Let A be the class of arithmetical sets. Call a set β "A-regressive" just in case β is γ-regressive for some $\gamma \in A$.

THEOREM 2. *Let α be any retraceable set such that $\alpha \notin A$. Then α splits an A-cohesive set β such that β is not decomposed by any finite collection of A-regressive sets.*

Proof. Let $\{p_i^A\}$ be an enumeration of all partial functions f such that $(\exists\alpha)(\alpha \in A \;\&\, f$ is partial recursive in $\alpha)$. Fix i. By relativization of an argument in **(4)**, if γ_1, γ_2 are distinct infinite sets both regressed by p_i^A, then $\gamma_1 \cap \gamma_2$ is finite and $\gamma_1, \gamma_2 - (\gamma_1 \cap \gamma_2)$ are separated by a member of A. By further relativization of observations in **(4)**, any A-regressive set is either arithmetical or A-immune. Finally, we claim that by Lemma 2^R we have $\bigcup_{1\leqslant i\leqslant k} \gamma_i$ A-immune provided $\gamma_1, \ldots, \gamma_k$ are all both A-regressive and A-immune. This last assertion requires a little argument: suppose $\bigcup_{1\leqslant i\leqslant k} \gamma_i$ were *not* A-immune; let α be a member of A which bears witness to the fact. Let $p_{i_1}{}^A, \ldots, p_{i_k}{}^A$ be functions regressing $\gamma_1, \ldots, \gamma_k$ respectively; and let $\alpha_1, \ldots, \alpha_k$ be arithmetical sets in which $p_{i_1}{}^A, \ldots, p_{i_k}{}^A$ are, respectively, partial recursive. Let α^* be an element of A such that each of $\alpha, \alpha_1, \ldots, \alpha_k$ is recursive in α^*. Then, applying Lemma 2^R with α^* for β, we see that $\bigcup_{1\leqslant i\leqslant k} \gamma_i$ is α^*-immune, which gives a contradiction. Thus, since A is a countable boolean algebra containing all the finite subsets of N, Theorem 2 is established as a special case of Theorem 1.

3. Retraceable sets with immune complements. In this section, we derive the principal results of **(7)**, that is, Propositions A, B, and D of that paper. It will not be necessary to give a separate proof for Proposition A, since that result follows as an almost immediate corollary to **(7**, Proposition B**)**; it is the latter theorem which we choose to prove below. Our proof of Theorem 3 is a priority argument of "classical" type; i.e., in the informal terminology favoured in **(8)**, each one of an infinite list of "requirements" is "injured" only a finite number of times prior to being permanently "met." We shall cast the argument in a casual "moving markers" form, in order to lay its modest conceptual content completely bare for all readers, some of whom may not relish interlocking inequalities. The length of the proof is due almost entirely to technical demands plus the use of ordinary English, rather than to peculiarity of idea.

THEOREM 3. *Let α, β be disjoint, infinite recursively enumerable sets. There is a recursive function f such that, for every i, $f(i)$ is an index of a general recursive basic retracing function which retraces a unique infinite set α_i with the following properties:*

(1) $j \neq k \Rightarrow \alpha_j \cap \alpha_k = \Box$;
(2) $|\alpha_j \cap \bar{\alpha}| = 1 \;\&\; \alpha_j \cap \bar{\alpha} \subset \beta$; *and*
(3) $\alpha - \alpha_i$ *is immune.*

Remark. A retracing function g is called *basic* **(4)** if and only if the following conditions are satisfied: $x \in \delta g \Rightarrow g(x) \leqslant x$; g is finite-to-one; $\rho g \subset \delta g$; and $|\{x | g(x) = x\}| < \aleph_0$.

Proof. We shall make use of a doubly indexed sequence $\{\Lambda_{ij}\}_{i,j=0}^{\infty}$ of "moving markers." We arrange these markers in a "priority" sequence, using a familiar diagonal enumeration of the subscribt pairs; each term of the sequence is to have higher priority than all those which follow it:

$$\Lambda_{00}, \Lambda_{01}, \Lambda_{10}, \Lambda_{02}, \Lambda_{11}, \Lambda_{20}, \Lambda_{03}, \Lambda_{12}, \ldots .$$

Let h be a 1–1 recursive function enumerating β; we assume, with no loss of generality, that $0 \in \beta$ and $h(0) = 0$. Let k be a strictly increasing recursive function such that

$$\{W_e | W_e \subset \alpha\} = \{W_{k(0)}, W_{k(1)}, W_{k(2)}, \ldots\}.$$

We assume as given some fixed procedure for uniform enumeration of $\{W_e | W_e \subset \alpha\}$ by stages; thus, at stage s, $W_{k(n)}{}^s$ is a finite subset of $W_{k(n)}$ whose contents are completely known to us; further, we must have $\bigcup_s W_{k(n)}{}^s = W_{k(n)}$, and there shall be a recursive function l such that, for every s,

$$t > l(s) \Rightarrow W_{k(t)}{}^s = \Box .$$

The sequence of approximating constructions on which the proof rests may now be described as follows.

Stage 0. Attach Λ_{00} to 0, and place $(0, 0)$ in f_0. Proceed to Stage 1.

Stage $s + 1$. We shall make a rather lengthy inductive hypothesis regarding the situation at the end of stage s, necessitated mainly by the condition $\alpha_i \cap \alpha_j = \Box$; it will be easy to see that this inductive hypothesis persists from stage to stage.

Inductive Assumption: At the conclusion of stage s, the markers which are *attached* include Λ_{00} and constitute an *initial segment*, $\Lambda_{00}, \ldots, \Lambda_{qt}$ of the priority listing; moreover, if Λ_{ru} is one of the attached markers and Λ_{kl} is any attached marker of higher priority, then (a) $u = 0 \Rightarrow$ the number to which Λ_{ru} is attached is greater than the number to which Λ_{kl} is attached, and (b) $u > 0 \Rightarrow$ every number in the $f_r{}^s$-chain from Λ_{ru} above $\Lambda_{r,u-1}$ is greater than the number to which Λ_{kl} is attached. (It will be made clear in the remainder of the description of Stage $s + 1$ what is meant by the "$f_r{}^s$-chain from Λ_{ru} above $\Lambda_{r,u-1}$.")

Now, letting Λ_{qt} be the attached marker of lowest priority at the end of stage s, we proceed as follows.

Case 1. Λ_{qt} is Λ_{00}.

Subcase 1a. There exist $t \leqslant l(s + 1)$ and $n \in W_{k(t)}{}^{s+1}$ such that $n > 0$ and n has not previously been placed in δf_0. Let t_0 be the smallest such t, and n_0 the smallest such n relative to t_0. Place $(n_0, 0)$ in f_0, attach Λ_{01} to n_0, and associate the index $k(t_0)$ with Λ_{01}. If there are numbers m such that $m \leqslant s + 1$ and m is not yet in δf_0, let $m_0, \ldots, m_r$ be a list of *all* such numbers. For each i such that $0 \leqslant i \leqslant r$, place (m_i, n_0) or $(m_i, 0)$ in f_0 according as $m_i > n_0$ or $m_i \leqslant n_0$. Then go to Stage $s + 2$.

Subcase 1b. Subcase 1a fails, but there exist $t \leqslant l(s+1)$ and $n \in W_{k(t)}^{s+1}$ such that, for some sequence of numbers $z_0, z_1, \ldots, z_r = 0$ $(r \geqslant 0)$, we have $(n, z_0), (z_0, z_1), \ldots, (z_{r-1}, 0) \in f_0^s$ (where "f_0^s" denotes the set of all pairs placed in f_0 by the end of stage s). Attach Λ_{01} to n_0, where t_0 is the least such t and n_0 the least such n relative to t_0. Designate $n_0, z_0, z_1, \ldots, z_{r-1}$ the *$f_0{}^s$-chain from Λ_{01} above Λ_{00}*. Associate with Λ_{01} the index $k(t_0)$. Next, add *new* members to δf_0 in the manner prescribed in Subcase 1a. Then go to Stage $s+2$.

Subcase 1c. Neither Subcase 1a nor Subcase 1b holds. If there are numbers m such that $m \leqslant s+1$ and m is not yet in δf_0, place $(m, 0)$ in f_0 for each such m, and then proceed to Stage $s+2$; otherwise, go directly to Stage $s+2$.

Case 2. Λ_{qt} is not Λ_{00}. The procedure under Case 1 was the "basis step" of the general procedure which we now describe. We shall suppose that all markers of priority greater than that of Λ_{pj} have been considered, where Λ_{pj} occurs in $\Lambda_{01}, \ldots, \Lambda_{qt}$; and we consider Λ_{pj} (our consideration of Λ_{00}, at the beginning of this inductively defined process, consists of doing nothing and passing at once to consideration of Λ_{01}). If any erasures occur in the consideration of a marker of higher priority than Λ_{pj}, we go on at once to consider the marker of priority next below that of Λ_{pj}; or, if Λ_{pj} is Λ_{qt}, and such erasures have occurred, we go directly on to Stage $s+2$. Otherwise, six main cases arise in the consideration of Λ_{pj}.

Case A. Λ_{pj} is not Λ_{qt}, and $j > 1$. We treat three subcases.

Subcase A1. There exist $t \leqslant l(s+1)$ and $n \in W_{k(t)}^{s+1}$ such that (i) $n \notin \delta f_p{}^s$, (ii) $k(t)$ is less than the index currently associated with Λ_{pj}, (iii) $k(t)$ is greater than the index currently associated with $\Lambda_{p,j-1}$, and (iv) $n > r$, where Λ_{uy} is the marker of priority one greater than that of Λ_{pj} and r is the current position of Λ_{uy}. Let t_0 be the smallest such t, and n_0 the smallest such n relative to t_0. Perform the following sequence of steps: (a) erase Λ_{pj} and all markers of lower priority, up to and including Λ_{qt}, and dissociate from each of these erased markers any index found associated with it; (b) place (n_0, w) in f_p, where w is the current position of $\Lambda_{p,j-1}$; (c) attach Λ_{pj} to n_0 and associate $k(t_0)$ with it; and (d) call a number w such that some marker Λ_{wu} is still attached after (c) a *relevant* number, and for each relevant w, let $u(w)$ be the greatest u such that Λ_{wu} is still attached after (c); then, if $m \leqslant s+1$ and w is relevant, and if $m \notin \delta f_w{}^s$ & $(w = p \Rightarrow m \neq n_0)$, place in f_w the pair (m, d) if

$$m > d = \max\{i \mid i \text{ is the position of a marker } \Lambda_{wg} \text{ with } g \leqslant u(w)\}$$

and place (m, m) in f_w otherwise. Then go to Stage $s+2$.

Subcase A2. Subcase A1 does not hold, but there exist $t \leqslant l(s+1)$ and $n \in W_{k(t)}^{s+1}$ such that, for some sequence $z_0, z_1, \ldots, z_i$ $(i \geqslant 0)$, we have: (i) z_i is the current position of $\Lambda_{p,j-1}$, (ii) $(n, z_0), (z_0, z_1), \ldots, (z_{i-1}, z_i) \in f_p{}^s$, (iii) $k(t)$ is less than the index currently associated with Λ_{pj}, (iv) $k(t)$ is greater than the index currently associated with $\Lambda_{p,j-1}$, and (v) z_{i-1} (or n, if $i = 0$) is greater than the current position of Λ_{uy}, where Λ_{uy} has priority one greater than the priority of Λ_{pj}. Let t_0 be the smallest such t, and n_0 the smallest such n relative to t_0. Designate the sequence $n_0, z_0, z_1, \ldots, z_{i-1}$ as the *f_p^{s+1}-chain from Λ_{pj} above $\Lambda_{p,j-1}$*, and perform the following sequence of steps: (a) Same as step (a) in Subcase A1; (b) attach Λ_{pj} to n_0 and associate $k(t_0)$ with Λ_{pj}; and (c) for any relevant w and any $m \leqslant s + 1$ such that $m \notin \delta f_w{}^s$, place in f_w the pair (m, d) or the pair (m, m) according to the prescription in Subcase A1. (w is *relevant*, in Subcase A2, just in case some marker Λ_{wu} is attached after step (b); "$u(w)$" has the same meaning as in Subcase A1.)

Subcase A3. Both of Subcases A1 and A2 fail to hold. Then proceed to the consideration of Λ_{ei}, where Λ_{ei} is the marker of priority one lower than that of Λ_{pj}.

Case B. Λ_{pj} is not Λ_{qt}, and $j = 1$.

Subcase B1. There exist $t \leqslant l(s + 1)$ and $n \in W_{k(t)}{}^{s+1}$ such that (i) $n \notin \delta f_p{}^s$, (ii) $k(t)$ is less than the index currently associated with Λ_{pj}, and (iii) n is greater than the current position of Λ_{uy}, where Λ_{uy} is the marker of priority one greater than that of Λ_{pj}. Let t_0 be the smallest such t, and n_0 the smallest such n relative to t_0. Perform the same sequence of steps as in Subcase A1; then go to Stage $s + 2$.

Subcase B2. Subcase B1 does not hold, but there exist $t \leqslant l(s + 1)$ and $n \in W_{k(t)}{}^{s+1}$ such that, for some sequence $z_0, z_1, \ldots, z_i$ ($i \geqslant 0$), we have: (i) z_i is the current position of $\Lambda_{p,j-1}$, (ii) $(n, z_0), (z_0, z_1), \ldots, (z_{i-1}, z_i) \in f_p{}^s$, (iii) $k(t)$ is less than the index currently associated with Λ_{pj}, and (iv) z_{i-1} (or n, if $i = 0$) is greater than the current position of Λ_{uy}, where Λ_{uy} has priority one greater than the priority of Λ_{pj}. Let t_0 be the smallest such t, and n_0 the smallest such n relative to t_0. Designate the sequence $n_0, z_0, z_1, \ldots, z_{i-1}$ as the *$f_p{}^{s+1}$-chain from Λ_{pj} above $\Lambda_{p,j-1}$*, and perform the same sequence of steps as in Subcase A2; then go to Stage $v + 2$.

Subcase B3. Both of Subcases B1 and B2 fail to hold. Then proceed to the consideration of Λ_{ei}, where Λ_{ei} is the marker of priority one lower than that of Λ_{pj}.

Case C. Λ_{pj} is not Λ_{qt}, and $j = 0$. Proceed directly to the consideration of Λ_{ei}, where Λ_{ei} is as in Subcase B3.

Case D. Λ_{pj} is Λ_{qt}, and $j > 1$.

Subcase D1. There exist $t \leqslant l(s + 1)$ and $n \in W_{k(t)}{}^{s+1}$ such that (i)–(iv) of Subcase A1 hold. Let t_0 be the least such t, n_0 the least such n relative to t_0. Perform the same sequence of steps as in Subcase A1; then go to Stage $s + 2$.

Subcase D2. Subcase D1 does not hold, but there exist $t \leqslant l(s + 1)$ and $n \in W_{k(t)}{}^{s+1}$, and a sequence $z_0, z_1, \ldots, z_i$ ($i \geqslant 0$), such that (i)–(v) of Subcase A2 hold. Let t_0 be the least such t, n_0 the least such n relative to t_0. Perform the same sequence of steps as in Subcase A2; then go to Stage $s + 2$.

Subcase D3. Both of Subcases D1 and D2 fail to hold. Here we get a ramification into secondary subcases. Let Λ_{my} be the marker of priority one less than the priority of Λ_{qt}. We strive to attach Λ_{my}. It is clear from the priority listing that $y > 0$, since $t > 1$.

Subcase D3a. $y = 1$ and there exist $t \leqslant l(s + 1)$ and $n \in W_{k(t)}{}^{s+1}$ such that (i) $n \notin \delta f_m{}^s$, and (ii) $n > b$, where b is the current position of Λ_{qt} (i.e., Λ_{pj}). Let t_0 be the smallest such t, and n_0 the least such n relative to t_0. Perform the following sequence of steps: (a) attach Λ_{my} to n_0 and associate $k(t_0)$ with Λ_{my}; (b) place (n_0, x) in f_m, where x is the current position of $\Lambda_{m,y-1}$; and (c) if w is relevant, $g \leqslant s + 1$, $g \notin f_w{}^s$, and $(w = m \Rightarrow g \neq n_0)$, place (g, d) in f_w if

$$g > d = \max\{i |\ i \text{ is the position of an attached marker } \Lambda_{wr} \text{ with } r \leqslant u(w)\}$$

and place (g, g) in f_w otherwise; here w is *relevant* just in case some marker Λ_{wu} is attached after step (b), and "$u(w)$" has the same meaning as in Case A. Then go to Stage $s + 2$.

Subcase D3b. $y = 1$ and Subcase D3a does not hold; but there exist $t \leqslant l(s + 1)$ and $n \in W_{k(t)}{}^{s+1}$ such that, for some sequence $z_0, z_1, \ldots, z_i$ $(i \geqslant 0)$, we have: (i) z_i is the current position of $\Lambda_{m,y-1}$, (ii) (n, z_0), $(z_0, z_1), \ldots,$ $(z_{i-1}, z_i) \in f_m{}^s$, and (iii) $z_{i-1} > b$, where b is as in Subcase D3a. (If $i = 0$, we require $n > b$.) Let t_0 be the smallest such t, and n_0 the smallest such n relative to t_0. Designate $n_0, z_0, z_1, \ldots, z_{i-1}$ as the $f_m{}^{s+1}$-*chain from* Λ_{my} *above* $\Lambda_{m,y-1}$. Perform the following sequence of steps: (a) same as step (a) under Subcase D3a; and (b) if w is relevant (i.e., some marker Λ_{wu} is attached after step (a)), $g \leqslant s + 1$, and $g \notin f_w{}^s$, add (g, d) or (g, g) to f_w according to the prescription in case (c) of subcase D3a. Then go to Stage $s + 2$.

Subcase D3c. $y = 1$ and both Subcase D3a and Subcase D3b fail to hold. Call w *relevant*, in this case, provided some Λ_{wu} is attached; and let "$u(w)$" have the same meaning as in Case A. For each relevant w and any $g \leqslant s + 1$ such that $g \notin f_w{}^s$, add (g, d) or (g, g) to f_w according to the prescription in case (c) of Subcase D3a. Then go to Stage $s + 2$.

Subcase D3d. $y > 1$ and there exist $t \leqslant l(s + 1)$ and $n \in W_{k(t)}{}^{s+1}$ such that (i) $n \notin \delta f_m{}^s$, (ii) $k(t)$ is greater than the index currently associated with $\Lambda_{m,y-1}$, and (iii) $n > b$, where b is the current position of Λ_{qt}. Let t_0 be the smallest such t, n_0 the least such n relative to t_0. Perform the same sequence of steps as in Subcase D3a; then go to Stage $s + 2$.

Subcase D3e. $y > 1$ and Subcase D3d fails to hold; but there exist $t \leqslant l(s + 1)$ and $n \in W_{k(t)}{}^{s+1}$ such that, for some sequence $z_0, z_1, \ldots, z_i$ $(i \geqslant 0)$, (i)–(iii) of Subcase D3b hold and, in addition, we have (iv) $k(t)$ is greater than the index currently associated with $\Lambda_{m,y-1}$. Let t_0 be the least such t, n_0 the least such n relative to t_0. Designate $n_0, z_0, z_1, \ldots, z_{i-1}$ as the $f_m{}^{s+1}$-*chain from* Λ_{my} *above* $\Lambda_{m,y-1}$. Perform the following sequence of steps, where w is *relevant* just in case some Λ_{wu} is attached: (a) same as step (a) under Subcase D3a; and (b) if w is relevant, $g \leqslant s + 1$, and $g \notin f_w{}^s$, add (g, d) or (g, g) to f_w according to the prescription in case (c) of Subcase D3a. Then go to Stage $s + 2$.

Subcase D3f. $y > 1$ and both of Subcases D3d, D3e fail to hold. Let "w is relevant" have the by-now-obvious meaning. (We shall henceforth not detail the minor variations, from case to case, in the meaning of "w is relevant"; the pattern has been established.) Add new pairs to f_w for relevant w as in Subcase D3c; then go to Stage $s + 2$.

Case E. Λ_{pj} is Λ_{qt}, and $j = 1$.

Subcase E1. There exist $t \leqslant l(s + 1)$ and $n \in W_{k(t)}{}^{s+1}$ such that (i) $n \notin \delta f_p{}^s$, (ii) $k(t)$ is smaller than the index currently associated with Λ_{pj}, and (iii) $n >$ the current position of the marker $\Lambda_{p-1,2}$ if $p > 0$, and $n > 0$ if $p = 0$. Let t_0 be the least such t, and n_0 the least such n relative to t_0. Perform the same sequence of steps as in Subcase A1; then go to Stage $s + 2$.

Subcase E2. Subcase E1 does not hold; but there exist $t \leqslant l(s + 1)$ and $n \in W_{k(t)}{}^{s+1}$ such that, for some sequence $z_0, z_1, \ldots, z_i$ $(i \geqslant 0)$, we have (i) and (ii) of Subcase D3b together with (iii) z_{i-1} (or n, if $i = 0$) is greater than the current position of the marker of priority one greater than that of Λ_{pj} (namely, $\Lambda_{p-1,2}$ if $p > 0$ and Λ_{00} otherwise) and (iv) $k(t)$ is smaller than the index currently associated with Λ_{pj}. Let t_0 be the least such t, and n_0 the least such n relative to t_0. Proceed exactly as in Subcase A2; then go to Stage $s + 2$.

Subcase E3. Neither Subcase E1 nor Subcase E2 holds. Since $j = 1$, the marker of priority one less than that of Λ_{pj} is $\Lambda_{p+1,0}$. Generate the sequence $h(1), h(2), h(3), \ldots$ until the smallest number m is found for which $h(m) \notin \delta f_{p+1}{}^s$ and $h(m)$ is greater than the current position of Λ_{pj}; let m_0 be this number. Attach $\Lambda_{p+1,0}$ to $h(m_0)$. If w is relevant and $g \leqslant s + 1$ and $g \notin \delta f_w{}^s$, add either (g, d) or (g, g) to f_w according to the prescription in Subcase D3a. Then go to Stage $s + 2$.

Case F. Λ_{pj} is Λ_{qt}, and $j = 0$. In this case, $\Lambda_{0,p+1}$ is the marker whose priority is one lower than that of Λ_{pj}. The following subcases arise as we attempt to attach $\Lambda_{0,p+1}$ (since Λ_{pj} is not Λ_{00}, $p > 0$).

Subcase F1. There are numbers $t \leqslant l(s + 1)$ and $n \in W_{k(t)}{}^{s+1}$ such that (i) $n \notin \delta f_0{}^s$, (ii) $n > b$, where b is the current position of Λ_{p0}, and (iii) $k(t)$ is greater than the index currently associated with Λ_{0p}. Let t_0 be the smallest such t, and n_0 the smallest such n relative to t_0. Perform the following sequence of steps: (a) attach $\Lambda_{0,p+1}$ to n_0, and associate $k(t_0)$ with $\Lambda_{0,p+1}$; (b) place (n_0, r) in f_0, where r is the current position of Λ_{0p}; and (c) if w is relevant, $g \leqslant s + 1$, $g \notin \delta f_w{}^s$, and $(w = 0 \Rightarrow g \neq n_0)$, place either (g, d) or (g, g) in f_w, according to the prescription given in Subcase A1. Then go to Stage $s + 2$.

Subcase F2. Subcase F1 does not hold, but there exist $t \leqslant l(s + 1)$ and $n \in W_{k(t)}{}^{s+1}$ such that, for some sequence $z_0, z_1, \ldots, z_i$ $(i \geqslant 0)$, we have (i) z_i is the current position of Λ_{0p}; (ii) $k(t)$ is greater than the current associate of Λ_{0p}; (iii) $(n, z_0), (z_0, z_1), \ldots, (z_{i-1}, z_i) \in f_0{}^s$, and (iv) $z_{i-1} > b$, where b is the current position of Λ_{p0} (if $i = 0$, we require $n > b$). Let t_0 be the smallest such t, and n_0 the smallest such n relative to t_0. Designate $n_0, z_0, z_1, \ldots, z_{i-1}$

as the *f_0^{s+1}-chain from $\Lambda_{0,p+1}$ above Λ_{0p}*. Perform the same sequence of steps as in Subcase D3b; then go to Stage $s + 2$.

Subcase F3. Neither Subcase F1 nor Subcase F2 holds. If w is relevant, $g \leqslant s + 1$, and $g \notin f_w^s$, place either (g, d) or (g, g) in f_w, according to the prescription in Subcase D3c; then go to Stage $s + 2$.

This finishes the description of Stage $s + 1$ of the construction. To complete the proof of Theorem 3, we establish a sequence of five lemmas, each a straightforward consequence of our construction.

LEMMA I. *Let Λ_{ij} be any marker with $j > 0$. Then there is a stage s and numbers n and t such that, throughout any stage $\tilde{s} \geqslant s$, we have:* (1) *Λ_{ij} is attached to n,* (2) *$k(t)$ is associated with Λ_{ij}, and* (3) *$n \in \delta f_i^s$. For any Λ_{i0}, there is a stage s and a number n such that, for all $\tilde{s} \geqslant s$, Λ_{i0} is constantly attached to n.*

Proof. Λ_{00} is attached to 0 at Stage 0, and never disturbed thereafter. We proceed by induction on the priority ordering. Suppose the lemma holds for the initial segment $\Lambda_{00}, \ldots, \Lambda_{qt}$ of this ordering; and let s_0 be a stage such that all of $\Lambda_{00}, \ldots, \Lambda_{qt}$ are permanently in place—and those which admit associated indices are with their final associates—for all $s \geqslant s_0$. Consider Λ_{mu}, the marker of priority one less than that of Λ_{qt}. It is clear from the construction that if Λ_{mu} becomes attached at a stage $s > s_0$, it continues to be attached to *some* number through all subsequent stages, though it may move from one position to another on occasion. If Λ_{mu} is of the form Λ_{m0}, it will in fact *not* move, once it is attached at a stage $s > s_0$; this is easily seen by examining the construction (note that every $W_{k(t)}^s$ is a subset of α, while every position of a marker Λ_{r0} is in β). If $u \neq 0$, Λ_{mu} may move subsequent to such attachment; however, we claim it can move only *finitely often*. For, if $u > 0$, then Λ_{mu}, once attached at a stage $s > s_0$, always thereafter appears in the company of an associated index; moreover, its movement (at a stage $s > s_0$) entails changing its associated index to a new and *smaller* associated index, since the index (if any) associated with $\Lambda_{m,u-1}$ is *fixed* after stage s_0. Thus, to complete the induction step, it clearly suffices to show that there must be a stage $s > s_0$ such that Λ_{mu} is attached at stage s.

Suppose this is not the case. First assume $u > 0$. If $u - 1 = 0$, let e be the smallest number such that $W_{k(e)}$ is infinite; otherwise, let e be the smallest number such that $W_{k(e)}$ is infinite *and* $k(e)$ is greater than the final associate of $\Lambda_{m,u-1}$. Let y be the largest number in $\bigcup_i \delta f_i^{s_0}$. (It is clear, from the construction, that the latter set is finite and its contents completely known). Let $\tilde{y}$ be an element of $W_{k(e)}$ such that $\tilde{y} > y$. Note, from the description of Stage $s + 1$ above, that since Λ_{mu} is never attached subsequent to Stage s_0, then *no* marker of priority no greater than that of Λ_{mu} can ever be attached after Stage s_0, so that after Stage s_0 numbers enter $\bigcup_i \delta f_i$ *only* by means of the clauses concerning "relevant" numbers. There are two possibilities.

(1) $\tilde{s} = \mu s$ ($s > s_0$ and $\tilde{y} \in W_{k(e)}{}^s$), and $\tilde{y} \geqslant \tilde{s}$. Then, at Stage $\tilde{s}$, if Λ_{mu} is not already attached, we are obliged to attach it, since we have

$$\tilde{y} \in W_{k(e)}{}^{\tilde{s}},\ \tilde{y} \notin \delta f_m{}^{\tilde{s}-1},\ \tilde{y} > \text{the final position of } \Lambda_{qt}, \text{ and } k(e) > \text{the associate (if any) of } \Lambda_{m,u-1} \text{ at Stage } \tilde{s}.$$

This gives a contradiction.

(2) $\tilde{s} = \mu s$ ($s > s_0$ and $\tilde{y} \in W_{k(e)}{}^s$), and $\tilde{y} < \tilde{s}$. Let $\tilde{y} = \tilde{s} - r$. Then at Stage $\tilde{s} - r$, $(\tilde{y}, z)$ is placed in f_m, where z is the final position of $\Lambda_{m,u-1}$. Since $\tilde{y}$ is greater than the final position of Λ_{qt} and $k(e)$ is greater than the associate (if any) of $\Lambda_{m,u-1}$ at Stage $\tilde{s}$, we are then obliged by the construction to attach Λ_{mu} to some number at Stage $\tilde{s}$, if it is not already attached by the end of Stage $\tilde{s} - 1$. Again, this gives a contradiction. Hence, assuming $u > 0$, Λ_{mu} must eventually be attached during a stage $s > s_0$. But if $u = 0$, matters are even simpler: then, at Stage $s_0 + 1$ at the latest, we must permanently attach Λ_{m0} to some element of β after listing sufficiently many values of h. Thus the induction step goes through, and the lemma follows.

LEMMA II. *For each i, let $f_i = \bigcup_s f_i{}^s$* (i.e., $f_i = \{(x, y) \mid (x, y)$ is *placed in* f_i at some stage $s\}$). *Then each f_i is a finite-to-one, general recursive function such that* (a) $(\forall x)(f_i(x) \leqslant x)$ *and* (b) $f(x) = x$ *for only finitely many x. Moreover, there is a recursive function ϕ such that, for all i, $\phi(i)$ is an index of f_i.*

Proof. First, it is clear from the construction that the f_i are functions, since no number is ever assigned more than once to the domain, δf_i, of f_i. It is, moreover, plain that the f_i are partial recursive and, in fact, *uniformly* so with respect to i; thus there is a recursive function ϕ such that $\phi(i)$ is an index of f_i for every i. Again, property (a) and the fact that f_i is defined on all numbers clearly follow from the construction. That f_i is finite-to-one with property (b) is a consequence of Lemma I; for it is plain from the construction that (1) once Λ_{ij} has achieved a permanent position p, all but finitely many x are mapped by f_i to a number $\geqslant p$, and (2) once Λ_{i0} is permanently in position, only finitely many x can be mapped to themselves by f_i. This finishes the proof of Lemma II.

LEMMA III. *Let $W_{k(e)}$ be infinite, and let i be any number. Then there exists a number $j > 0$ such that the final position of Λ_{ij} is a member of $W_{k(e)}$.*

Proof. Suppose this is not the case. For each $j > 0$, let k^j be the final associate of Λ_{ij}; let -1 be taken by convention as the "final associate" of Λ_{i0}. Since, as is clear from the construction, we have $k^j < k^{j+1}$ for all j, there is a unique j_0 such that $k^{j_0} < k(e) < k^{j_0+1}$. (If $k(e)$ were the final associate of some $\Lambda_{ij}, j > 0$, then, as is evident from the construction, the final *position* of Λ_{ij} would be a member of $W_{k(e)}$.) Let s_0 be a stage such that, for all $j \leqslant j_0 + 1$, Λ_{ij} is in a final position with final associate by the end of Stage s_0. Let m_0 be the largest number belonging to $\bigcup_i \delta f_i{}^{s_0}$. Let $\tilde{m}$ be an element of $W_{k(e)}$ such that $\tilde{m} > m_0$. Now, $\tilde{m}$ must eventually enter δf_i; and, in view of our choice of s_0, it is clear that it must enter in such a way that there is a sequence $\tilde{m}, z_0, z_1, \ldots, z_{r-1}$ $(r \geqslant 1)$ such that $(\tilde{m}, z_0), (z_0, z_1), \ldots, (z_{r-2}, z_{r-1}) \in f_i$ (where $z_{r-2} = \tilde{m}$ if $r = 1$) and z_0 is the final position of Λ_{i,j_0+1}. But there is a stage $s > s_0$ such that $\tilde{m} \in W_{k(e)}{}^s - W_{k(e)}{}^{s-1}$; and so it follows from the construction that the associate of Λ_{i,j_0+1} must change at some stage $s > s_0$, which is a contradiction. Lemma III follows.

It follows from Lemma III that f_i retraces the infinite set $\bigcup_{p_j} \hat{f}_i(p_j)$, where p_j is the final position of Λ_{ij} and "$\hat{f}_i(p_j)$" denotes, as in **(2)**, the set

$$\{p_j, f_i(p_j), f_i(f_i(p_j)), \ldots\}.$$

We define: $\alpha_i = \bigcup_{p_j} \hat{f}_i(p_j)$.

Lemma IV. *For all i and j, we have:*
(1) $i \neq j \Rightarrow \alpha_i \cap \alpha_j = \square$.
(2) $|\alpha_i \cap \beta| = 1$; *and*
(3) $\alpha_i - \beta \subset \alpha$.

Proof. (2) and (3) are obvious from the construction. For each i and s, define

$$\alpha_i{}^s = \bigcup_{p_j^s \in S(i)} \hat{f}_i(p_j{}^s),$$

where $S(i)$ is the set of all markers Λ_{ij} which are attached at the end of Stage s, and $p_j{}^s$ is the position, at the end of Stage s, of Λ_{ij}, where $\Lambda_{ij} \in S(i)$. We prove (1) by showing that $\alpha_i{}^s \cap \alpha_j{}^s = \square$ for all s, provided $i \neq j$. If $s = 0$, this is obvious. Assume it true for $s < \tilde{s}$, and consider Stage $\tilde{s}$; it is easily seen by examination of the construction that there are exactly three cases. (1) $\alpha_i{}^{\tilde{s}} \subset \alpha_i{}^{\tilde{s}-1}$ and $\alpha_j{}^{\tilde{s}} \subset \alpha_j{}^{\tilde{s}-1}$. Here, obviously, we have $\alpha_i{}^{\tilde{s}} \cap \alpha_j{}^{\tilde{s}} = \square$. The other two cases are (2) $\alpha_i{}^{\tilde{s}} \subset \alpha_i{}^{\tilde{s}-1}$ and $\alpha_j{}^{\tilde{s}} \not\subset \alpha_j{}^{\tilde{s}-1}$, (3) $\alpha_i{}^{\tilde{s}} \not\subset \alpha_i{}^{\tilde{s}-1}$ and $\alpha_j{}^{\tilde{s}} \subset \alpha_j{}^{\tilde{s}-1}$. It suffices, "by symmetry," to consider only (2). Two subcases must be considered.

(2a) No marker contributing to either $\alpha_i{}^{\tilde{s}-1}$ or $\alpha_j{}^{\tilde{s}-1}$ is erased in passing to $\alpha_i{}^{\tilde{s}}$ and $\alpha_j{}^{\tilde{s}}$. This implies that a new marker Λ_{ju} has been attached at Stage $\tilde{s}$, and Λ_{ju} has lower priority than any marker contributing to either $\alpha_i{}^{\tilde{s}-1}$ or $\alpha_j{}^{\tilde{s}-1}$. But therefore, by the requirements of the construction, we have: if $u > 0$, then every element of the $f_j{}^{\tilde{s}}$-chain from Λ_{ju} above $\Lambda_{j,u-1}$ is greater than the largest member of $\alpha_i{}^{\tilde{s}-1}$; and if $u = 0$ then the new position of Λ_{ju} is greater than the largest element of $\alpha_i{}^{\tilde{s}-1}$. (In those clauses of the construction in which a marker Λ_{rw}, $w > 0$, becomes attached at Stage s to a number n not previously in δf_r, the one-term sequence n is what is meant by the "$f_r{}^s$-chain from Λ_{rw} above $\Lambda_{r,w-1}$.") Thus $\alpha_i{}^{\tilde{s}} \cap \alpha_j{}^{\tilde{s}} = \square$.

(2b) Markers are erased in passing from $\alpha_i{}^{\tilde{s}-1}, \alpha_j{}^{\tilde{s}-1}$ to $\alpha_i{}^{\tilde{s}}, \alpha_j{}^{\tilde{s}}$. Then, since $\alpha_j{}^{\tilde{s}} \not\subset \alpha_j{}^{\tilde{s}-1}$, it can only be the case that some marker Λ_{ju} which contributes to $\alpha_j{}^{\tilde{s}-1}$ is both erased and restored during Stage $\tilde{s}$. Then $u > 0$ and Λ_{ju} has lower priority than any marker Λ_{ir} which contributes to $\alpha_i{}^{\tilde{s}}$. Now the remarks under (2a) apply, beginning with the words "But therefore"; and so we have $\alpha_i^{\tilde{s}} \cap \alpha_j^{\tilde{s}} = \square$. Lemma IV follows.

LEMMA V. *For each i, $\alpha - \alpha_i$ is immune and α_i is the unique infinite set retraced by f_i.*

Proof. Since $\{W_{k(t)} | t = 0, 1, 2, \ldots\}$ is the class of recursively enumerable subsets of α, the immunity of $\alpha - \alpha_i$ follows from Lemma III. But we claim that the immunity of $\alpha - \alpha_i$ implies that f_i can retrace no infinite set other than α_i. For, it is clear from the construction that no non-fixed point of f_i is in the *range* of f_i unless it is in α; hence, if f_i retraces a second infinite set, say γ, then all non-fixed points of γ under f_i lie in α. But hence, by a simple argument in **(4)**, there would be recursively enumerable subsets τ_1, τ_2 of α such that $\alpha_i \cap \alpha \subset \tau_1$, $\gamma \cap \alpha \subset \tau_2$, and $\tau_1 \cap \tau_2$ is finite. Since this contradicts the immunity of $\alpha - \alpha_i$, the rest of Lemma V follows.

Theorem 3 now obviously results from the conjunction of Lemmas I–V.

Remarks. (1) Proposition A of **(7)** is an easy corollary to Theorem 3. Just decompose $\alpha - \bigcup_i \alpha_i$ into singletons and distribute these singletons among the α_i, assigning not more than one new number to each α_i. Then, after trivially readjusting each f_i, we have **(7**, Proposition A**)**. Of course, we necessarily lose the recursive enumerability of the class $\{f_i | i = 0, 1, 2, \ldots\}$.

(2) As we remarked in **(7)**, our proof of Theorem 3 is such that each α_i is not only immune but *hyperimmune*. We are still unable to settle the question: does there exist a non-hyperimmune, co-immune retraceable set?

As a lemma to Theorem 4, we require **(6**, Lemma 2**)**. We shall prove the latter result once more because the proof in **(6)** made use of the axiom of choice; however, we can easily avoid using that axiom, by appealing instead to a special case of Lemma 1.

LEMMA 3. *Every infinite set of numbers has a supercohesive subset.*

Proof. Let α be an infinite set of natural numbers. We begin by constructing in the well-known way, a merely *cohesive* subset τ of α: letting $\{W_e\}$ be a standard enumeration of the recursively enumerable sets, we define

$$\alpha_0 = \alpha,$$

$$\alpha_{n+1} = \begin{cases} \alpha_n \cap W_n & \text{if } \alpha_n \cap W_n \text{ is infinite,} \\ \alpha_n & \text{otherwise.} \end{cases}$$

Then, if $t_0, t_1, t_2, \ldots$ is a non-repeating sequence such that $t_n \in \alpha_n$ for all n, the set $\tau = \{t_0, t_1, t_2, \ldots\}$ is cohesive. We now extract a *supercohesive* set from τ. By a trivial generalization of the observation from **(4)** used in the proof of Lemma V above, we obtain: if p is a partial recursive function regressing two *distinct* infinite sets γ_1 and γ_2, then (modulo a finite subset of γ_1) γ_1 and γ_2 are separable by disjoint recursively enumerable sets. It follows that if $\{p_i\}$ is an enumeration of all the one-place partial recursive functions, then each p_i regresses at most one infinite set γ such that γ splits τ. This enables us to define a sequence $\{\tau_n\}$ of subsets of τ as follows:

$$\tau_0 = \tau,$$
$$\tau_{n+1} = \begin{cases} \gamma_n \cap \tau_n & \text{if } p_n \text{ regresses } \gamma_n,\ \gamma_n \text{ is infinite, and } \gamma_n \text{ splits } \tau_n, \\ \tau_n & \text{otherwise.} \end{cases}$$

Let $s_0, s_1, s_2, \ldots$ be a non-repeating sequence such that $s_n \in \tau_n$ for all n; then the set $\zeta = \{s_0, s_1, s_2, \ldots\}$ is a supercohesive subset of τ and so of α.

Let α be an infinite number set; and let C be a collection of regressive sets each of which splits α. We shall say that C is a *reduced regressive decomposition* of α if (i) $\alpha \subset \bigcup_{\beta \in C} \beta$ and

(ii) $(\beta_1, \beta_2 \in C$ and $\beta_1 \neq \beta_2) \Rightarrow$ (Both $(\alpha \cap \beta_1) - (\alpha \cap \beta_2)$ and $(\alpha \cap \beta_2) - (\alpha \cap \beta_1)$ are infinite).

Let α be an infinite retraceable set. Then $\alpha \in UGB_1 \Leftrightarrow_{\mathrm{df}} \alpha$ is the unique infinite set retraced by a general recursive, basic retracing function. If α is an infinite regressive set, then $\alpha \in UGB_2 \Leftrightarrow_{\mathrm{df}} \alpha$ is the unique infinite set regressed by a general recursive, basic regressing function. Clearly, $UGB_1 \subset UGB_2$. It can be shown that the inclusion is proper. Let K be a cardinal such that $2 \leqslant K \leqslant \aleph_0$. We define six classes of sets:

$\mathrm{Ret}^*(K) = \{\alpha \mid \alpha$ has a reduced regressive decomposition C, consisting of K pairwise disjoint elements of $UGB_1\}$;

$\mathrm{Reg}^*(K) = \{\alpha \mid \alpha$ has a reduced regressive decomposition C, consisting of K pairwise disjoint elements of $UGB_2\}$;

$\mathrm{Ret}(K) = \{\alpha \mid \alpha$ has a reduced regressive decomposition C, consisting of K retraceable sets$\}$;

$\mathrm{Reg}(K) = \{\alpha \mid \alpha$ has a reduced regressive decomposition C, consisting of K regressive sets$\}$;

$\mathrm{Ret}^+(K) = \{\alpha \mid \alpha$ has a reduced regressive decomposition of cardinality $>K$, each member of which is retraceable$\}$;

$\mathrm{Reg}^+(K) = \{\alpha \mid \alpha$ has a reduced regressive decomposition of cardinality $>K\}$.

A proof of the following lemma (due in essence to C. E. M. Yates) may be found in **(5)**:

LEMMA 4 (**5**, Lemma 3). *If β is the unique infinite set regressed by a basic regressing function, then β has degree $\leqslant 0'$.*

It follows at once from Lemma 4 that every element of UGB_2—and, in particular, all the sets α_i of Theorem 3— have degree $\leqslant 0'$. This does not work the other way, however: although Yates has shown that any retraceable set of degree $\leqslant 0'$ is the unique set retraced by some basic retracing function, there need be no *general recursive*, *basic regressing* function which regresses it, and no other infinite set. We remark, finally, that $2 \leqslant K < \aleph_0$ implies that $\mathrm{Ret}(K)$ is a proper subset of $\mathrm{Reg}(K)$; this follows from Theorem 6 below.

We are now ready to state and prove Proposition D of **(7)** in a form slightly stronger than the version stated in **(7)**.

THEOREM 4. *For each cardinal K such that $2 \leqslant K \leqslant \aleph_0$, there exists a cohesive set β such that $\beta \in \mathrm{Ret}(K) - \mathrm{Reg}^+(K)$; indeed, we can require that*

$$\beta \in \mathrm{Ret}^*(K) - \mathrm{Reg}^+(K).$$

Proof. Applying Theorem 3 to, say, the set of all even numbers, let $\alpha_0, \alpha_1, \alpha_2, \ldots$ be an infinite sequence of infinite sets of even numbers such that for all i and j we have: (1) α_i is the unique infinite set retraced by a certain general recursive, basic retracing function (and hence, by Lemma 4, has degree $\leqslant 0'$); (2) $i \neq j \Rightarrow \alpha_i \cap \alpha_j = \square$; and (3) $\{n | n \text{ is even}\} - \alpha_i$ is immune. Let r be a (non-recursive) function such that $\{W_{r(n)} | n = 0, 1, 2, \ldots\}$ is the class of all infinite recursively enumerable sets of even numbers. Define:

$$W^*_{r(0)} = W_{r(0)},$$
$$W^*_{r(n+1)} = \begin{cases} W_{r(n+1)} \cap W^*_{r(n)} & \text{if } W_{r(n+1)} \cap W^*_{r(n)} \text{ is infinite,} \\ W^*_{r(n)} & \text{otherwise.} \end{cases}$$

We shall construct a cohesive set β of even numbers such that $\beta \cap \alpha_i$ is infinite for every i. This is done by a minor modification of the standard cohesive set construction. By property (3) of the sets α_i, $W^*_{r(n)} \cap \alpha_i$ must be infinite for every n and i. Define a sequence $\{b_n\}$ as follows:

$$b_0 = \mu y \,(y \in \alpha_0 \cap W^*_{r(0)}), \qquad b_{n+1} = \mu y (y > b_n \;\&\; y \in \alpha_{(n)_0} \cap W^*_{r(n)}).$$

(As usual, "$(n)_j$" denotes the power of the jth prime in the prime-power factorization of n.) Then, plainly, $\{b_0, b_1, b_2, \ldots\}$ is an infinite set having infinite intersection with each α_i; moreover, it is easy to see from the definition of the sequence $\{W^*_{r(n)}\}$ that $\{b_0, b_1, b_2, \ldots\}$ is cohesive. Let $\beta = \{b_0, b_1, b_2, \ldots\}$. Applying Lemma 3, let γ_i be a supercohesive subset of $\beta \cap \alpha_i$, for each i. Now let K be a cardinal such that $1 \leqslant K \leqslant \aleph_0$. If $K = \aleph_0$, then $\gamma = \bigcup_i \gamma_i$ will satisfy the requirements of the theorem. Indeed, all is obvious except perhaps that there is no uncountable reduced regressive decomposition of γ. But here again, we need only appeal to the special case of Lemma 1 used in the proof of Lemma 3. For if there *were* an uncountable reduced regressive decomposition C of γ, C would of necessity contain *distinct* elements λ_1, λ_2 both of which split the cohesive set γ. (Thus, in fact, *no cohesive set admits an uncountable reduced regressive decomposition.*) If $2 \leqslant K < \aleph_0$, let

$$\gamma = \bigcup_{i \leqslant K-1} \gamma_i;$$

then it is clear from the supercohesion of the γ_i that γ satisfied the requirements of the theorem relative to K.

4. On Theorems 2 and 3 of (6). We have already obtained, in §2, a substantial improvement of **(6**, Theorem 2**)**. In that version of the result, however, the splitting set was found outside the arithmetical hierarchy. We shall next prove, by using Theorem 3, Lemma 2′, and the method of proof of Theorem 1, that the splitting set can be required to be arithmetical; indeed it can be taken of degree $\leqslant 0'$. Even more, we can arrange that our cohesive set be split by all the members of an infinite disjoint family of retraceable sets belonging to UGB_1.

THEOREM 5. *There exists a cohesive set α such that:* (i) *there is an infinite family of pairwise disjoint elements of UGB_1 each of which splits α, and* (ii) *α cannot be decomposed by any finite family of regressive sets.*

Proof. Applying Theorem 3, let $\alpha_0, \alpha_1, \alpha_2, \ldots$ be an infinite sequence of infinite sets of even numbers such that for all i and j we have: (1) α_i is the unique infinite set retraced by a certain general recursive, basic retracing function; (2) $i \neq j \Rightarrow \alpha_i \cap \alpha_j = \square$; and (3) $\{n|n \text{ is even}\} - \alpha_i$ is immune. Let $\{W^*_{r(n)}\}$ be the same as in the proof of Theorem 4. Let P be the family of all partial recursive functions of one variable; and let $\mathfrak{P}$ be the family of all non-empty finite subsets of P; enumerate $\mathfrak{P}$ as a sequence $F_0, F_1, F_2, \ldots$. We shall make use of the precise specialization of Lemma 1 to the ordinary cohesive case: if $p \in P$, β is cohesive, p regresses λ_1 and λ_2, and λ_1, λ_2 are distinct infinite sets, then at most one of the sets λ_1, λ_2 can split some cohesive superset of β. Let γ be a cohesive subset of α_0 such that $\gamma - W^*_{r(n)}$ is finite for all n; it is clear from the proof of Theorem 4 that such a set γ exists. We obtain the required set α as an extension of γ. The extension is made in stages, according to the following procedure:

Stage 0. Set $\alpha^0 = \square$.

Stage s, $s > 0$. We consider $\alpha_{(s)_0}$ and $F_{(s)_1}$; let $F_{(s)_1} = \{p_1, \ldots, p_m\}$.

Case 1. Each p_i, $1 \leqslant i \leqslant m$, regresses a set which splits some cohesive extension of γ. Then, for each p_i, there is *exactly one* such set; let $\beta_1, \ldots, \beta_m$ be these uniquely determined sets. By Lemma 2′, $\bigcup_{1\leqslant i\leqslant m} \beta_i$ misses infinitely much of $W^*_{r(s)}$. (Obviously each β_i must be immune.) Let

$$n_1 = \mu y\ (y \in W^*_{r(s)} - \textstyle\bigcup_{1\leqslant i\leqslant m} \beta_i), \quad n_2 = \mu y\ (y \in (W^*_{r(s)} \cap \alpha_{(s)_0}) - \alpha^{s-1}).$$

Set $\alpha^s = \alpha^{s-1} \cup \{n_1, n_2\}$.

Case 2. Otherwise. Let

$$n = \mu y\ (y \in (W^*_{r(s)} \cap \alpha_{(s)_0}) - \alpha^{s-1}) \qquad \text{and set } \alpha^s = \alpha^{s-1} \cup \{n\}.$$

This completes Stage s, $s > 0$.

Set $\alpha = \gamma \cup \bigcup_s \alpha^s$. α is cohesive since, as is plain from the construction, $\alpha \subset \{n|\ n \text{ is even}\}$ and $\alpha - W^*_{r(n)}$ is finite for all n. Since α^s contains a member of $\alpha_{(s)_0}$ for all $s > 0$, each of $\alpha_0, \alpha_1, \alpha_2, \ldots$ has infinite intersection with α.

Finally, we claim α is not decomposed by any finite collection of regressive sets. For suppose that, to the contrary, $\{\beta_1, \ldots, \beta_r\}$ were such a collection. Since $\gamma \subset \alpha$, each of $\beta_1, \ldots, \beta_r$ therefore splits a cohesive extension of γ. Let $p_1, \ldots, p_r$ be partial recursive functions which respectively regress $\beta_1, \ldots, \beta_r$; and let $F_j = \{p_1, \ldots, p_r\}$. Then if $s > 0$ and $(s)_1 = j$, Case 1 is in force at

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Stage s and so at Stage s a number enters α which is not in $\bigcup_{1\leqslant i\leqslant r}\beta_i$, which is a contradiction. Theorem 5 follows.

In our last theorem, we extend **(6**, Theorem 3**)** to the case of indecomposability by *any finite* collection of sets belonging to $R - (F - E)$. Here, as in **(6)**, R is the class of regressive sets, E is the class of recursive sets, and F is the class of recursively enumerable sets.

THEOREM 6. *There exists a set α of natural numbers such that* (i) *α is sequentially decomposable* (i.e., there is a recursive function f such that

$$\alpha \subset \bigcup_n W_{f(n)}, \qquad j \neq k \Rightarrow W_{f(j)} \cap W_{f(k)} = \square,$$

and, for all n, $\alpha \cap W_{f(n)} \neq \square$), *and* (ii) *$\alpha$ is not decomposed by any finite collection of sets each of which belongs to $R - (F - E)$.*

Proof. In outline, the proof is similar to the proof of Theorem 3 in **(6)**; however, the exact details of the argument are rather different. We shall freely cite certain lemmas stated and used in **(6)** but not explicitly stated in the present paper. Let W_e be a simple set; and, applying **(6**, Lemma 8**)**, let f and g be one- and two-place recursive functions, respectively, such that

(i) $j \neq k \Rightarrow W_{f(j)} \cap W_{f(k)} = \square$,

(ii) $\bigcup_k W_{f(k)} = W_e$, and

(iii) $W_j \cap W_{f(k)} = \square \Rightarrow W_{g(j,k)} = (W_j - W_e) \cup$ (a finite set).

Applying **(6**, Lemma 7**)**, let β be an infinite subset of $W_{f(0)}$ whose intersection with any recursive subset of $W_{f(0)}$ is finite. We can and do assume, additionally, that β is cohesive. (Note that, by (iii) above, the sets $W_{f(n)}$ are pairwise recursively inseparable, and hence are individually non-recursive.) Let $\rho_0, \rho_1, \rho_2, \ldots$ be a listing of all of the recursive supersets of β; and define $\lambda_n = \bigcap_{j\leqslant n}\rho_j$, for each n. For each j, λ_j is a recursive superset of β and must therefore have infinite intersection with $\bar{W}_{f(0)}$. In fact, $\lambda_j \cap \bar{W}_e$ must be infinite; for, as is easily seen, there would otherwise be an effective test for membership in $W_{f(0)} \cap \lambda_j$. Since W_e is simple, it follows from condition (iii) above that, for all j and k, $\lambda_j \cap W_{f(k)}$ is infinite. We define as follows a nested sequence $\{\tau_n\}$ of infinite sets:

$$\tau_0 = \begin{cases} W_{f(1)} \cap \lambda_0 \cap W_0 & \text{if } W_{f(1)} \cap \lambda_k \cap W_0 \text{ is infinite for all } k,\\ W_{f(1)} \cap \lambda_0 & \text{otherwise;}\end{cases}$$

$$\tau_{n+1} = \begin{cases} \lambda_{n+1} \cap \tau_n \cap W_{n+1} & \text{if } \lambda_k \cap \tau_n \cap W_{n+1} \text{ is infinite for all } k,\\ \lambda_{n+1} \cap \tau_n & \text{otherwise.}\end{cases}$$

It is clear that (1) $\tau_j \subset \tau_{j+1}$ for all j, (2) each τ_j is an infinite subset of $W_{f(1)} \cap \lambda_j$, and (3) each τ_j is recursively enumerable. Moreover, it is easy to see that if $W_{f(1)} \cap \lambda_k \cap \tau_{n-1} \cap W_n$ is *finite* for some k, then there is a number k_0 such that $k \geqslant k_0 \Rightarrow \tau_k \cap W_n = \square$. Let $t_0, t_1, t_2, \ldots$ be a non-repeating sequence of numbers such that, for all n, $t_n \in \tau_n$, and set $\tau = \{t_0, t_1, t_2, \ldots\}$; then it is easy to see that τ is a cohesive set. Applying **(6**, Lemma 1**)**, let

$\xi_0, \xi_1, \xi_2, \ldots$ be a listing of the $\aleph_0$ *immune* regressive supersets of τ. (It is safe to assume that τ has immune regressive supersets; if need be, one could guarantee this by (a) requiring τ to be *super*-cohesive and, (b) decomposing $W_{f(1)}$ into the union of $\aleph_0$ disjoint immune retraceable sets as in (**7**, Proposition A). Alternatively, at the cost of splitting the proof into cases, the assumption could simply be dropped.) Let P, $\mathfrak{P}$, and the sequence $F_0, F_1, F_2, \ldots$ be as in the proof of Theorem 5. We construct one-third of the required set α by stages, as follows (the remaining two-thirds are β and τ):

Stage 3s. If $s = 0$, set $\alpha^s = \square$; if $s \neq 0$ but $(s)_0 = 0$ or 1, set $\alpha^s = \alpha^{s-1}$. Otherwise, let $n = \mu y\ (y \in \lambda_s \cap W_{f((s)_0)})$; then set $\alpha^s = \alpha^{s-1} \cup \{n\}$.

Stage $3s + 1$. Consider $\xi_{(s)_0}$. $\xi_{(s)_0}$ is immune; hence $\tau_s - \xi_{(s)_0}$ is infinite. Let $n = \mu y\ (y \in \tau_s - (\alpha^{s-1} \cup \xi_{(s)_0})$. Set $\alpha^s = \alpha^{s-1} \cup \{n\}$.

Stage $3s + 2$. Consider $F_{(s)_0} = \{p_1, \ldots, p_r\}$. If some $p_i \in F_{(s)_0}$ fails to regress a set which splits a cohesive extension of τ, set $\alpha^s = \alpha^{s-1}$. Otherwise, let $\zeta_1, \ldots, \zeta_r$ be the *unique* such sets regressed, respectively, by $p_1, \ldots, p_r$. By Lemma 2′, $\tau_s - \bigcup_{1\leqslant i\leqslant r} \zeta_i$ is infinite. Let

$$n = \mu y(y \in \tau_s - (\alpha^{s-1} \cup \textstyle\bigcup_{1\leqslant i\leqslant r} \zeta_i)).$$

Set $\alpha^s = \alpha^{s-1} \cup \{n\}$. This completes the construction.

Now put $\alpha = \beta \cup \tau \cup \bigcup_n \alpha^n$. We claim that α has the required properties. First of all, since $\beta \subset W_{f(0)}$ and $\tau \subset W_{f(1)}$, it is clear from Stage 3s that $W_{f(k)} \cap \alpha \neq \square$ holds for all k; thus α is sequentially decomposable. Suppose there is a finite collection $\{\gamma_1, \ldots, \gamma_r\} \subset R - (F - E)$ which decomposes α; we shall obtain a contradiction. First, none of the γ_i can be recursive. For if γ_i is recursive, then (modulo adjustment on a finite set, which has no essential effect on the argument) we have either $\beta \subset \gamma_i$ or $\beta \subset \bar{\gamma}_i$ (since β is cohesive). In either case, it is clear (since eventually all work is done *inside* any given λ_j, and since $\tau - \lambda_j$ is finite for all j) that γ_i cannot split α. Thus $\gamma_1, \ldots, \gamma_r$ are all immune. Now, we claim that each of $\gamma_1, \ldots, \gamma_r$ must either *split* $\alpha \cap W_{f(1)}$ or else have *finite* intersection with $\alpha \cap W_{f(1)}$. For suppose that, for some i with $1 \leqslant i \leqslant r$, $(\alpha \cap W_{f(1)}) - \gamma_i$ is finite. Allowing for a harmless adjustment on a finite set of numbers, we may as well assert that $\alpha \cap W_{f(1)} \subset \gamma_i$. Hence, $\tau \subset \gamma_i$; so $\gamma_i = \xi_t$ for some t. But now it is clear from the description of Stage $3s + 1$ that *infinitely many* elements of $\alpha \cap W_{f(1)}$ lie *outside* γ_i; this is a contradiction. With no loss of generality, assume the γ_i ordered so that each of $\gamma_1, \ldots, \gamma_l$ splits $\alpha \cap W_{f(1)}$ while each of $\gamma_{l+1}, \ldots, \gamma_r$ does not. Let q be a number greater than any member of

$$\bigcup_{l+1\leqslant i\leqslant r} \gamma_i \cap W_{f(1)} \cap \alpha.$$

Since $\alpha \cap W_{f(1)}$ was constructed in such a way as to be a *cohesive* extension of τ, there is a set $F_m = \{p_1, \ldots, p_l\}$ such that, for $1 \leqslant i \leqslant l$, γ_i is *the unique* set which is regressed by p_i and splits $\alpha \cap W_{f(1)}$. Examination of Stage $3s + 2$ now shows that *infinitely many* elements of $\alpha \cap W_{f(1)}$ are $>q$ and lie outside $\bigcup_{1\leqslant i\leqslant l} \gamma_i$: we merely consider all stages $3s + 2$ for which $F_{(s)_0} = \{p_1, \ldots, p_l\}$.

This again is a contradiction, and Theorem 6 is now fully proved.

COROLLARY (cf. **6**, Corollary 3). *There exists a set α of natural numbers such that* (1) *α is decomposed by a pair of recursively enumerable sets, and* (2) *α is not decomposed by any finite collection of retraceable sets.*

5. Questions. We conclude by listing a pair of open problems which appear to require for their solution something other than (or additional to) the methods used in this paper.

(P1) Theorem 4 as it stands does not assure us that (for $K \geqslant 3$) there is no reduced regressive decompostion of β of cardinality *less than* K. Can such a requirement be added, showing that for each K from 3 to $\aleph_0$ there is a cohesive set which has a reduced regressive decomposition *precisely* at cardinal K? For $K = \aleph_0$, the proof of Theorem 5 shows that the answer is yes. What about *finite* $K > 2$?

(P2) Is there a set α of natural numbers such that α can be split by a regressive set but cannot be split by a retraceable set?

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