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UNIFORM FINITE GENERATION OF THE ISOMETRY GROUPS OF EUCLIDEAN AND NON-EUCLIDEAN GEOMETRY

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I. Introduction. A connected Lie group H is generated by a pair of one-parameter subgroups if every element of H can be written as a finite product of elements chosen alternately from the two one-parameter subgroups. If, moreover, there exists a positive integer n such that every element of H possesses such a representation of length at most n, then H is said to be uniformly finitely generated by the pair of one-parameter subgroups. In this case, define the order of generation of H as the least such n; otherwise define it as infinity.

For the isometry group of the spherical geometry, or equivalently for the rotation group SO(3), the order of generation is always finite. In fact, if ψ denotes the angle between the axes of rotation of the two one-parameter subgroups, the order of generation is 3 for $\psi = \pi/2$ and $k + 2$ for

$$\pi/(k+1) \leqq \psi < \pi/k \qquad (k \geqq 2)$$

[**1**]. In contrast, it is shown in this paper that the order of generation of the isometry group of the Euclidean geometry is either 3 or ∞ and that the order of generation of the isometry group of the hyperbolic geometry is either 3, 4, 6 or ∞.

The isometry group of the Euclidean geometry is a subgroup of the affine group $w = \alpha z + \beta$, $\alpha \neq 0$, α, β complex. This latter group has order of generation equal to either 4 or 5 while all its other proper connected Lie subgroups (except $w = az + \beta$, $a > 0$, β complex, which is not generated by any pair of one-parameter subgroups) have order of generation always equal to their dimension. In particular, the subgroups $w = e^{(1+bi)t}z + \beta$, b a fixed, non-zero real number, t real, β complex, all have order of generation equal to 3 [**2**].

It is of interest to observe that for all the above groups the minimum order of generation is equal to their dimension. This is the best possible result since it is a simple consequence of Sard's Theorem [**5**, pp. 45–55] that the order of generation of a Lie group must be $\geqq$ its dimension. For the groups considered here, this will be proved directly.

II. Preliminaries. The isometry group of the Euclidean geometry, denoted by G, consists of all transformations $w = \alpha z + \beta$, α, β complex, $|\alpha| = 1$. Choosing the Poincaré half-plane as the model for the hyperbolic geometry,

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one finds that its isometry group, denoted by H, consists of all transformations $w = (az + b)/(cz + d)$, a, b, c, d real, $ad - bc > 0$. H is isomorphic to SL$(2, R)$, the group of all 2×2 real matrices having determinant equal to $+1$, provided that we identify the matrices A and $-A$.

The infinitesimal transformations, i.e. elements of the Lie algebra, of G are of the form $\epsilon = \gamma w + \delta$, γ, δ complex, Re$\gamma = 0$. This means that the one-parameter subgroups of G are the solutions of the differential system

$$(1) \qquad \frac{dw}{dt} = \gamma w + \delta, \qquad w(0, z) = z,$$

i.e. for each t, $-\infty < t < +\infty$, $w(t, z)$ is an element of G and the set of all solutions of (1) forms a one-parameter subgroup. It is worth noting that these one-parameter subgroups can be determined by means of functional equations instead of using Lie group theory. In fact, if $w_t = \alpha_t z + \beta_t$ is a one-parameter subgroup of the *affine* group, then the condition $w_t \circ w_s = w_{t+s}$ yields $\alpha_t \alpha_s = \alpha_{t+s}$ and $\alpha_t \beta_s + \beta_t = \beta_{t+s}$. These equations imply that $\alpha_t = e^{\gamma t}$ and $\beta_t = \delta t$ if $\gamma = 0$ and $\beta_t = \lambda(1 - e^{\gamma t})$ if $\gamma \neq 0$ which agrees with the solution of (1) (provided $\lambda = -\delta/\gamma$).

The infinitesimal transformations of H are given by $\epsilon = pw^2 + 2qw + r$, p, q, r real. Under the transformations of the Lie algebra induced by inner automorphisms of the group, the discriminant of the infinitesimal transformation is an absolute invariant. An infinitesimal transformation and the one-parameter subgroup that it generates are classified as elliptic, parabolic or hyperbolic depending upon whether its discriminant is negative, zero or positive, respectively. Thus $\epsilon = \gamma w + \delta$ is elliptic if $\gamma^2 < 0$, parabolic if $\gamma = 0$, and never hyperbolic while $\epsilon = pw^2 + 2qw + r$ is elliptic if $q^2 - pr < 0$, parabolic if $q^2 - pr = 0$, and hyperbolic if $q^2 - pr > 0$.

Given a pair of infinitesimal transformations ϵ and η, denote by T_t and S_s, respectively, the generated one-parameter subgroups. The orbit under ϵ or T_t of the point z_0 is defined as $\{T_t(z_0), -\infty < t < +\infty\}$; similarly, one defines the orbit of z_0 under η.

A pair of distinct infinitesimal transformations of G may be simultaneously transformed into one and only one of the normal forms

$$(2) \qquad \begin{array}{llll} \text{(a)} & \epsilon = 1, & \eta = \sigma, & \text{Im}\, \sigma \neq 0, \text{ both parabolic,} \\ \text{(b)} & \epsilon = iw, & \eta = 1, & \epsilon \text{ elliptic, } \eta \text{ parabolic,} \\ \text{(c)} & \epsilon = iw, & \eta = i(w - a), & a > 0, \text{ both elliptic,} \end{array}$$

by means of a suitably chosen inner automorphism of G. In case (a), the subgroup generated by T_t and S_s is not G but rather the two-dimensional subgroup of G consisting of all translations $w = z + \beta$, β complex. In both cases (b) and (c), the subgroup generated by T_t and S_s is G since it is easily established that ϵ, η, and

$$[\epsilon, \eta] = \eta \frac{d\epsilon}{dw} - \epsilon \frac{d\eta}{dw}$$

are linearly independent over the reals [**3**].

If ϵ and η are infinitesimal transformations of H, then T_t and S_s generate H if and only if ϵ and η have no common roots [**4**]. A pair of infinitesimal transformations of H with no common roots may be simultaneously transformed by means of an inner automorphism of H into one and only one of the normal forms [**4**].

(3) (a)	$\epsilon = w^2 + 1$,	$\eta = w^2 + c^2$,	$c > 1$,	both elliptic,
(b)	$\epsilon = w^2 + 1$,	$\eta = w^2 - c^2$,	$c \geqq 1$,	ϵ elliptic, η hyperbolic,
(c)	$\epsilon = w^2 + 1$,	$\eta = 1$		ϵ elliptic, η parabolic,
(d)	$\epsilon = 1$,	$\eta = w^2$,		both parabolic,
(e)	$\epsilon = 1$,	$\eta = w^2 - 1$		ϵ parabolic, η hyperbolic,
(f)	$\epsilon = w$,	$\eta = (w - 1)(w - r)$,	$r > 1$,	both hyperbolic with roots separating,
(g)	$\epsilon = w$,	$\eta = (w - 1)(w + r)$,	$r \geqq 1$,	both hyperbolic with roots interlacing.

III.

THEOREM 1. *If the whole group G is generated by two one-parameter subgroups, then the order of generation is* 3 *or* ∞. *It is* 3 *if one is elliptic and the other parabolic, and it is* ∞ *if both are elliptic.*

Proof. If both T_t and S_s are elliptic, then for every positive integer n the subset of G consisting of all finite products of T_t and S_s of length at most n is compact. But G is not compact.

If T_t is elliptic, S_s parabolic, assume that their infinitesimal transformations have the form (2)(b), i.e. $\epsilon = iw$, $\eta = 1$. Let $W(z) \in G$ and suppose that $W(\gamma) = 0$, $\gamma \neq 0$. Choose the rotation T_t so that $\operatorname{Im} T_t(\gamma) = 0$: there are, in fact, two possible choices for T_t. Next, select the translation S_s so that $S_sT_t(\gamma) = 0$. Since $|W(0)| = |S_sT_t(0)|$, it is possible to find a rotation T_u such that $T_uS_sT_t(0) = W(0)$. Since W and $T_uS_sT_t$ agree on 0, γ, and ∞, $W = T_uS_sT_t$. Since the transformation $W(z) = z - i$ cannot be represented as a product of length 2, the order of generation of G is 3; it is worth observing that, in fact, all transformations of G of the form $w = \pm z + \beta$, $\operatorname{Im} \beta \neq 0$, cannot be represented as a product $S_uT_tS_s$ so that the roles of the two one-parameter subgroups are not interchangeable.

IV.

THEOREM 2. *If the whole group H is generated by two one-parameter subgroups, then the order of generation is* 3, 4, 6 *or* ∞. *It is* ∞ *if both are elliptic,* 3 *if exactly one is elliptic, and* 4 *in all other cases except that it is* 6 *if both are hyperbolic with interlacing fixed points.*

Proof. Refer to the list of normal forms in (3).

(a) *Both elliptic.* Since H is not compact, the order of generation of H must be infinite.

(b) $\epsilon = w^2 + 1$, $\eta = w^2 - c^2$, $c \geqq 1$. The orbits under the elliptic infinitesimal transformation ϵ are just the circles of Appollonius with respect to its fixed points i and $-i$, while the orbits under the hyperbolic infinitesimal transformation η are the arcs of the family of all circles through c and $-c$ that lie in the upper and lower half planes, respectively, together with the two intervals of the real axis determined by c and $-c$.

Let $W(z) \in H$ and assume that $W(\gamma) = i$, $\gamma \neq i$ (note that $\operatorname{Im} \gamma > 0$). The orbit of γ under ϵ intersects the orbit of i under η at exactly two points. Choose T_t so that $T_t(\gamma)$ is one of these two points and then select S_s so that $S_sT_t(\gamma) = i$. Now both $W(0)$ and $S_sT_t(0)$ are on the real axis which is an orbit under ϵ so that it is possible to find T_u such that $T_uS_sT_t(0) = W(0)$. Since W and $T_uS_sT_t$ agree on 0, γ, and $\bar{\gamma}$ (complex conjugate of γ), $W = T_uS_sT_t$. It is clear that the order of generation of H is at least 3; in fact, all hyperbolic transformations in H exactly one of whose fixed points is c or $-c$ together with all parabolic transformations in H with either $+c$ or $-c$ as fixed point cannot even be represented as a product $S_uT_tS_s$. For if T_t is not the identity, the product $S_uT_tS_s$ leaves neither c nor $-c$ fixed.

(c) $\epsilon = w^2 + 1$, $\eta = 1$. The orbits under the parabolic infinitesimal transformation η are just the family of parallel lines $\operatorname{Im} z = \text{const}$. Let $W(z) \in H$ be as above. Observe that the orbit of γ under ϵ intersects the orbit of i under η at two points and then proceed as above. Note that all hyperbolic transformations in H one of whose fixed points is ∞ cannot be represented as a product $S_uT_tS_s$.

In the remaining cases the order of generation of H must be at least 4 since any hyperbolic transformation of H one of whose fixed points is a root of ϵ, the other a root of η, cannot be represented as a product of length 3.

(d) $\epsilon = 1$, $\eta = w^2$. In this case it will be convenient to work with $\mathrm{SL}(2, R)$, identifying the matrices A and $-A$. One easily finds that $T_t(z) = z + t$, $S_s(z) = z/(sz + 1)$ or in matrix form

$$T_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad S_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + st & v + t + stv \\ s & 1 + sv \end{pmatrix}, \tag{4}$$

every matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}(2, R)$ with $c \neq 0$ can be represented as a product of length 3: choose $s = c$, $t = (a - 1)/c$, $v = (d - 1)/c$. Similarly, every matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}(2, R)$ with $b \neq 0$ can be represented as a product

$$\begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} a & 0 \\ 1 & 1/a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}, \tag{5}$$

where $\left(\begin{smallmatrix} a & 0 \\ 1 & 1/a \end{smallmatrix}\right)$ is representable as a product of T_t and S_s of length 3, the order of generation of H is 4.

(e) $\epsilon = 1, \eta = w^2 - 1$. Let (x, y) be an ordered pair of distinct points on the real axis, one of which could be ∞. Define order(x, y) (with respect to -1 and 1) as the least positive integer k for which there exists a transformation $W(z) \in H$ expressible as a product of T_t and S_s of length k with $W(x) = -1$, $W(y) = 1$. Since -1 and 1 are the fixed points of $S_s(z)$, the last (reading from right to left) element of any product of T_t and S_s of length $k =$ order(x, y) representing $W(z)$ must always be a T_t. If order$(x, y) = k$, then every transformation $V(z) \in H$ such that $V(x) = -1$, $V(y) = 1$, can be expressed as a product of T_t and S_s of length at most $k + 1$ since for some s, $-\infty < s < +\infty$,

$$V(z) = S_s W(z). \tag{6}$$

Hence, to prove that the order of generation of H is 4, it suffices to prove that order$(x, y) \leqq 3$ for all (x, y).

Since $T_t(z) = z + t$ is the one-parameter translation subgroup, it is clear that order$(x, y) = 1$ if and only if $y = x + 2$. Hence order$(x, y) \leqq 2$ if and only if there is an element $S_s(z)$ such that $S_s(y) = S_s(x) + 2$. In particular, if $|x| < 1$ and if either $|y| > 1$ or $y = \infty$, then order$(x, y) \leqq 2$; there are three cases to consider:

(a′) $x < y \leqq x + 2$,

(b′) $x + 2 < y$,

(c′) $y = \infty$ or $y < -1$.

The continuous function $f(s) = S_s(y) - S_s(x)$ satisfies

$$f(0) = S_0(y) - S_0(x) = y - x.$$

For $y > 1$, $y \neq \infty$, there is a $v > 0$ such that $S_v(y) = \infty$. The function $f(s)$ is strictly increasing on $(-\infty, v)$, and

$$\lim_{s\to-\infty} f(s) = \lim_{s\to-\infty} S_s(y) - \lim_{s\to-\infty} S_s(x) = 1 - 1 = 0, \qquad \lim_{s\to v-} f(s) = +\infty.$$

In case (a′), $f(0) \leqq 2$ and hence there exists an $s \geqq 0$ such that $f(0) = 2$; in case (b′), $f(0) > 2$ and hence there exists an $s < 0$ such that $f(s) = 2$. In case (c′), first choose $S_u(z)$ such that $S_u(y) > 1$, $S_u(y) \neq \infty$. Since $-1 < x < 1$, $-1 < S_u(x) < 1$, and hence either case (a′) or case (b′) applies to the pair $(S_u(x), S_u(y))$ (there exists an $S_s(z)$ such that $S_sS_u(y) = S_sS_u(x) + 2$ or $S_{s+u}(y) = S_{s+u}(x) + 2$). A similar argument shows that if $|y| < 1$, and if either $|x| > 1$ or $x = \infty$, then order$(x, y) \leqq 2$.

If $x \neq \infty$, then there exists a translation $T_t(z)$ such that $|T_t(x)| < 1$, and $|T_t(y)| > 1$ or $T_t(y) = \infty$ and hence order$(x, y) \leqq 3$. If $x = \infty$, $y \neq \infty$,

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there exists a translation $T_t(z)$ such that $|T_t(y)| < 1$, $T_t(x) = T_t(\infty) = \infty$ and again it follows that order$(x, y) \leqq 3$.

In the last two cases define order(x, y) (with respect to 0 and ∞) as the least positive integer k for which there exists a transformation $W(z) \in H$ expressible as a product of T_t and S_s of length k with $W(x) = 0$, $W(y) = \infty$. In both cases, $\epsilon = z$, and thus the fixed points of $T_t(z)$ are 0 and ∞. The last element of any product of T_t and S_s of length k = order(x, y) representing $W(z)$ must always be an S_s. If order$(x, y) = k$, then every transformation $V(z) \in H$ such that $V(x) = 0$, $V(y) = \infty$ can be expressed as a product of T_t and S_s of length at most $k + 1$ since for some t, $-\infty < t < +\infty$,

$$V(z) = T_t W(z). \tag{7}$$

Finally, observe that order$(\infty, 0)$ must be odd since both the first and last elements of the product of length k = order$(\infty, 0)$, representing $W(z)$, must be an S_s.

(f) $\epsilon = w$, $\eta = (w - 1)(w - r)$, $r > 1$. Since $T_t(z) = e^t z$, all transformations of this one-parameter subgroup preserve the ratio x/y, i.e.

$$T_t(x)/T_t(y) = x/y$$

for all t. The fixed points of the one-parameter subgroup $S_s(z)$ are 1 and r; $\lim_{s\to+\infty} S_s(x) = 1$ for $x \neq r$ and $\lim_{s\to-\infty} S_s(x) = r$ for $x \neq 1$. Let I be the open interval $1 < x < r$ and J the open interval $\{x > r\} \cup \{x < 1\} \cup \{\infty\}$. Note that if $S_s(x) = 0$, then $x \in J$ and if $S_s(y) = \infty$, then $y \in J$.

If order$(x, y) = 1$, then clearly $x \in J$, $y \in J$. Further, for each $x \in J$, there is a unique $y \in J$, denoted by $g(x)$, such that order$(x, g(x)) = 1$; in fact, given $x \in J$, choose $s(x)$ such that $S_{s(x)}(x) = 0$; then

$$y = g(x) = S_{-s(x)}(\infty) = S_{s(x)}^{-1}(\infty).$$

Clearly g is a continuous function from J into the projective line. If $y_0 = g(\infty)$, then $r < y_0 < +\infty$ and if s_0 is the unique solution of $S_s(0) = \infty$, then $x_0 = S_{s_0}^{-1}(0)$ satisfies $0 < x_0 < 1$. The behaviour of g is as follows: on $(r, +\infty)$, g is strictly increasing, $r < g(x) < x$, $\lim_{x\to r+} g(x) = r$, and $\lim_{x\to+\infty} g(x) = y_0$; on $(-\infty, 0]$, g is strictly increasing, $\lim_{x\to-\infty} g(x) = y_0$, and $\lim_{x\to 0-} g(x) = g(0) = +\infty$; finally, on $(0, 1)$, g is again strictly increasing, $-\infty < g(x) < x$, $\lim_{x\to 0+} g(x) = -\infty$, $g(x_0) = 0$, and $\lim_{x\to 1-} g(x) = 1$. Hence the range of $x/g(x)$ on $(r, +\infty)$ is $(1, +\infty)$; on $(-\infty, 0)$, $(-\infty, 0)$; on $(0, x_0)$, $(-\infty, 0)$; on $(x_0, 1)$, $(1, +\infty)$.

It follows from the above characterization of the points of order 1 that if $x/y < 0$ or if both $x/y > 1$ and $x > 0$, then order$(x, y) \leqq 2$. Assume first that $x/y < 0$ and $x < 0$. Choose $x' < 0$ such that $x'/g(x') = x/y$ and then select $T_t(z)$ such that $T_t(x) = x'$; clearly $T_t(y) = g(x')$ and order$(x', g(x')) = 1$. If $x/y < 0$ but $x > 0$, choose $x' > 0$ such that $x'/g(x') = x/y$ and then proceed as above. If $x/y > 1$ and $x > 0$, choose $x' > 0$ such that $x'/g(x') = x/y$ and then again proceed as above. Further, order$(\infty, y) \leqq 2$ if $y > 0$: choose $T_t(z)$

such that $T_t(y) = y_0$ and then note that order$(\infty, y_0) = 1$. Since T_t preserves the ratio x/y and takes both the positive and negative real axes into themselves, it follows that if $0 < x/y < 1$ or if both $x < 0$ and $y < 0$ or if $x = 0, y \neq \infty$ or if $x = \infty, y \leqq 0$, then order$(x, y) > 2$.

Next, it will be shown that for all (x, y) except those satisfying $1 \leqq x < y \leqq r$,

$$\text{order}(x, y) \leqq 3.$$

If $x \in J$, then there exists an $S_s(z)$ such that $-\infty < S_s(x) < 0$ and

$$0 < S_s(y) < +\infty.$$

Since $S_s(x)/S_s(y) < 0$, order$(S_s(x), S_s(y)) \leqq 2$. Similarly, if $y \in J$, choose $S_s(z)$ such that $-\infty < S_s(y) < 0$ and $0 < S_s(x) < +\infty$. If $1 \leqq x \leqq r$, $1 \leqq y \leqq r$, and $x > y$, then order$(x, y) = 2$. However, if $1 \leqq x < y \leqq r$, then since $1 \leqq S_s(x) < S_s(y) \leqq r$ for all s, it follows that order$(S_s(x), S_s(y)) > 2$ for all s and hence order$(x, y) > 3$. In this case, since there does exist a $T_t(z)$ such that $T_t(y) \in J$, order$(x, y) = 4$.

Equation (7) implies that unless $1 \leqq x < y \leqq r$, all transformations $V(z) \in H$ such that $V(x) = 0$, $V(y) = \infty$, can be expressed as a product of T_t and S_s of length at most 4; in the exceptional case, (7) provides the estimate 5 which is unsatisfactory since the more careful analysis given below shows that $V(z)$ can also in this case be expressed as a product of length 4. Note that if $V(x) = 0$, $V(y) = \infty$, where $1 \leqq x < y \leqq r$, then $V(0)$ lies on the negative real axis and, conversely, all such $V(z) \in H$.

Note that all $W(z) \in H$, $W(0) = \infty$, and $W(\infty) > 0$ can be represented as a product $T_t S_s T_u$; in fact, choose $s = s_0$, T_t such that $T_t S_{s_0}(\infty) = W(\infty)$ (possible as $S_{s_0}(\infty) > 0$) and finally T_u such that $T_u(W^{-1}(0)) = x_0$ (possible since $W^{-1}(0) > 0$). Then $W(z)$ and $T_t S_s T_u(z)$ agree on $0, \infty$, and $W^{-1}(0)$ and hence are identical. Assume that $1 \leqq x < y \leqq r$. Any transformation $W(z) \in H$ such that $W(0) = \infty$, $W(x) < 0$, and $W(y) > 0$ satisfies $W(\infty) > 0$ and hence is expressible as a product $T_t S_s T_u$ of length 3. Next, observe that if for $x' < 0$, $s(x')$ is such that $S_{s(x')}(x') = 0$, then the function $h(x') = S_{s(x')}(\infty)$ is strictly decreasing and

$$\lim_{x' \to -\infty} h(x') = 0, \qquad \lim_{x' \to 0-} h(x') = -\infty.$$

Given $V(z) \in H$ such that $V(x) = 0$, $V(y) = \infty$, $1 \leqq x < y \leqq r$, choose $x' < 0$ such that $h(x') = V(0)$ and then choose $W(z)$ such that $W(x) = x'$, $W(y) = g(x')$, and $W(0) = \infty$. Then $V(z)$ and $S_{s(x')}W(z)$ agree on x, y and on 0 and therefore $V(z) = S_{s(x')}W(z)$. But $W(z)$ is expressible as a product of length 3 so that the order of generation of H is again 4.

(g) $\epsilon = w$, $\eta = (w - 1)(w + r)$, $r \geqq 1$. The fixed points of the one-parameter subgroup $S_s(z)$ are 1 and $-r$; $\lim_{s \to +\infty} S_s(x) = -r$ for $x \neq 1$ and $\lim_{s \to -\infty} S_s(x) = 1$ for $x \neq -r$. Let I be the open interval $-r < x < 1$ and J the open interval $\{x < -r\} \cup \{x > 1\} \cup \{\infty\}$. Note that if $S_s(x) = 0$, then $x \in I$ and if $S_s(y) = \infty$, then $y \in J$.

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If order$(x, y) = 1$, then $x \in I$, $y \in J$. Further, for each $x \in I$, there is a unique $y \in J$, denoted by $g(x)$, such that order$(x, g(x)) = 1$. Clearly g is a continuous function from I into J whose behaviour is as follows: on $(0, 1)$, g is strictly decreasing, $\lim_{x\to 0+} g(x) = +\infty$, and $\lim_{x\to 1-} g(x) = 1$; on $(-r, 0)$, g is again strictly decreasing, $\lim_{x\to -r+} g(x) = -r$, and $\lim_{x\to 0-} g(x) = -\infty$. Therefore the range of $x/g(x)$ on $(0, 1)$ is $(0, 1)$, and on $(-r, 0)$ it is again $(0, 1)$. It easily follows that order$(x, y) \leqq 2$ if and only if $0 < x/y < 1$ (except that order$(0, \infty) = 1$).

Next it will be established that order$(x, y) \leqq 3$ if and only if $x \in I$ or $y \in J$. Assume first that $x \in I$ and $y \in \bar{I}$ (closure of I). If $x < y$, there is an $S_s(z)$ such that $0 < S_s(x) < S_s(y)$ since $\lim_{s\to-\infty} S_s(x) = 1$ and $S_s(x) < S_s(y) \leqq 1$ for all s. If $x > y$, there is an $S_s(z)$ such that $S_s(y) < S_s(x) < 0$. In either case, $0 < S_s(x)/S_s(y) < 1$ and hence order$(S_s(x), S_s(y)) \leqq 2$. If $x \in I$ and $y \in J$, there is an $S_s(z)$ such that $0 < S_s(x) < 1 < S_s(y)$ since

$$\lim_{s\to-\infty} S_s(x) = \lim_{s\to-\infty} S_s(y) = 1;$$

again order$(S_s(x), S_s(y)) \leqq 2$. Finally, if $x \in \bar{J}$ and $y \in J$, there exists either an $S_s(z)$ such that $1 \leqq S_s(x) < S_s(y)$ or an $S_s(z)$ such that

$$S_s(y) < S_s(x) \leqq -r$$

and in both cases order$(S_s(x), S_s(y)) \leqq 2$. Conversely, if $x \in \bar{J}$ and $y \in \bar{I}$, then for all s, either $S_s(x)/S_s(y) > 1$ or $S_s(x)/S_s(y) < 0$ or $S_s(x) = \infty$ or $S_s(y) = 0$ and in all these instances, order$(S_s(x), S_s(y)) > 2$ for all s. In particular, order$(\infty, 0) > 3$; since order$(\infty, 0)$ must be odd, it follows that order$(\infty, 0) \geqq 5$. In fact, order$(\infty, 0) = 5$; this is a simple consequence of the fact proved below that order$(x, y) \leqq 4$ for all $(x, y) \neq (\infty, 0)$.

If $(x, y) \neq (\infty, 0)$, then either $x \neq \infty$ and there exists a $T_t(z)$ such that $T_t(x) \in I$ or $y \neq 0$ and there exists a $T_t(z)$ such that $T_t(y) \in J$. In either case, it follows that order$(T_t(x), T_t(y)) \leqq 3$ and thus order$(x, y) \leqq 4$.

It follows that the order of generation of H is $\leqq 6$; to complete the proof of Theorem 2 it suffices to show that the transformation $w = -r/z \in H$ cannot be expressed as a product of T_t and S_s of length $\leqq 5$. This transformation takes 0 into ∞, ∞ into 0, 1 into $-r$, and $-r$ into 1. Since order$(\infty, 0) = 5$, $w = -r/z$ cannot be expressed as a product of length < 5, and, moreover if it is expressible as a product of length 5, such a product must begin and end with an $S_s(z)$. Since 1 and $-r$ are the fixed points of $S_s(z)$, it would follow that there exists a product $T_t S_s T_u$ that takes 1 into $-r$ and $-r$ into 1. This implies that

$$S_s T_u(1) < 0 \quad \text{and} \quad S_s T_u(-r) > 0.$$

Clearly $T_u(1) \neq 1$; if $T_u(1) > 1$, then $T_u(-r) < -r$ and hence $S_s T_u(1) < 0$ implies that

$$S_s T_u(1) < S_s T_u(-r) < -r; \tag{8}$$

if $T_u(1) < 1$, then $T_u(-r) > -r$ and hence $S_sT_u(1) < 0$ implies that

$$S_sT_u(-r) < S_sT_u(1) < 0. \tag{9}$$

In either case, $S_sT_u(-r) < 0$, and a contradiction is obtained.

Remark 1. It can be shown that every $V(z) \in H$ such that $V(0) = \infty$, $V(\infty) = 0$ except $V(z) = -r/z$ can be expressed as a product of T_t and S_s of length 5.

Remark 2. To understand why $w = -r/z$ could not be expressed as a product of length 5, consider the inner automorphism of H induced by

$$V(z) = (z + r)/(z - 1).$$

Note that although $V(z) \notin H$, $V^{-1}HV = H$. Further, since $V(\infty) = 1$, $V(0) = -r$, $V(1) = \infty$, and $V(-r) = 0$, it is clear that V transforms the pair of infinitesimal transformations ϵ, η into the pair η, ϵ. Suppose that there were a product $T_tS_sT_u$ that took 1 into $-r$, $-r$ into 1; then

$$V^{-1}T_tVV^{-1}S_sVV^{-1}T_uV = V^{-1}T_tS_sT_uV \tag{10}$$

would take 0 into ∞ and ∞ into 0. Hence order$(\infty, 0) = 3$, a contradiction. What has really been shown is that order$(\infty, 0)$ (with respect to 0 and ∞) must equal order$(-r, 1)$ (with respect to 1 and $-r$).

Remark 3. Observe that in both cases (e) and (g), the order of generation of H is equal to the maximum of the order$(x, y) + 1$ while in (f) the order of generation of H is equal to the maximum of order(x, y).

It is of interest to compare the situation for the isometry group of the spherical geometry with that for the isometry groups of the hyperbolic and Euclidean geometry. For the latter two groups the order of generation is uniquely determined by the nature of the infinitesimal transformations, i.e., elliptic, parabolic or hyperbolic, except that if both are hyperbolic, then the order of generation depends upon whether or not the roots separate or interlace. For the isometry group of the spherical geometry, both infinitesimal transformations must be elliptic and the order of generation depends upon the magnitude of the cross-ratio of the roots.

V. Let $n \neq \infty$ be the order of generation of a connected Lie group by T_t and S_s. It is of interest to determine whether every element of the group can, in fact, be represented as a product of length n whose last element is a T_t; a dual question may be asked of S_s. Note that any element that can be expressed as a product of length $< n$ can be expressed both as a product of length n whose last element is a T_t and one whose last element is an S_s by inserting the identity $I = T_0 = S_0$ an appropriate number of times.

If there is an inner automorphism of the group (or even if there is any automorphism of the group) that interchanges the two one-parameter subgroups, then both questions must have the same answer. The same conclusion holds under the quite different assumption that n is even; if an element is not representable as a product of length n ending in a T_t, then its inverse is not representable as a product of length n ending in an S_s.

For SO(3), the answer to both questions is in the affirmative (both questions must have the same answer since every pair of one-parameter subgroups of SO(3) can be interchanged by some inner automorphism) [**1**]. From the proof of Theorem 1 it follows that every element of G can be expressed as a product $T_uS_sT_u$ but not all elements of G can be expressed as a product $S_uT_tS_s$ (T_t is elliptic; S_s is parabolic). From the proof of Theorem 2 the same conclusion follows for H in case T_t is elliptic and S_s parabolic or hyperbolic. If the order of generation of H is 4 or 6, then both questions must have the same answer. It is in the affirmative except if both one-parameter subgroups are hyperbolic with roots separating. If both are parabolic, this is clear from the proof of Theorem 2. The assertion in the remaining cases follows directly from the observation in Remark 3 that if one is parabolic and the other hyperbolic or if both are hyperbolic with roots interlacing, then all pairs (x, y) have order less than the order of generation of H while if both are hyperbolic with roots separating, then there exist pairs (x, y) whose order equals the order of generation of H.

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