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THE BINOMIAL EDGE IDEAL OF A PAIR OF GRAPHS

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Abstract. We introduce a class of ideals generated by a set of 2-minors of an $(m \times n)$-matrix of indeterminates indexed by a pair of graphs. This class of ideals is a natural common generalization of binomial edge ideals and ideals generated by adjacent minors. We determine the minimal prime ideals of such ideals and give a lower bound for their degree of nilpotency. In some special cases we compute their Gröbner basis and characterize unmixedness and Cohen–Macaulayness.

Introduction

The study of ideals generated by minors of a generic matrix, mostly motivated by geometric questions, has a long tradition (see the fundamental papers [13] and [8] and the survey [2]). Classically, these are ideals generated by all minors of a given size. More recently, research has focused on ideals generated by arbitrary sets of minors of a generic matrix. Perhaps the first article in this direction is that of Andrade [1] from 1981 in which regular sequences of minors are considered. In the last years, due to techniques used in algebraic statistics, it proves necessary to study certain classes of binomial and determinantal ideals. This includes ideals generated by adjacent minors, as introduced by Diaconis, Eisenbud, and Sturmfels [4] and further studied in [9], [10], and [6], as well the binomial edge ideals, first considered in [7] and [11] and recently generalized in [12]. The algebraic properties of this class of ideals are widely open, although several partial results are known (see, e.g., [5]). From an algebraic point of view, we are interested in the following questions: What are the associated primes of these ideals and,

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in particular, their minimal primes? What is their Gröbner basis? When are these ideals reduced or prime? When are they Cohen–Macaulay or Gorenstein?

In this article, we introduce binomial edge ideals J_{G_1,G_2} attached to a pair (G_1, G_2) of finite graphs. This class of ideals generalizes the versions of binomial edge ideals, considered in [7], [11], and [12], but also includes ideals generated by adjacent minors which turn out to be the ideals attached to a pair of line graphs.

In Section 1 we study the Gröbner basis of these ideals. A general description of these Gröbner bases seems to be extremely difficult. However, in Theorem 1.3 we succeed in classifying those pairs of graphs for which J_{G_1,G_2} has a quadratic Gröbner basis. Unlike the classical binomial edge ideals, the binomial edge ideals attached to a pair of graphs are never radical, unless G_1 or G_2 is complete (see Theorem 1.2).

In Theorem 2.7 we describe quite explicitly the minimal prime ideals of J_{G_1,G_2}. They are essentially determined by the so-called *admissible sets of variables* which are determined by data of the two graphs. The results obtained in Section 2 are applied in Section 3 to give a detailed description of all minimal prime ideals in the case that G_1 is a line graph of length 2 and G_2 is an arbitrary graph. The information on the minimal prime ideals is also used in Section 4, where the unmixed binomial edge ideals of pairs of graphs are characterized in Proposition 4.1. The condition for being unmixed is that one of the graphs is complete and the other graph satisfies certain numerical conditions related to its sets having the cut-point property. In the case that one graph is complete and the other one is a cycle, we fully classify in Proposition 4.2 the unmixed binomial edge ideals. Though the conditions guaranteeing that the binomial edge ideals of a pair of graphs are unmixed are already pretty restrictive, for them to be Cohen–Macaulay is even more restrictive. Under the assumption that G_1 is complete and that G_2 is closed in the sense of [7, Section 1] and $|V(G_2)| \geq |V(G_1)| \geq 3$, the unmixedness of J_{G_1,G_2} is characterized and the depth of $S/J_{G_1,G_2}$ is computed (see Theorem 4.4). It follows that, under the assumptions of the theorem, J_{G_1,G_2} is Cohen–Macaulay only if both graphs are complete.

For an ideal I with radical $\sqrt{I}$, the least number k with the property that $(\sqrt{I})^k \subset I$ is called the *index of nilpotency* of I and is denoted $\mathrm{nilpot}(I)$. It is clear that $\mathrm{nilpot}(I) = 1$ if and only if I is a radical ideal. Thus, as noticed before, $\mathrm{nilpot}(J_{G_1,G_2}) = 1$ if and only if G_1 or G_2 is complete. In the last section of this paper we give in Theorem 5.1 a lower bound for the index of nilpotency of J_{G_1,G_2} in terms of data of graphs G_1 and G_2. Applying this result to an $(m \times n)$-matrix of adjacent minors, one obtains that this lower bound is approximately $mn/16$.

It would be interesting to investigate how the construction we develop in this paper could be used to study minors of matrices that are not necessarily generic, for instance, scrolls.

§1. Binomial edge ideals of pairs of graphs and their Gröbner basis

Let G_1 be a graph on the vertex set $[m]$, and let G_2 be a graph on the vertex set $[n]$. We fix a field K; let $X = (x_{ij})$ be an $(m \times n)$-matrix of indeterminates, and denote by $K[X]$ the polynomial ring in the variables x_{ij}, $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Let $e = \{i, j\}$ for some $1 \le i < j \le m$, and let $f = \{k, l\}$ for some $1 \le k < l \le n$. To the pair (e, f) we assign the following 2-minor of X:

$$p_{e,f} = [i, j \mid k, l] = x_{ik}x_{jl} - x_{il}x_{jk}.$$

The ideal

$$J_{G_1,G_2} = \big(p_{e,f} : e \in E(G_1), f \in E(G_2)\big)$$

is called the *binomial edge ideal* of the pair (G_1, G_2).

EXAMPLES 1.1.

(a) If G_1 and G_2 are complete graphs, then $J_{G_1,G_2} = I_2(X)$, the ideal of all 2-minors of X.

(b) If G_1 is the graph consisting of exactly one edge, then J_{G_1,G_2} is the binomial edge ideal J_{G_2} introduced in [7].

(c) If G_1 is a complete graph, then J_{G_1,G_2} is the generalized binomial edge ideal attached to G_2, as considered in [12].

(d) If G_1 and G_2 are line graphs, then J_{G_1,G_2} is the ideal of adjacent 2-minors of the matrix X, studied in [4], [9], and [10].

THEOREM 1.2. *Let J_{G_1,G_2} be the binomial edge ideal of the pair of graphs (G_1, G_2). Then the following conditions are equivalent:*

(a) *J_{G_1,G_2} is a radical ideal, that is, $J_{G_1,G_2} = \sqrt{J_{G_1,G_2}}$;*

(b) *J_{G_1,G_2} has a square-free Gröbner basis with respect to the lexicographic order induced by*

$$x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > x_{22} > \cdots > x_{mn};$$

(c) *either G_1 or G_2 is a complete graph.*

Proof. The implication (c) $\Rightarrow$ (b) is shown in [12, Theorem 18], and (b) $\Rightarrow$ (a) is a general fact (see, e.g., [7, proof of Corollary 2.2]). Thus, it remains to be shown that (a) implies (c). Suppose that neither G_1 nor G_2 is a complete graph. Then there exist subsets $T_1 \subset [m]$ and $T_2 \subset [n]$ such that the restrictions $L_1 = (G_1)_{T_1}$ and $L_2 = (G_2)_{T_2}$ are line graphs, each of them with two edges, say, $E(L_1) = \{\{i,j\},\{j,k\}\}$ and $E(L_2) = \{\{r,s\},\{s,t\}\}$. Then the element

$$f_{L_1,L_2} = x_{it}x_{jr}x_{ks} - x_{ir}x_{js}x_{kt} \tag{1}$$

does not belong to the ideal

$$I = \big(p_{e,f} \colon e \in E(L_1), f \in E(L_2)\big),$$

and hence $f_{L_1,L_2} \notin J_{G_1,G_2}$ because I is obtained from J_{G_1,G_2} by substituting by 0 all the variables which do not appear among the generators of I. On the other hand, $f^2_{L_1,L_2} \in I$, and hence $f^2_{L_1,L_2} \in J_{G_1,G_2}$. This shows that J_{G_1,G_2} is not a radical ideal. ∎

In [7] the concept of a closed graph is introduced. Recall that a graph G on the vertex set $[n]$ is called *closed* if there exists a labeling of its vertices such that for all edges $\{i,j\}$ and $\{k,l\}$ of G with $i<j$ and $k<l$, one has $\{j,l\} \in E(G)$ if $i=k$, and $\{i,k\} \in E(G)$ if $j=l$. Closed graphs are exactly those for which the classical associated binomial edge ideals have a quadratic Gröbner basis (see [7] and [3]).

The next result shows that only in exceptional cases do the binomial generators of J_{G_1,G_2} form a Gröbner basis of J_{G_1,G_2}.

THEOREM 1.3. *Let J_{G_1,G_2} be the binomial edge ideal of the pair of graphs (G_1,G_2). Then the following conditions are equivalent:*

(a) *J_{G_1,G_2} has a quadratic Gröbner basis with respect to the monomial order introduced in Theorem 1.2,*
(b) *G_1 is complete and G_2 is closed, or vice versa.*

Proof. The proofs consists of two parts.

(a) $\Rightarrow$ (b): Since the quadratic Gröbner basis of J_{G_1,G_2} consists of binomials with square-free terms, it follows that J_{G_1,G_2} is a radical ideal. Therefore, by Theorem 1.2, one of the graphs must be complete. Let us assume that G_1 is complete and show that G_2 is closed. Let $\{i,j\}$ be an edge of G_1, and let $\{k,l\}$, $\{k,q\}$ be two edges of G_2 with $k<l$ and $k<q$. Then the

S-polynomial $S(f,g)$ for $f = x_{ik}x_{jl} - x_{jk}x_{il}$ and $g = x_{ik}x_{jq} - x_{iq}x_{jk}$ has the initial monomial $x_{jq}x_{jk}x_{il}$, and since J_{G_1,G_2} has quadratic Gröbner basis, we must have the edge $\{l,q\}$ in G_2.

(b) $\Rightarrow$ (a): Let $p_{e,f}, p_{e',f'} \in J_{G_1,G_2}$. We show that $S(p_{e,f}, p_{e',f'})$ reduces to zero. If the initial terms of $p_{e,f}, p_{e',f'}$ are coprime, then there is nothing to prove. Let $e = \{i,j\}$, let $f = \{k,l\}$, let $e' = \{i',j'\}$, and let $f' = \{k',l'\}$, with $i < j$, $k < l$, $i' < j'$, $k' < l'$. The initial terms of $p_{e,f}, p_{e',f'}$ have a common factor if and only if (1) $i = i'$ and $k = k'$, (2) $j = j'$ and $l = l'$, or (3) $j = i'$ and $l = k'$. It is straightforward to verify in all three cases that $S(p_{e,f}, p_{e',f'})$ reduces to zero. For example, in case (1), if $j < j'$ and $l < l'$, then completeness of G_1 gives $g = \{j,j'\} \in E(G_1)$, and closedness of G_2 implies that $h = \{l,l'\} \in E(G_2)$. Then $S(p_{e,f}, p_{e',f'})$ reduces to zero with respect to $p_{e',h}$ and $p_{g,f}$. ∎

§2. The minimal prime ideals

Let $J_{G_1,G_2} \subset K[X]$ be the binomial edge ideal of the pair of graphs (G_1, G_2). Our aim is to describe the minimal prime ideals of J_{G_1,G_2}. This will be done in several steps. Throughout this section we will assume that G_1 and G_2 are both connected. This hypothesis is not restrictive. Indeed, let $G_{1i}, 1 \le i \le r$, be the connected components of G_1, and let $G_{2j}, 1 \le j \le s$, be the connected components of G_2. Then $J_{G_1,G_2} = \sum_{i,j} J_{G_{1i},G_{2j}}$, and the generators of $J_{G_{1i},G_{2j}}$ are binomials involving disjoint sets of variables of X. If P is a minimal prime ideal of J_{G_1,G_2}, then for any i,j there exists a minimal prime P_{ij} of $J_{G_{1i},G_{2j}}$ contained in P. This implies that $J_{G_1,G_2} \subset \sum_{i,j} P_{ij} \subset P$. But since $\sum_{i,j} P_{ij}$ is a prime ideal, we must have $P = \sum_{i,j} P_{ij}$.

LEMMA 2.1. *The ideal $I_2(X)$ of all 2-minors of X is a minimal prime ideal of J_{G_1,G_2}, and if P is a minimal prime ideal of J_{G_1,G_2} containing no variable, then $P = I_2(X)$.*

Proof. Let $x = \prod_{\substack{i=1,\dots,m \\ j=1,\dots,n}} x_{ij}$. We claim that $J_{G_1,G_2} : x^\infty = I_2(X)$. This will then imply the assertions of the lemma, because if P is a minimal ideal of J_{G_1,G_2} not containing a variable, then $J_{G_1,G_2} \subset I_2(X) = J_{G_1,G_2} : x^\infty \subset P : x^\infty = P$, and hence P is equal to $I_2(X)$.

In order to prove the claim, let $\delta = [i,j|k,l]$ be an arbitrary 2-minor of X. We will show that $\delta \in J_{G_1,G_2} : x^\infty$. Assuming this, we conclude that $I_2(X) : x^\infty = J_{G_1,G_2} : x^\infty$. However, since $I_2(X)$ is a prime ideal, we then have $I_2(X) : x^\infty = I_2(X)$, and the claim is proved.

To see that $\delta \in J_{G_1,G_2} : x^\infty$, we observe that there is a path P_1 in G_1 from i to j, that is, a sequence $i = i_0, i_1, \ldots, i_{r-1}, i_r = j$ such that $\{i_s, i_{s+1}\} \in E(G_1)$ for $s = 0, \ldots, r-1$. The number r is called the *length* of the path. Similarly, there exists a path $P_2 \colon k = k_0, k_1, \ldots, k_{t-1}, k_t = l$ in G_2 from k to l. We will show by induction on $r + t$ that $\delta \in J_{G_1,G_2} : x^\infty$. Notice that $r + t \geq 2$. If $r + t = 2$, then $\delta \in J_{G_1,G_2}$, and the assertion is trivial. Suppose now that $r + t > 2$. We may assume that $r > 1$. By applying the induction hypothesis, we have that $\delta_1 = [i, i_{r-1}|k, l]$ and $\delta_2 = [i_{r-1}, j|k, l]$ belong to $J_{G_1,G_2} : x^\infty$. Since $x_{i_{r-1}k}\delta = x_{ik}\delta_2 + x_{jk}\delta_1$, it follows that $\delta \in J_{G_1,G_2} : x^\infty$, as desired. ∎

Corollary 2.2. *J_{G_1,G_2} is a prime ideal if and only if G_1 and G_2 are complete graphs.*

Next we will study minimal prime ideals of J_{G_1,G_2} which contain variables. In this context, the following definition turns out to be useful.

Definition 2.3. A subset $W \subset [m] \times [n]$ is called *admissible* with respect to (G_1, G_2) if it satisfies the following property: whenever $(i, j) \in e \times f \cap W$ for some $e \in E(G_1)$ and some $f \in E(G_2)$, then $\{i\} \times f \subset W$ or $e \times \{j\} \subset W$.

Obviously, the empty set and the set $[m] \times [n]$ are admissible.

If $e = \{i, j\} \in E(G_1)$ and $f = \{k, l\} \in E(G_2)$, then the sets $\{i\} \times f$, $\{j\} \times f$, $e \times \{k\}$, and $e \times \{l\}$ are called the *edges* of $e \times f$. An admissible set W with respect to (G_1, G_2) is characterized by the property that if $W \cap (e \times f) \neq \emptyset$, then one of the edges of $e \times f$ is contained in W.

The significance of admissible sets for the study of the minimal prime ideals of J_{G_1,G_2} becomes apparent by the next result.

Lemma 2.4. *Let P be a prime ideal containing J_{G_1,G_2}, and let $W = \{(i, j) \colon x_{ij} \in P\}$. Then W is an admissible set.*

Proof. Let $(i, j) \in W$. Then $x_{ij} \in P$. Assume that $(i, j) \in e \times f$, with $e = \{i, k\}$ and $f = \{j, l\}$. Then $x_{ij}x_{kl} - x_{il}x_{kj} \in J_{G_1,G_2} \subset P$. This implies that $x_{il}x_{kj} \in P$. Since P is prime, we have either $x_{il} \in P$ or $x_{kj} \in P$. If $x_{il} \in P$, then $\{i\} \times f \subset W$. Otherwise, we have $x_{kj} \in P$, and then $e \times \{j\} \subset W$. ∎

We call a subset $E \subset E(G_1) \times E(G_2)$ *connected* if for all $e \times f$ and $e' \times f'$ in E there exist $e_i \times f_i \in E$, $i = 1, \ldots, r$ such that $e \times f = e_1 \times f_1$, $e' \times f' = e_r \times f_r$, and $(e_i \times f_i) \cap (e_{i+1} \times f_{i+1}) \neq \emptyset$ for $i = 1, \ldots, r-1$.

An arbitrary subset $E \subset E(G_1) \times E(G_2)$ can be uniquely written as a disjoint union of connected subsets of $E(G_1) \times E(G_2)$, called the *connected components* of E.

LEMMA 2.5. *Let $W \subset [m] \times [n]$ be an admissible set with respect to (G_1, G_2). Then the connected components of*

$$W^c = \{e \times f : e \in E(G_1), f \in E(G_2), W \cap (e \times f) = \emptyset\}$$

are of the form $E_1 \times E_2$, where $E_1 \subset E(G_1)$ and $E_2 \subset E(G_2)$.

Proof. Let $e \times f$ and $e' \times f'$ belong to the same connected component C of W^c. Then there exist $e_i \times f_i \in C$, $i = 1, \ldots, r$, such that $e \times f = e_1 \times f_1$, $e' \times f' = e_r \times f_r$, and $(e_i \times f_i) \cap (e_{i+1} \times f_{i+1}) \neq \emptyset$ for $i = 1, \ldots, r-1$.

We have to show that $e \times f' \in C$ and that $e' \times f \in C$. We show this by induction on r. The assertion is trivial if $r = 1$. Now let $r > 1$, and assume that the assertion is already shown for $r-1$. Then, since $e_2 \times f_2$ is connected in C to $e_r \times f_r$ by a chain of length $r-1$, the induction hypothesis implies that $e_2 \times f_r$ belongs to C. Similarly, since $e_{r-1} \times f_{r-1}$ is connected in C to $e_1 \times f_1$ by a chain of length $r-1$, we have $e_1 \times f_{r-1}$ in C. Suppose that $e_1 \times f_r \notin C$. Then $e_1 \neq e_2$ and $f_{r-1} \neq f_r$, and moreover, $(e_1 \times f_r) \cap W \neq \emptyset$, say, $(i, j) \in (e_1 \times f_r) \cap W$. Since W is admissible, it follows that either $\{i\} \times f_r \in W$ or $e_1 \times \{j\} \in W$. This implies that $(e_2 \times f_r) \cap W \neq \emptyset$ or $e_1 \times f_{r-1} \neq \emptyset$. It follows that $e_2 \times f_r \notin C$ or $e_1 \times f_{r-1} \notin C$, a contradiction. Hence, we conclude that $e \times f' = e_1 \times f_r \in C$. Similarly, one can show that $e' \times f \in C$. ∎

Let W be an admissible subset of $G_1 \times G_2$, and let $C_1, \ldots, C_r$ be the connected components of W^c in the graph $G_1 \times G_2$. The set of edges of $G_1 \times G_2$ is defined to be the set $\{\{\{i,j\},\{k,l\}\} : \{i,j\} \in E(G_1), \{k,l\} \in E(G_2)\}$. By Lemma 2.5, there exist subgraphs $G_{1i} \subset G_1$ and $G_{2i} \subset G_2$ such that $C_i = E(G_{1i}) \times E(G_{2i})$. Since all C_i are connected, it follows that the graphs G_{1i} and G_{2i} are connected and that

$$W^c = \bigsqcup_i E(G_{1i}) \times E(G_{2i}),$$

where $\bigsqcup$ denotes the disjoint union.

For a graph G, we define $\widehat{G}$ to be the complete graph on the vertex set $V(G)$. By using this notation, we define

$$\widehat{W}^c = \bigsqcup_i E(\widehat{G}_{1i}) \times E(\widehat{G}_{2i}).$$

Obviously, the ideal

$$P_W = \left(\{x_{ij} : (i,j) \in W\}, Q_W\right) \quad \text{with } Q_W = (p_{e,f} : e \times f \in \widehat{W}^c)$$

is a prime ideal.

PROPOSITION 2.6. *Let V and W be two admissible sets with respect to (G_1, G_2). Then the following conditions are equivalent:*

(a) $P_V \subsetneq P_W$,

(b) $V \subsetneq W$*, and for all* $e \times f \subset \widehat{V}^c \setminus \widehat{W}^c$*, an edge of* $e \times f$ *belongs to* W.

Proof. We divide the proof into two parts.

(a) $\Rightarrow$ (b): Let $(i, j) \in V$. Then $x_{ij} \in P_V \subset P_W$. This implies that $(i, j) \in W$. Therefore, $V \subset W$. The inclusion must be proper; otherwise, $P_V = P_W$. Assume that $e \times f \subset \widehat{V}^c \setminus \widehat{W}^c$. Then $p_{e,f} \in Q_V \setminus Q_W$. This implies that $p_{e,f} \in P_W \setminus Q_W$. Therefore, some corner of $e \times f$ belongs to W. Since P_W is a prime ideal, an edge of $e \times f$ belongs to W.

(b) $\Rightarrow$ (a): The inclusion $V \subsetneq W$ implies that $\{x_{ij} : (i, j) \in V\} \subsetneq \{x_{ij} : (i, j) \in W\}$. If there exist $p_{e,f} \in Q_V \setminus Q_W$, then $e \times f \subset \widehat{V}^c \setminus \widehat{W}^c$. By our assumption, this implies that an edge of $e \times f$ belongs to W. Therefore, $p_{e,f} \in (x_{ij}, \{i, j\} \in W)$. This shows that $P_V \subsetneq P_W$. ∎

THEOREM 2.7. *We have the following.*

(a) *Let P be a minimal prime ideal of the binomial edge ideal J_{G_1,G_2} of the pair (G_1, G_2). Then there exists an admissible set $W \subset G_1 \times G_2$ such that $P = P_W$.*

(b) *Let $W \subset G_1 \times G_2$ be an admissible set. Then P_W is a minimal prime ideal of J_{G_1,G_2} if and only if for any admissible set $V \subset G_1 \times G_2$ properly contained in W there exists $e \times f \in \widehat{V}^c \setminus \widehat{W}^c$ such that no edge of $e \times f$ belongs to W.*

Proof. The proof is divided as follows.

(a) Let $W = \{(i, j) \colon x_{ij} \in P\}$. Then $(\{x_{ij} : (i, j) \in W\}, J_{G_1,G_2}) \subset P$, and $(\{x_{ij} : (i, j) \in W\}, J_{G_1,G_2}) = (\{x_{ij} : (i, j) \in W\}, Q)$, where Q is generated by all minors $p_{e,f}$ such that W does not contain an edge of $e \times f$. Hence, since W is admissible, as is shown in Lemma 2.4, it follows that $Q = (\{p_{e,f} \colon e \times f \in \widehat{W}^c\})$. Now we apply Lemma 2.5 and conclude that $Q = \sum_{i=1}^r J_{G_{1i},G_{2i}}$, where $C_1, \ldots, C_r$ are the connected components $\widehat{W}^c$ and $C_i = E(G_{1i}) \times E(G_{2i})$, as described in Lemma 2.5 and the comments following it.

Thus, our discussion so far shows that P is a minimal prime ideal of

$$Q = \Big(\{x_{ij} : (i, j) \in W\}, \sum_{i=1}^r J_{G_{1i},G_{2i}}\Big).$$

Since the summands $J_{G_{1i},G_{2i}}$ in Q are ideals in pairwise different sets of variables, it follows that $P = (\{x_{ij} : (i, j) \in W\}, \sum_{i=1}^r P_i)$, where each P_i

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is a minimal prime ideal of $J_{G_{1i},G_{2i}}$. None of the P_i contains a variable. It follows therefore from Lemma 2.1 that $P_i = I_2((x_{kl})_{\substack{k\in V(G_{1i})\\ l\in V(G_{2i})}})$ for $i = 1,\ldots,r$, as desired.

(b) This follows from Proposition 2.6. ∎

Among the minimal prime ideals of J_{G_1,G_2} are those which are determined only by the data of G_1 (resp., data of G_2). To explain this, let G be a finite simple graph on the vertex set $[n]$. A subset $S \subset G$ is said to have the *cut-point property* if each $i \in S$ is a cut point of the graph $G_{[n]\setminus S}$. In other words, S has the cut-point property if, for all $i \in S$, the number of connected components of $G_{([n]\setminus S)\cup\{i\}}$ is smaller than that of $G_{[n]\setminus S}$.

PROPOSITION 2.8. *Let $S_1 \subset V(G_1) = [m]$ and $S_2 \subset V(G_2) = [n]$ be subsets with the cut-point property. Then $W_1 = S_1 \times [n]$ and $W_2 = [m] \times S_2$ are admissible sets, and P_{W_1} and P_{W_2} are minimal prime ideals of J_{G_1,G_2}.*

Proof. By symmetry, it is enough to show that W_1 is admissible and that P_{W_1} is a minimal prime ideal. The set W_1 being admissible is obvious. Now let $V \subset W_1$ be an admissible set which is a proper subset of W_1. Then $V = T \times [n]$, where $T \subset S_1$ is a proper subset of S. Since S has the cut-point property, it follows that $(G_1)_{[n]\setminus T}$ has fewer connected components than does $(G_1)_{[n]\setminus S}$. Let G be a connected component of $(G_1)_{[n]\setminus T}$ which is not a connected component of $(G_1)_{[n]\setminus S}$. Then there exist two vertices $i,j \in V(G)$ which are not connected in $(G_1)_{[n]\setminus S}$. Therefore, for any $f \in E(G_2)$, the set $\{i,j\} \times f$ is contained in $\widehat{V}^c \setminus \widehat{W}^c$ and does not have any edge in W. Thus, it follows from Theorem 2.7(b) that P_{W_1} is a minimal prime ideal of J_{G_1,G_2}. ∎

§3. The case $3 \times n$

In this section we aim to describe explicitly the minimal prime ideals of J_{G_1,G_2} in the case that $|V(G_1)| = 3$.

Let G_1 be a connected graph on vertex set $[3]$, and let G_2 be a connected graph on vertex set $[n]$. The graph G_1 is either a path graph or a complete graph. In the case of a complete graph, the minimal prime ideals are known by [12]. Here we want to analyze the case when G_1 is a line graph with edges $\{1,2\}$ and $\{2,3\}$.

Let T be any subset of $[n]$, and let $C_1, \dots, C_r$ be the connected components of $(G_2)_{[n]\setminus T}$. Furthermore, let B be a subset of $[r]$. We set

$$W_{T,B} = \big([3] \times T\big) \cup \bigcup_{j \in B} \big(\{2\} \times V(C_j)\big). \tag{2}$$

Note that $W_{T,B}$ is an admissible set with respect to (G_1, G_2). We are going to prove that any admissible set W for which P_W is a minimal prime ideal of J_{G_1,G_2}, is of the form $W_{T,B}$, where T and B satisfy some extra conditions.

We first show the following.

Lemma 3.1. *Let P_W be a minimal prime ideal of J_{G_1,G_2}. Suppose that there exists some $(i,s) \in W$ with $i \in \{1,3\}$ and $s \in [n]$. Then $[3] \times s \subset W$.*

Proof. Let

$$W' = \big\{(2,r) \colon (2,r) \in W\big\} \cup \bigcup_{[3]\times\{r\}\subset W} [3] \times \{r\}.$$

We first show that W' is an admissible set with respect to (G_1, G_2). Let $(i,r) \in e \times f \cap W'$ for some $e \in E(G_1)$ and for some $f \in E(G_2)$. If $[3] \times \{r\} \subset W$, then $[3] \times \{r\} \subset W'$; in particular, $e \times \{r\} \subset W'$. Otherwise, we may assume that $i = 2$ and that $[3] \times \{r\} \not\subset W$. Then $\{2\} \times f \in W$ because W is admissible, and hence $\{2\} \times f \in W'$. Therefore, W' is admissible.

Assume that $W' \neq W$. We claim that in this case $P_{W'}$ is properly contained in P_W, contradicting the assumption that P_W is a minimal prime ideal. Indeed, W' is a proper subset of W. Let $e \times f \in \widehat{W'^c} \setminus \widehat{W^c}$. We may assume that $e = \{1,2\}$. Then $\{1\} \times f \subset W$ because $\{2\} \times f \not\subset W$ and W is admissible. ∎

In the following, we will have to refer to the following operations on graphs. Let G be a graph and let H be a subgraph of G. Then $G \setminus \{i\}$ denotes the subgraph of G which is obtained by removing the vertex i along with all the edges incident to i, and $H \cup \{i\}$ denotes the subgraph of G which is obtained by adding to H the vertex i and all the edges of G which connect i with H.

Lemma 3.2. *Let P_W be a minimal prime ideal of J_{G_1,G_2}, and let $T = \{a \in [n] : [3] \times \{a\} \in W\}$. Then T has the cut-point property.*

Proof. Assume that T does not have the cut-point property. Then there exists an element $a \in T$ such that $(G_2)_{[n]\setminus T}$ has the same number of connected components as $(G_2)_{([n]/T)\cup\{a\}}$. This implies that there exists a unique connected component D of $(G_2)_{[n]/T\cup\{a\}}$ which contains a and such that $C = D \setminus \{a\}$ is connected.

We set $W' = W \setminus ([3] \times \{a\})$ if $W \cap ([3] \times V(C)) = \emptyset$; otherwise, we set $W' = W \setminus \{(1,a),(3,a)\}$. By using Lemma 3.1, it follows that W' is of the form $W_{T,B}$ as described in (2). Therefore, W' is admissible.

We claim that $P_{W'} \subsetneq P_W$. By using Proposition 2.6, it is enough to show that for all $e \times f \subset \widehat{W'^c} \setminus \widehat{W^c}$, an edge of $e \times f$ is contained in W. In the case where $W' = W \setminus ([3] \times \{a\})$, any $e \times f \subset \widehat{W'^c} \setminus \widehat{W^c}$ has an edge in $[3] \times \{a\}$. In the case where $W' = W \setminus \{(1,a),(3,a)\}$, we have $\widehat{W'^c} = \widehat{W^c}$. Therefore, our claim holds and we obtain a contradiction to the minimality of P_W. □

Now we are ready to describe the minimal prime ideals of J_{G_1,G_2}.

THEOREM 3.3. *Let W be an admissible set with respect to (G_1, G_2). Then the following conditions are equivalent:*

(a) *P_W is a minimal prime ideal of J_{G_1,G_2},*

(b) *$W = W_{T,B}$, where T and B satisfy the following conditions:*
  - (i) *T has the cut-point property with respect to G_2,*
  - (ii) *let $C_1, \ldots, C_r$ be the connected components of $(G_2)_{[n]\setminus T}$; then*
    - (α) *$|V(C_j)| \geq 2$ for $j \in B$,*
    - (β) *for all $k, l \in B$ with $k \neq l$, $(C_k \cup C_l) \cup \{a\}$ is disconnected for all $a \in T$.*

Proof. The proof is divided into two parts.

(a) ⇒ (b): We know from Lemmas 3.1 and 3.2 that $W = W_{T,B}$, where T has the cut-point property with respect to G_2. Suppose that $|V(C_j)| = 1$ for some $j \in B$; then $C_j = \{a\}$ for some $a \in V(G_2)$, and $W' = W \setminus \{(2,a)\}$ is admissible with $P_{W'} \subsetneq P_W$, a contradiction. This proves condition (α). Then suppose that there exists $a \in T$ such that $(C_k \cup C_l) \cup \{a\}$ is connected in G_2 for some $k, l \in B$ with $k \neq l$. Let $W' = W \setminus \{(1,a),(3,a)\}$. Then W' is admissible and $P_{W'} \subsetneq P_W$, a contradiction. This proves (β).

(b) ⇒ (a): Assume that $P_{W_{T,B}}$ is not a minimal prime ideal of J_{G_1,G_2}. Then there exists a minimal prime ideal $Q \subsetneq P_{W_{T,B}}$ of J_{G_1,G_2}. By the implication (a) ⇒ (b), which is already shown, it follows that $Q = P_{W_{T',B'}}$, with $T' \subset T$ and $B' \subset B$. Suppose that $T' \subsetneq T$. Since T has the cut-point property, there exist two connected components C_k, C_l of $(G_2)_{[n]\setminus T}$ and $a \in T \setminus T'$ such that $(C_k \cup C_l) \cup \{a\}$ is connected. Let $i \in V(C_k)$, let $j \in V(C_l)$, and let $e \in E(G_1)$. Then $e \times \{i,j\}$ is contained in $\widehat{W^c_{T',B'}} \setminus \widehat{W^c_{T,B}}$. It is clear

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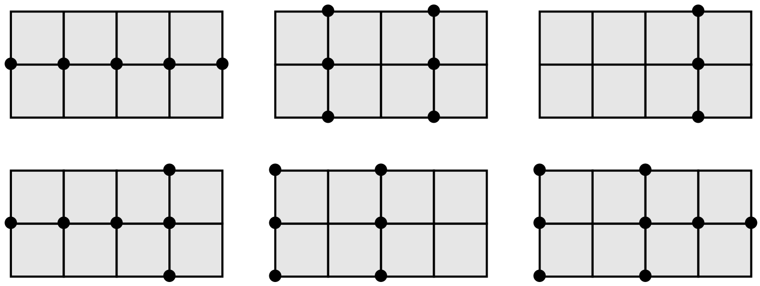

Figure 1: Admissible sets.

that the edges $e \times \{i\}$, $e \times \{j\}$, and $\{1\} \times \{i,j\}$, if $e = \{1,2\}$, respectively, and $\{3\} \times \{i,j\}$, if $e = \{2,3\}$, are not contained in $W_{T,B}$. But also the edge $\{2\} \times \{i,j\}$ is not contained in $W_{T,B}$ because of condition (β). Therefore, it follows from Proposition 2.6 that $P_{W_{T',B'}} \not\subseteq P_{W_{T,B}}$, a contradiction. Hence, we have $T' = T$. Therefore, we must have $B' \subsetneq B$. Then there exists $k \in B \setminus B'$ such that $([3] \times V(C_k)) \cap W_{T,B'} = \emptyset$. By condition (α), there exist $i, j \in V(C_k)$ with $i \neq j$. Therefore, $\{1,3\} \times \{i,j\}$ is contained in $\widehat{W}^c_{T,B'} \setminus \widehat{W}^c_{T,B}$ and has no edge in $W_{T,B}$. It again gives a contradiction to our assumption that $P_{W_{T,B'}} \subsetneq P_{W_{T,B}}$. ∎

In [9, Theorem 3.1], Hoşten and Shapiro describe the minimal prime ideals of the ideal of adjacent 2-minors of a $(3 \times n)$-matrix. In our language, these are the minimal prime ideals of J_{G_1,G_2} where G_1 and G_2 are line graphs with $|V(G_1)| = 3$ and $|V(G_2)| = n$. By using the fact that in this particular case the subsets $T = \{a_1, \ldots, a_r\}$ of $V(G_2) = [n]$ with the cut-point property are of the form $1 < a_1$, $a_r < n$, and $a_i < a_{i+1} - 1$ for $i = 1, \ldots, r-1$, we obtained the result of Hoşten and Shapiro as a special case of Theorem 3.3.

In Figure 1, we display the admissible sets, marked by fat dots, attached with the minimal prime ideals of J_{G_1,G_2}, where G_1 is a line graph of length 2 and G_2 is a graph on vertex set [5] with edge set $\{\{1,2\},\{2,3\},\{3,4\},\{1,4\}, \{4,5\}\}$.

§4. Unmixed binomial ideals of pairs of graphs

In this section we classify all pairs of graphs (G_1, G_2) such that J_{G_1,G_2} is unmixed, and those for which J_{G_1,G_2} is Cohen–Macaulay, under the additional assumption that the graphs are closed.

For Cohen–Macaulayness, we may assume as well that G_1 and G_2 are connected since, with the same notation as at the beginning of Section 2,

$$S/J_{G_1,G_2} \cong \bigoplus_{ij} S_{ij}/J_{G_{1i},G_{2j}},$$

where, for any i, j, the ring S_{ij} is the polynomial ring over K in the variables $X_{ij} = \{x_{k\ell} : k \in V(G_{1i}), \ell \in V(G_{2j})\}$. Then, J_{G_1,G_2} is Cohen–Macaulay if and only if $J_{G_{1i},G_{2j}}$ is Cohen–Macaulay for any i, j.

PROPOSITION 4.1. *Let $n \geq m \geq 3$ be integers, and let G_1 and G_2 be connected simple graphs with $V(G_1) = [m]$ and $V(G_2) = [n]$. Then the binomial edge ideal J_{G_1,G_2} is unmixed if and only if G_1 is complete and for all subsets $T \subset [n]$ with the cut-point property for G_2 one has*

$$(3) \qquad \big(c(T) - 1\big)(m - 1) = |T|.$$

Proof. Assume that J_{G_1,G_2} is unmixed, and let us suppose that G_1 is not complete. Since $I_2(X)$ is one of the minimal primes of J_{G_1,G_2} with height $(m-1)(n-1)$, all the other minimal prime ideals of J_{G_1,G_2} must have the same height. By Proposition 2.8, any prime ideal P_W, where $W = S \times [n]$ and $\emptyset \neq S \subset [m]$ has the cut-point property for G_1, is a minimal prime of J_{G_1,G_2}. Let $G'_1, \ldots, G'_{c(S)}$ be the connected components of $(G_1)_{([m]\setminus S)}$, and let $g_i = |V(G'_i)|$ for $i = 1, \ldots, c(s)$. Then $\sum_{i=1}^{c(S)} g_i = m - |S|$ and

$$\text{height}\, P_W = n|S| + \sum_{i=1}^{c(S)} (g_i - 1)(n-1) = n|S| + \big(m - |S| - c(S)\big)(n-1).$$

Hence, since J_{G_1,G_2} is unmixed, we get $(c(S) - 1)(n - 1) = |S|$. Moreover, we have $|S| \geq n - 1 \geq m - 1$. But it is obvious that no $(m-1)$-subset of $[m]$ has the cut-point property for G_1; therefore, G_1 must be complete. By using arguments as in the first part of the proof for the graph G_2, one gets condition (3).

For the converse, we use [12, Theorem 23], which says that if G_1 is complete, then the minimal prime ideals of J_{G_1,G_2} are exactly the prime ideals P_W with $W = [m] \times T$, where $T \subset [n]$ is a set with the cut-point property for G_2. The numerical condition (3) shows that these prime ideals have all the same height; hence, J is unmixed. ∎

Here Proposition 4.1 and Theorem 1.2 show, in particular, that an unmixed ideal associated with a pair of graphs is radical. It is very easy to see that the converse is not true. For instance, one may take G_1 the complete graph on [3] and G_2 the line graph with the edges $\{1,2\},\{2,3\}$. The ideal J_{G_1,G_2} is radical, by Theorem 1.2, and it is not unmixed, since its minimal prime ideals have different heights.

Proposition 4.1 shows also that J_{G_1,G_2} is not unmixed for any connected graph G_2 which has a nonempty set T with the cut-point property such that $m-1$ does not divide $|T|$. In particular, if G_2 is a tree, the ideal J_{G_1,G_2} is not unmixed, since we may find subsets $T \subset [n]$ with the cut-point property of cardinality 1. The next proposition addresses the unmixedness for the case when G_2 is a cycle.

PROPOSITION 4.2. *Let $n \geq m \geq 3$, let G_1 be the complete graph on $[m]$, and let G_2 be the cycle on the set $[n]$. Then J_{G_1,G_2} is unmixed if and only if $m = n = 3$, or $n = 4, m = 3$, or $n = 5, m = 3$.*

Proof. By Proposition 4.1, J_{G_1,G_2} is unmixed if and only if, for every subset $T \subset [n]$ which has the cut-point property for G_2, we have

$$\big(c(T)-1\big)(m-1) = |T|. \tag{4}$$

If $n \geq 6$, then there exist subsets T of $[n]$ with the cut-point property such that $c(T) = |T| = 3$. Hence, we get $2(m-1) = 3$, which is impossible. Therefore, for unmixedness we must restrict to $n = 3$, 4, or 5. If $m = n = 3$, then the claims are obvious since J_{G_1,G_2} is the ideal of all 2-minors of the matrix X.

Let $n = 4$, and assume that G_2 has the edges $\{1,2\},\{2,3\},\{3,4\}$, and $\{4,1\}$. Then the sets with the cut-point property for G_2 are $\emptyset, \{1,3\}$, and $\{2,4\}$. By using (4) for a set T with two elements, we get $m-1 = 2$; hence, $m = 3$. In this case all the minimal prime ideals of J_{G_1,G_2} have the same height equal to 6.

Let $n = 5$. In this case we see again that the nonempty subsets of [5] with the cut-point property for G_2 are of cardinality 2, and, as in the case $n = 4$, we obtain $m = 3$. ∎

REMARK 4.3. By using the computer, one easily sees that, in the hypotheses of the above proposition, J_{G_1,G_2} is Cohen–Macaulay if and only if $m = n = 3$.

Closed graphs form an interesting class of graphs G_2 for which one may discuss the unmixedness property. We recall that the collection of cliques of a graph G forms a simplicial complex, called the *clique complex* of G. We denote it $\Delta(G)$. In [5, Theorem 2.2] it is shown that a graph G on the vertex set $[n]$ is closed if and only if there exists a labeling of G such that all the facets of $\Delta(G)$ are intervals $[a,b] \subset [n]$. Moreover, if one labels the facets $F_1,\dots,F_r$ of $\Delta(G)$ such that $\min(F_1) < \min(F_2) < \cdots < \min(F_r)$, then $F_1,\dots,F_r$ is a leaf order of $\Delta(G)$.

THEOREM 4.4. *Let $n \ge m \ge 3$ be integers, let G_1 be the complete graph on $[m]$, and let G_2 be a connected closed graph on $[n]$. The following conditions are equivalent:*

(i) *J_{G_1,G_2} is unmixed,*
(ii) *there exists a leaf order $F_1,\dots,F_r$ of the facets of $\Delta(G_2)$ such that, for $1 \le i \le r$, $F_i = [a_i,b_i]$, where a_i, b_i are positive integers with $a_i < a_{i+1} < b_i < b_{i+1}$ and $b_i - a_{i+1} = m-2$ for $1 \le i \le r-1$.*

Moreover, in the above conditions,

$$\operatorname{depth}(S/J_{G_1,G_2}) = n - (r-2)m + 2r - 3,$$

where r is the number of the facets of the clique complex $\Delta(G_2)$. Consequently, $S/J_{G_1,G_2}$ is Cohen–Macaulay if and only if G_2 is a complete graph.

Proof. By [5, Theorem 2.2], the clique complex $\Delta(G_2)$ has the facets $F_1,\dots,F_r$, where each facet is an interval—that is, $F_i = [a_i,b_i]$, and $1 = a_1 < a_2 < \cdots < a_r \le b_r = n$. Since G_2 is connected, it follows that $a_{i+1} \le b_i$ for all i.

For (i) $\Rightarrow$ (ii), we proceed by induction on r. Let $T = [a_r, b_{r-1}]$. Then T has the cut-point property, and $c(T) = 2$; thus, by (4), we get

$$b_{r-1} - a_r + 1 = m - 1.$$

Let G_2' be the graph whose clique complex $\Delta(G_2')$ has the facets $F_1,\dots,F_{r-1}$, and let $P_{W'}$ be a minimal prime of $J_{G_1,G_2'}$, where $W' = [m] \times T'$, with $T' \subset V(G_2')$ a set with the cut-point property for G_2'. Then $b_{r-1} \notin T'$; thus, $c_{G_2'}(T') = c_{G_2}(T')$. It follows that T' has the cut-point property for G_2 as well. Therefore, T' satisfies condition (4), so we may apply induction.

For (ii) $\Rightarrow$ (i) and for the formula of the depth, we apply again induction on r. For $r = 1$ there is nothing to prove since $J_{G_1,G_2} = I_2(X)$. In particular, $S/J_{G_1,G_2}$ is Cohen–Macaulay of depth $m + n - 1$.

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Now let $r > 1$, and let G_2 be a closed graph whose clique complex has r facets, $F_1, \ldots, F_r$. For each subset T of $[n]$ with the cut-point property for G_2, we denote by $P_T(J)$ the minimal prime ideal of $J = J_{G_1,G_2}$ which corresponds to the admissible set $W = [m] \times T$. Let $T_0 = [a_r, b_{r-1}]$, and set

$$J' = \bigcap_{\substack{P_T(J)\in \mathrm{Min}(J)\\ T\not\supset T_0}} P_T(J), \qquad J'' = \bigcap_{\substack{P_T(J)\in \mathrm{Min}(J)\\ T\supset T_0}} P_T(J),$$

where $\mathrm{Min}(J)$ is the set of the minimal prime ideals of J. Then $J = J' \cap J''$; hence, in order to prove the unmixedness of J, we have to show that J' and J'' are unmixed of the same height equal to $(m-1)(n-1)$.

We note that $J' = J_{G_1,G_2'}$, where G_2' is obtained from G_2 by replacing the facets F_{r-1} and F_r of $\Delta(G_2)$ with the clique on the set $[a_{r-1}, n]$. Therefore, G_2' has $r-1$ cliques and J' is unmixed, by induction. In addition, again by induction, we get

$$\mathrm{depth}(S/J') = n - (r-3)m + 2r - 5.$$

On the other hand, $J'' = (\{x_{ij} : (i,j) \in [m] \times T_0\}) + J_{G_1,G_2''}$, where G_2'' is the restriction of G_2 to the vertex set $[n] \setminus T_0$. It follows that G_2'' has two connected components; let us denote them H_1 and H_2, where H_1 is given by $r-1$ cliques on the vertex set $[a_r - 1]$ and H_2 is the clique on the vertex set $[b_{r-1}+1, n]$. Therefore, by the inductive hypothesis, it follows that J_{G_1,H_1} is unmixed of height $(m-1)(a_r-2)$. This implies that every minimal prime of J'' has height equal to $m|T_0| + (m-1)(a_r-2) + (m-1)(n-b_{r-1}-1) = (m-1)(n-1)$; thus, J'' is also unmixed. This ends the proof of unmixedness of J_{G_1,G_2}.

In order to finish the proof of depth's formula, we use the following exact sequence:

$$0 \to \frac{S}{J} \to \frac{S}{J'} \oplus \frac{S}{J''} \to \frac{S}{J'+J''} \to 0. \tag{5}$$

It is clear from the decomposition of J'' that

$$\frac{S}{J''} \cong \frac{S_1}{J_{G_1,H_1}} \otimes_K \frac{S_2}{J_{G_1,H_2}}, \tag{6}$$

where S_1 is the polynomial ring in the variables $x_{ij}, (i,j) \in [m] \times [a_r - 1]$ and S_2 is the polynomial ring in the variables $x_{ij}, (i,j) \in [m] \times [b_{r-1}+1, n]$. Since

J_{G_1,H_1} is unmixed and H_1 has $r-1$ cliques, by induction, it follows that $\operatorname{depth}(S_1/J_{G_1,H_1}) = a_r - 1 - (r-3)m + 2r - 5 = a_r - (r-3)m + 2r - 6$. Since H_2 is a clique, we get $\operatorname{depth}(S_2/J_{G_1,H_2}) = n - b_{r-1} + m - 1$. Consequently, by (6), we obtain

$$\operatorname{depth}(S/J'') = n - (r-3)m + 2r - 5.$$

Therefore,

$$\operatorname{depth}(S/J' \oplus S/J'') = n - (r-3)m + 2r - 5. \tag{7}$$

Now we observe that $J' + J'' = J' + (\{x_{ij} : (i,j) \in [m] \times T_0\}) + J_{G_1,G_2''} = J' + (\{x_{ij} : (i,j) \in [m] \times T_0\})$ since $J_{G_1,G_2''}$ is obviously contained in J'. But this shows that $S/(J' + J'')$ is nothing else than $S/J_{G_1,H}$, where H is the graph obtained from G_2' by replacing its last clique on the vertex set $[a_{r-1}, n]$ by the clique on the set $[a_{r-1}, n] \setminus T_0$. Therefore, $J_{G_1,H}$ is again unmixed and has $r-1$ cliques, so we may apply the inductive hypothesis. We then get

$$\begin{aligned} \operatorname{depth}\Big(\frac{S}{J' + J''}\Big) &= \operatorname{depth}\Big(\frac{S'}{J_{G_1,H}}\Big) \\ &= n - |T_0| - (r-3)m + 2r - 5 \\ &= n - (r-2)m + 2r - 4, \end{aligned} \tag{8}$$

where we denote by S' the polynomial ring in the variables x_{ij} with $(i,j) \in [m] \times ([n] \setminus T_0)$. Finally, by applying the depth lemma in sequence (5), we get

$$\operatorname{depth}(S/J) = \operatorname{depth}\big(S/(J' + J'')\big) + 1 = n - (r-2)m + 2r - 3.$$

The argument for the last claim in our theorem follows easily. If G_2 has r cliques and if J_{G_1,G_2} is Cohen–Macaulay, then the equality $m + n - 1 = n - (r-2)m + 2r - 3$ must hold. Then we get $(r-1)m = 2r - 2$, which implies that $m = 2$ or $r = 1$. Hence, for $m \geq 3$, G_2 must be complete. □

§5. A lower bound for the nilpotency index of J_{G_1,G_2}

Let I be an ideal in a Noetherian ring. Then there exists an integer k such that $(\sqrt{I})^k \subset I$, where $\sqrt{I}$ denotes the radical of I. We call the minimal number k with this property the *index of nilpotency* of I and denote it by

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nilpot(I). We have seen in Theorem 1.2 that nilpot$(J_{G_1,G_2})=1$ if and only if either G_1 or G_2 is complete. In this section we want to give a lower bound for nilpot(J_{G_1,G_2}).

In the proof of the next result we will need the following concept. Let I be an ideal in a polynomial ring S over a field, and let X be a set of variables of S. We say that I is *supported in* X if there exists a system of generators $f_1,\ldots,f_l$ of I such that $X=\bigcup_{i=1}^{l}\operatorname{supp}(f_i)$, where for a polynomial f, supp(f) denotes the set of variables which appear in f. If I is supported in X, we call X a *supporting set* of I.

THEOREM 5.1. *Let $T_1\subset V(G_1)$, let $T_2\subset V(G_2)$, and let $C_{11},\ldots,C_{1r}$ and $C_{21},\ldots,C_{2s}$ be those connected components of $(G_1)_{T_1}$ and $(G_2)_{T_2}$, respectively, which contain as an induced subgraph a line graph of length at least 2. Then* nilpot$(J_{G_1,G_2})\geq rs+1$.

Proof. In each C_{ij} we choose a line graph L_{ij} of length 2 which is an induced subgraph of C_{ij}, and let $f_{ij}=f_{L_{1i}L_{2j}}$, as defined in (1). Then, since L_{1i} and L_{2j} are also induced subgraphs of G_1 (resp., G_2), it follows that $f_{ij}\notin J_{G_1,G_2}$, but $f_{ij}^2\in J_{G_1,G_2}$, as shown in the proof of Theorem 1.2. Let I be the ideal generated by the f_{ij}. Then $I\subset\sqrt{J_{G_1,G_2}}$. We claim that $f=\prod_{\substack{i=1,\ldots,r\\ j=1,\ldots,s}} f_{ij}$ does not belong to J_{G_1,G_2}. This then implies that $I^{rs}\not\subseteq J_{G_1,G_2}$, and we obtain the desired inequality for the nilpotency index for J_{G_1,G_2}.

In order to prove the claim, let $L=(\{x_{kl}\colon (k,l)\in T_1\times T_2\})$, and mark by the *overline* symbol reduction modulo L. Then $f=\bar{f}$ and

$$\bar{J}_{G_1,G_2}=\sum_{\substack{i=1,\ldots,r\\ j=1,\ldots,s}} J_{C_{1i},C_{2j}}+J_0,$$

where J_0 is the sum of the ideals of the form $J_{C,D}$ for the remaining connected components C of $(G_1)_{T_1}$ and D of $(G_2)_{T_2}$, which are different from the C_{ij}. Moreover, there exist supporting sets X_{ij} for $J_{C_{1i},C_{2j}}$ and X_0 for J_0 (resulting from the generating 2-minors of these ideals) such that supp$(f_{ij})\subset X_{ij}$ for all i,j, and such that all the supporting sets, including X_0, are pairwise disjoint.

Now suppose that $f\in J_{G_1,G_2}$. Then $f\in\bar{J}_{G_1,G_2}$ because $f=\bar{f}$. The next lemma, however, shows that $f\notin\bar{J}_{G_1,G_2}$, a contradiction. Thus, $f\notin J_{G_1,G_2}$. This proves the claim and the theorem. ∎

LEMMA 5.2. *Let* $I_1, \ldots, I_r$ *be ideals in a polynomial ring* S *with supporting sets* $X_1, \ldots, X_r$, *and let* $f_1, \ldots, f_r$ *be polynomials in* S *such that* $f_j \notin I_j$ *for* $j = 1, \ldots, r$. *Let* $I = \sum_{i=1}^r I_j$, *let* $f = \prod_{i=1}^r f_i$, *and suppose that*

(i) $X_i \cap X_j = \emptyset$ *for all* $i \neq j$, *and*

(ii) $\mathrm{supp}(f_i) \subset X_i$ *for all* i.

Then $f \notin I$.

Proof. Choose any monomial order $<$ on S. It follows from (i) that $\mathrm{in}_<(I) = \sum_{j=1}^r \mathrm{in}_<(I_j)$. Let g_j be the remainder of f_j with respect to a Gröbner basis of I_j. Since $f_j = g_j + h_j$ with $h_j \in I_j$, it follows that $f = \prod_{i=1}^r g_j + h$, where $h \in I$. Hence, we see that $f \notin I$ if and only if $\prod_{i=1}^r g_j \notin I$. Thus, we may replace the f_j by the g_j and hence may assume from the very beginning that $\mathrm{in}_<(f_j) \notin \mathrm{in}_<(I_j)$.

Suppose that $f \in I$. Then $\mathrm{in}_<(f) \in \mathrm{in}_<(I)$, and therefore, $\prod_{i=1}^r \mathrm{in}_<(f_i) \in \sum_{j=1}^r \mathrm{in}_<(I_j)$. This implies that for some j there exists a monomial generator $u \in \mathrm{in}_<(I_j)$ such that u divides $\prod_{i=1}^r \mathrm{in}_<(f_i)$. Since $\mathrm{supp}(u) \subset X_j$, it follows from (i) and (ii) that u divides $\mathrm{in}_<(f_j)$. This is a contradiction, since $\mathrm{in}_<(f_j) \notin \mathrm{in}_<(I)$. ∎

We give a concrete example of Theorem 5.1 in the form of the following corollary.

COROLLARY 5.3. *Let* J *be the ideal of adjacent minors of an* $(m \times n)$*-matrix, and let* k *and* l *be integers such that* $m = 4k + p$ *and* $n = 4l + q$, *with* $0 \leq p, q < 4$. *Then*

$$\mathrm{nilpot}(J) \geq \left(k + \left\lfloor \frac{p}{3} \right\rfloor\right)\left(l + \left\lfloor \frac{q}{3} \right\rfloor\right) + 1 \approx \frac{mn}{16}.$$

In particular, the index of nilpotency of the binomial edge of a pair of graphs can be arbitrarily big.

Proof. We apply Theorem 5.1, in the case that G_1 is a line graph on $[m]$ and G_2 is a line graph on $[n]$. We choose $T_1 = \{4a \colon a \in [m], 4a \leq m\}$ and $T_2 = \{4b \colon b \in [n], 4b \leq n\}$. Then Theorem 5.1 yields the desired conclusion. ∎

The bound given in Corollary 5.3 is definitely not the best possible. Calculations by computer show that for $k = 2, 3, 4$ the index of nilpotency of the ideal of adjacent 2-minors of a $(3 \times 3k)$-matrix is at least $k + 1$, whereas our Corollary 5.3 gives only k as the lower bound for the index of nilpotency in these cases.

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