THE NORMAL CURVATURE OF TOTALLY REAL SUBMANIFOLDS OF $S^6(1)$

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Abstract. We prove the pointwise inequality $0 \geq \rho + \rho^{\perp} - 1$ involving the normalized scalar curvature ρ and normal scalar curvature $\rho^{\perp}$ of a totally real 3-dimensional submanifold of the nearly Kaehler 6-sphere. Further we classify submanifolds realizing the equality in this inequality.

1. Introduction. Let M^n be an n–dimensional (immersed) submanifold of an m-dimensional real is space form $N^m(c)$. If $R^{\perp}$ is the curvature tensor of the normal connection, $\{e_1, ..., e_n\}$ an orthonormal basis of tangent vector fields and $\{\xi_1, ..., \xi_{m-n}\}$ an orthonormal basis of normal vector fields, then the (normalized) normal scalar curvature $\rho^{\perp}$ is defined in **[7]** by

$$\rho^{\perp} = \frac{2}{n(n-1)} \left(\sum_{i<j=1}^{n} \sum_{r<s=1}^{(m-n)} \langle R^{\perp}(e_i, e_j)\xi_r, \xi_s \rangle^2 \right)^{1/2}.$$

For $n = 2$, this definition is the same as the definition of "normal curvature", given in **[11]**. The following was conjectured in **[7]**.

Conjecture. *Let $\phi : M^n \to N^m(c)$ be an isometric immersion. Then at every point p, we have*

$$|H|^2 \geq \rho + \rho^{\perp} - c,$$

where ρ is the normalized scalar curvature, and H is the mean curvature vector of M^n.

This conjecture was proved for $n = 2$, $m = 4$ and $c = 0$ by Wintgen **[13]**; for $n = 2$ and $m \geq 4$ by Guadalupe and Rodriguez **[11]**; for $n \geq 2$ and $m = n + 2$ by the authors in **[7]**. For $n \geq 2$ and $m = m + 1$, in which case $\rho^{\perp}$ is trivially zero, the conjecture follows from a more general result of Chen in **[4]** stating that for arbitrary submanifolds of real space forms, $|H|^2 \geq \rho - c$.

In this paper, we prove the conjecture for 3-dimensional totally real submanifolds of S^6. Note that such submanifolds are always minimal **[10]**. In particular, we prove the following theorem.

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THEOREM 1. *Let $x : M^3 \to S^6(1)$ be a totally real isometric immersion. Then*

$$0 \geq \rho + \rho^{\perp} - 1. \tag{1.1}$$

If we define, following Chen [3], a Riemannian invariant δ_M by

$$\delta_M(p) = \frac{n(n-1)}{2}\rho(p) - \inf K(p),$$

where $\inf K$ is the function assigning to each $p \in M$ the infimum of $K(\pi)$, π running over all planes in T_pM, then it is proved in [3] that any minimal submanifold of a unit sphere satisfies $\delta_M \leq \frac{1}{2}(n+1)(n-2)$. For totally real 3-dimensional submanifolds of $S^6(1)$, this becomes $\delta_M \leq 2$.

THEOREM 2. *Let $X : M^3 \to S^6(1)$ be a totally real isometric immersion, and let $p \in M$. Then the following statements are equivalent.*

(1) $\rho + \rho^{\perp} = 1$ *at* p,
(2) $\delta_M = 2$ *at* p,
(3) *There exists an orthonormal basis* $\{e_1, e_2, e_3\}$ *at* p *such that*

$$\begin{array}{lll} h(e_1, e_1) = \lambda J e_1 & h(e_2, e_2) = -\lambda J e_1 & h(e_1, e_2) = -\lambda J e_2, \\ h(e_2, e_3) = 0 & h(e_1, e_3) = 0 & h(e_3, e_3) = 0. \end{array}$$

Totally real immersions with $\delta_M = 2$ at every point are completely classified, see [5], [6] and [9]. The classification is summarized in the following theorems from [9].

THEOREM 3. *Let $\phi : N_1 \to CP^2(4)$ be a holomorphic curve in $CP^2(4)$. Let PN_1 be the circle bundle over N_1 induced by the Hopf fibration $p : S^5(1) \to CP^2(4)$ and let ψ be the isometric immersion such that the following diagram commutes.*

$$\begin{array}{ccc} PN_1 & \xrightarrow{\psi} & S^5(1) \\ \downarrow & & \downarrow p \\ N_1 & \xrightarrow{\phi} & CP^2(4) \end{array}$$

Then there exists a totally geodesic embedding i of $S^5(1)$ into the nearly Kähler 6-sphere such that the immersion $i \circ \psi : PN_1 \to S^6(1)$ is a 3-dimensional totally real immersion in $S^6(1)$ with $\delta_{PN_1} = 2$. Conversely let $F : M^3 \to S^6(1)$ be a totally real immersion which is in linearly full in $S^6(1)$. Then M^3 automatically satisfies $\delta_M = 2$ and there exists a totally geodesic $S^5(1)$ and a holomorphic curve $\phi : N_1 \to CP^2(4)$ such that F is congruent to ψ, which is obtained from ϕ as above.

THEOREM 4. *Let $\overline{\phi} : N_2 \to S^6(1)$ be an almost complex curve (with second fundamental form α) without totally geodesic points. Denote by UN_2 the unit tangent bundle over N_2 and define a map*

$$\overline{\psi} : UN_2 \to S^6(1) : v \mapsto \overline{\phi}_*(v) \times \frac{\alpha(v, v)}{\|\alpha(v, v)\|}.$$

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Then $\overline{\psi}$ is a (possibly branched) totally real immersion into $S^6(1)$ satisfying $\delta_{UN_2} = 2$. Moreover, the immersion is linearly full in $S^6(1)$. Conversely let $F: M^3 \to S^6(1)$ be a linearly full totally real immersion of a 3-dimensional manifold satisfying $\delta_M = 2$. Let p be a non totally geodesic point of M^3. Then there exists a (possibly branched) almost complex curve $\overline{\phi}: N_2 \to S^6(1)$ such that, around p, F is congruent to $\overline{\psi}$, which is obtained from $\overline{\phi}$ as above.

So the following corollary follows immediately from the previous theorems.

COROLLARY 1. *If $f: M^3 \to S^6(1)$ is a totally real isometric immersion, satisfying $\rho + \rho^\perp = 1$, then f is one of the immersions given in Theorem 3 or Theorem 4 above.*

2. The nearly Kaehler structure on S^6. We briefly describe how the standard nearly Kähler structure on $S^6(1)$ arises in a natural manner from Cayley multiplication. The multiplication on the Cayley numbers $\mathcal{O}$ may be used to define a vector cross product $\times$ on the purely imaginary Cayley numbers R^7. The standard nearly Kähler structure on $S^6(1) \subset R^7$ is then obtained as follows.

$$Ju = x \times u, \qquad u \in T_xS^6(1), x \in S^6(1).$$

Then J is an orthogonal almost complex structure on $S^6(1)$. In fact J is a nearly Kähler structure in the sense that the (2,1)-tensor field G on $S^6(1)$, defined by $G(X, Y) = (\tilde{\nabla}_X J)(Y)$, where $\tilde{\nabla}$ is the Levi-Civita connection on $S^6(1)$, is skew-symmetric. For more information on the properties of $\times$, J and G, we refer to **[2]**, **[1]** and **[8]**.

3. Totally real immersions with $\delta_M = 2$. An immersion $F: M^3 \to S^6(1)$ is called totally real if the almost complex structure J maps the tangent space into the normal space. In **[10]** Ejiri proved that a 3-dimensional totally real submanifold of $S^6(1)$ is orientable and minimal and that $G(X, Y)$ is orthogonal to M, for tangent vectors X and Y. We denote that Levi-Civita connection of M by ∇. The formulas of Gauss and Weingarten are then respectively given by

$$\tilde{\nabla}_X F_* Y = F_*(\nabla_X Y) + h(X, Y), \tag{3.1}$$

$$\tilde{\nabla}_X \eta = -F_*(A_\eta X) + \nabla_X^\perp \eta, \tag{3.2}$$

for tangent vector fields X and Y and normal vector field η. The second fundamental form h is related to A_η by $\langle h(X, Y), \eta \rangle = \langle A_\eta X, Y \rangle$. From (3.1) and (3.2), we find that

$$\nabla_X^\perp JF_*(Y) = JF_*(\nabla_X Y) + G(F_*X, F_*Y), \tag{3.3}$$

$$F_*(A_{JY}X) = -Jh(X, Y). \tag{3.4}$$

The above formulas immediately imply that $\langle h(X, Y), JF_*Z \rangle$ is totally symmetric.

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The fundamental equations of Gauss, Codazzi and Ricci then respectively state that

$$\begin{aligned}\langle R(X,Y)Z,W\rangle &= \langle Y,Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,W\rangle \\ &\quad + \langle h(Y,Z),h(X,W)\rangle - \langle h(X,Z),h(Y,W)\rangle, \\ (\nabla h)(X,Y,Z) &= (\nabla h)(Y,X,Z), \\ \langle R^{\perp}(X,Y)\xi,\eta\rangle &= \langle [A_\xi, A_\eta]X, Y\rangle],\end{aligned}$$

where X, Y, Z, W are tangent vector fields and ξ and η are normal vector fields. From these equations, one obtains easily that

$$\langle R^{\perp}(X,Y)JZ,JW\rangle = \langle R(X,Y)Z,W\rangle - \langle Y,Z\rangle\langle X,W\rangle + \langle X,Z\rangle\langle Y,W\rangle. \tag{3.5}$$

4. Proofs. Let $F: M^3 \to S^6$ be a totally real immersion. We identify M^3 with its image in S^6. Let $p \in M$ and assume that p is not a totally geodesic point. Following **[10]**, see also **[12]**, there exists an orthonormal basis $\{e_1, e_2, e_3\}$ at the point p such that $G(e_1,e_2) = J_{e_3}$, $G(e_2,e_3) = Je_1$, $G(e_3,e_1) = J_{e_2}$, and

$$\begin{aligned} h(e_1,e_1) &= (a+b)Je_1, & h(e_2,e_2) &= -aJe_1 + cJe_2 - dJe_3, \\ h(e_1,e_2) &= -aJe_2, & h(e_2,e_3) &= -dJe_2 - cJe_3, \\ h(e_1,e_3) &= -bJe_3, & h(e_3,e_3) &= -bJe_1 - cJe_2 + dJe_3,\end{aligned}$$

where $a + b > 0$. A straightforward computation using the Gauss equation gives that

$$\begin{aligned} K(e_1\wedge e_2) &= 1 - 2a^2 - ab, \\ K(e_1\wedge e_3) &= 1 - 2b^2 - ab, \\ K(e_2\wedge e_3) &= 1 + ab - 2(c^2+d^2).\end{aligned}$$

Hence we obtain that

$$3\rho = \sum_{i<j} K(e_i\wedge e_j) = 3 - (2a^2 + 2b^2 + 2c^2 + 2d^2 + ab). \tag{4.1}$$

Since $a + b > 0$, we have $-2ab < a^2 + b^2$ and from the Ricci equation we find

$$\begin{aligned} 9(\rho^{\perp})^2 &= \frac{1}{2}\sum_{r<s} \|[A_{Je_r}, A_{Je_s}]\|^2 \\ &= 2(c^2+d^2)(a-b)^2 + a^2(2a+b)^2 + b^2(a+2b)^2 + (ab - 2(c^2+d^2))^2, \\ &= 4a^4 + 4b^4 + 4c^4 + 4d^4 + a^2b^2 + 2a^2c^2 - 8abc^2 + 2b^2c^2 \\ &\quad + 2a^2d^2 - 8abd^2 + 2b^2d^2 + 4a^3b + 2a^2b^2 + 4ab^3 + 8c^2d^2 \\ &= 4a^4 + 4b^4 + 4c^4 + 4d^4 + a^2b^2 + 2a^2c^2 - 12abc^2 + 2b^2c^2 \\ &\quad + 2a^2d^2 - 12abd^2 + 2b^2d^2 + 4a^3b + 2a^2b^2 + 4ab^3 + 8c^2d^2 \\ &\quad + 4abc^2 + 4abd^2\end{aligned}$$

$$\leq 4a^4 + 4b^4 + 4c^4 + 4d^4 + a^2b^2 + 2a^2c^2 + 6(a^2+b^2)c^2 + 2b^2c^2 + 2a^2d^2 + 6(a^2+b^2)d^2 + 2b^2d^2 + 4a^3b + 2a^2b^2 + 4ab^3 + 8c^2d^2 + 4abc^2 + 4abd^2 \tag{4.2}$$

$$= 4a^4 + 4b^4 + 4c^4 + 4d^4 + a^2b^2 + 2a^2b^2 + 8a^2c^2 + 8b^2c^2 + 8a^2d^2 + 8b^2d^2 + 4a^3b + 4ab^3 + 8c^2d^2 + 4abc^2 + 4abd^2$$

$$\leq 4a^4 + 4b^4 + 4c^4 + 4d^4 + a^2b^2 + 8a^2b^2 + 8a^2c^2 + 8a^2d^2 + 4a^3b + 8b^2c^2 + 8b^2d^2 + 4ab^3 + 8c^2d^2 + 4abc^2 + 4abd^2 \tag{4.3}$$

$$= (2a^2 + 2b^2 + 2c^2 + 2d^2 + ab)^2.$$

Since $2a^2 + 2b^2 + ab \geq 0$, we deduce from this that

$$3\rho^{\perp} \leq 2a^2 + 2b^2 + 2c^2 + 2d^2 + ab.$$

Hence, using (4.1), this becomes

$$3\rho^{\perp} \leq 3 - 3\rho,$$

which proves the inequality. This finishes the proof of Theorem 1.

Let us assume now that the equality is realized in (1.1) at the point p. Equality in (4.2) implies that $c = d = 0$ and equality in (4.3) that $ab = 0$. So, if necessary, by replacing e_2 and e_3 by $-e_3$ and e_2, we may assume that $a \neq 0$ and $b = 0$. This proves that (1) and (3) of Theorem 2 are equivalent. The equivalence of (2) and (3) is proved in [**5**]. This completes the proof of Theorem 2.

REMARK 1. Using (3.5), one can always express the normal scalar curvature in terms of intrinsic invariants. After a straightforward computation, we obtain that

$$3(\rho^{\perp})^2 = \frac{1}{12}\|R\|^2 - 2\rho + 1.$$

So (1.1) is in fact an intrinsic obstruction.

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