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MULTIVARIATE SAMPLING THEOREMS ASSOCIATED WITH MULTIPARAMETER DIFFERENTIAL OPERATORS

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Abstract We investigate the multivariate sampling theory associated with multiparameter eigenvalue problems. A several-variable counterpart of the classical sampling theorem of Whittaker, Kotel'nikov and Shannon is given. It arose when the multiparameter system has order one. Two-dimensional sampling theorems associated with two-parameter systems of second-order differential operators will be established. The sampling formulae are of multivariate non-uniform Lagrange interpolation type. Unlike many of the known formulae, the interpolating functions are not necessarily products of single variable functions.

Keywords: multiparameter eigenvalue problems; sampling theory; Lagrange interpolations

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1. Introduction

In the following $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of integers, real and complex numbers, respectively. For a positive integer n, the sets $\mathbb{Z}^n$, $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the sets of all n integer tuples, n real tuples and n complex tuples. Let $\mathcal{E} \subset \mathbb{R}^2$ be compact and symmetric with respect to the origin. A function $f \in L_2(\mathbb{R}^2)$ is called band-limited to $\mathcal{E}$ if

$$f(x,y) = \frac{1}{2\pi}\int_{\mathcal{E}} \hat{f}(u,v)\exp(\mathrm{i}(ux+vy))\,\mathrm{d}(u,v), \quad (x,y) \in \mathbb{R}^2, \tag{1.1}$$

where $\hat{f}(u,v)$ is the Fourier transform

$$\hat{f}(u,v) := \lim_{\eta\to\infty}\frac{1}{2\pi}\int_{-\eta}^{\eta}\int_{-\eta}^{\eta} f(x,y)\exp(-\mathrm{i}(ux+vy))\,\mathrm{d}x\,\mathrm{d}y, \tag{1.2}$$

and the limit converges in the $L_2(\mathbb{R}^2)$-norm, cf. [**7**, p. 54]. Before the end of this section we will give a general definition of n-dimensional band-limited functions and state a multivariate counterpart of the well-known Paley–Wiener theorem [**27**] established by Plancherel and Pólya in [**30**]. A two-dimensional sampling theorem for functions band-limited to $\mathcal{E} = [-\pi,\pi] \times [-\pi,\pi]$ (see, for example, [**23**, **28**, **29**, **31**]) reads as follows.

Theorem A. *Let $f(x,y)$ be band-limited to $[-\pi,\pi]\times[-\pi,\pi]$. Then*

$$f(x,y)=\sum_{n,m=-\infty}^{\infty} f(n,m)\frac{\sin\pi(x-n)}{\pi(x-n)}\frac{\sin\pi(y-m)}{\pi(y-m)},\quad (x,y)\in\mathbb{R}^2. \tag{1.3}$$

The series (1.3) converges uniformly on compact subsets of $\mathbb{R}^2$.

Theorem A is a two-dimensional version of the classical sampling theorem of Whittaker, Kotel'nikov and Shannon (see [**34**, **38**] and [**10**, **11**]). The theory of non-uniform sampling of two-dimensional band-limited signals is established by Butzer and Hinsen in [**7**, **8**] (see also [**24**]). According to the multivariate Paley–Wiener theorem, functions band-limited to $[-\pi,\pi]^n$, $n\in\mathbb{Z}^+$ are entire functions of exponential type (see the definitions below). In [**23**] the convergence of the sampling representation of such functions is shown to be absolute and uniform on compact subsets of $\mathbb{C}^2$. In [**32**] under restrictive conditions, multidimensional reconstruction formulae were given for multidimensional signals (functions) which are not necessarily band-limited. The error in these formulae is proportional to the energy carried in the *tail* of the function. In case of multidimensional functions band-limited to $[-\pi,\pi]^n$, $n\in\mathbb{Z}^+$, the energy carried in the *tail* of the function is zero. However, there are still restrictive conditions.

In view of the Kramer analytic theorem, derived by Everitt *et al*. [**17**, **18**], and its applications in differential equations, the classical sampling theorem of Whittaker, Kotel'nikov and Shannon can be derived by using the first-order differential operator

$$-\mathrm{i}y'(x)=\lambda y(x),\quad |x|\leqslant\pi,\ \lambda\in\mathbb{C},\ y(-\pi)=y(\pi)$$

(see, for example, [**15**]). Also, Kramer's theorem is applied to higher-order eigenvalue problems to derive sampling formulae (see, for example, [**3**–**5**, **9**, **14**, **16**]).

In the present article we discuss the possibility of deriving multidimensional sampling representations for several-variable transforms arising from multiparameter systems of differential equations. Multivariate sampling theorems appear when investigating the theory associated with partial differential operators (see, for example, [**2**]). The separation of variables of the partial differential equation split the problem into several Sturm–Liouville problems, i.e. a multiparameter system where only one parameter appears in every equation. This is why the kernels of the sampled integral transforms of [**2**] are products of functions of two variables, one real and one complex. Moreover, the interpolating functions are products of single variable functions. In the following we study the situation when we have a system of multiparameter problems where all eigenvalue parameters may appear in every differential equation. This will lead to sampling representations of transforms whose kernels are products of functions of more than two variables and the interpolating functions are more general than those of [**2**] or (1.3) above. We derive two-dimensional sampling representations of two-dimensional integral transforms whose kernels are solutions of two-parameter systems. The spectral analysis of these systems has a long history and has been extensively studied (see, for example, [**6**, **19**–**22**, **33**, **35**, **36**]). In § 3 we introduce these systems as well as the properties we need to derive the sampling theorems. Section 4 is devoted to the derivation of the multivariate sampling theorems.

We have four different results according to the distribution of the eigenvalues. Section 5 exhibits all the derived sampling representations with illustrative examples. All results in this setting are two-dimensional Lagrange interpolation series. Moreover, the sampling representations are in general non-uniform in the sense of [**7**, **8**]. Section 2 contains a first-order example that leads to a multivariate counterpart of the classical sampling theorem of Whittaker, Kotel'nikov and Shannon. In this example, all necessary spectral properties can be easily checked. We end this introduction by defining several-variable entire functions of exponential type, their order and by stating the multivariate analogue of the Paley–Wiener theorem of [**30**]. The definitions and the theorem are taken from [**26**, Chapter 3] and [**37**, Chapter 1]. A function $f(\boldsymbol{z})$ is said to be entire in $\boldsymbol{z} \in \mathbb{C}^n$ if it decomposes into an absolutely convergent power series

$$f(\boldsymbol{z}) = \sum_{\boldsymbol{K} \geqslant \boldsymbol{0}} \boldsymbol{a_K} \boldsymbol{z}^{\boldsymbol{K}} = \sum_{k_1,\dots,k_n \geqslant 0} a_{k_1,\dots,k_n} z_1^{k_1} \cdots z_1^{k_1}, \tag{1.4}$$

where $\boldsymbol{a_K} = (a_{k_1}, \dots, a_{k_n})$ are constant coefficients. This definition of entire functions is in the sense of Weierstrass (see [**37**, p. 28] and [**37**, p. 30]) it is equivalent to say that $f(\boldsymbol{z})$ is entire in every variable z_i. The coefficients $\boldsymbol{a_K} = (a_1, \dots, a_n)$ are determined via [**36**, pp. 30 and 31]

$$\boldsymbol{a_K} = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{f(\boldsymbol{r}\mathrm{e}^{\mathrm{i}\boldsymbol{t}})}{\boldsymbol{r}^{\boldsymbol{K}}} \mathrm{e}^{-\mathrm{i}\boldsymbol{tK}} \, \mathrm{d}\boldsymbol{t}. \tag{1.5}$$

From now on when we say an entire function we mean entire in Weierstrass's sense. An entire function $f(\boldsymbol{z})$ is called of exponential type $\sigma := (\sigma_1, \dots, \sigma_n) \geqslant 0$ if for every $\varepsilon > 0$ there exists a positive constant A_ε such that

$$|f(\boldsymbol{z})| \leqslant A_\varepsilon \exp\biggl[\sum_{j=1}^{n} (\sigma_j + \varepsilon)|z_j|\biggr], \quad \boldsymbol{z} = (z_1, \dots, z_n) \in \mathbb{C}^n. \tag{1.6}$$

If inequality (1.6) is satisfied with $|z_j|^{\rho_j}$, $\rho_j > 0$, instead of $|z_j|$ for an entire function $f(\boldsymbol{z})$ of exponential type σ, then it is said to have order $\rho = (\rho_1, \dots, \rho_n) > 0$. A several-variable analogue of the Paley–Wiener theorem [**27**] is given in [**30**] and [**26**, pp. 109 and 110] (see also [**1**, p. 134 ff.]). Let $\mathfrak{M}_{\sigma p}$, $p > 1$, denote the space of all entire functions of exponential type σ which belong to $L_p(\mathbb{R}^n)$ when restricted to $\mathbb{R}^n$. The space $\mathfrak{M}_{\sigma 2}$ will be called the space of n-variable functions band-limited to $\Delta_\sigma := \{\boldsymbol{x} \in \mathbb{R}^n : |x_j| \leqslant \sigma_j, j = 1, \dots, n\}$. A several-variable analogue of the Paley–Wiener theorem reads as follows.

Theorem B. *If $f \in \mathfrak{M}_{\sigma 2}, \sigma = (\sigma_1, \dots, \sigma_n)$, then the function*

$$\tilde{f}(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\boldsymbol{z}) \exp(-\mathrm{i}\boldsymbol{xz}) \, \mathrm{d}\boldsymbol{z}, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n, \tag{1.7}$$

where the integral converges in the mean, is an $L_2(\mathbb{R}^n)$-function which vanishes outside $\Delta_\sigma := \{\boldsymbol{x} \in \mathbb{R}^n : |x_j| \leqslant \sigma_j, j = 1, \dots, n\}$. Conversely, the function

$$f(\boldsymbol{z}) = \frac{1}{(2\pi)^{n/2}} \int_{\Delta_\sigma} g(\boldsymbol{x}) \exp(\mathrm{i}\boldsymbol{xz}) \, \mathrm{d}\boldsymbol{x}, \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n, g(\boldsymbol{x}) \in L_2(\Delta_\sigma), \tag{1.8}$$

lies in $\mathfrak{M}_{\sigma 2}$ and $g = \tilde{f}$ almost everywhere.

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2. A multivariate classical result

Let $\Omega_\pi := [-\pi,\pi]^n$, with $\underline{x} = (x_1,\dots,x_n) \in \Omega_\pi$, $n \in \mathbb{Z}^+$. Let $A := (\alpha_{ij})_{1\leqslant i,j\leqslant n}$ be an $n\times n$ non-singular matrix of complex entries. Let $\mathfrak{H}$ denote the Hilbert space of all Lebesgue measurable functions on Ω_π which are square integrable. The inner product and norm are defined in $\mathfrak{H}$ to be

$$\langle f,g\rangle_{\mathfrak{H}} := \int_{\Omega_\pi} f(\underline{x})\bar{g}(\underline{x})\,\mathrm{d}\underline{x}, \quad \|f\|_{\mathfrak{H}} := \biggl(\int_{\Omega_\pi} |f(\underline{x})|^2\,\mathrm{d}\underline{x}\biggr)^{1/2}. \tag{2.1}$$

Consider the first-order multiparameter eigenvalue problem

$$-\mathrm{i}y_j'(x_j) = (\alpha_{j1}\lambda_1 + \alpha_{j2}\lambda_2 + \cdots + \alpha_{jn}\lambda_n)y_j(x_j), \quad |x_j| \leqslant \pi, \tag{2.2}$$

$$y_j(-\pi) = y_j(\pi), \quad j = 1,\dots,n, \tag{2.3}$$

in the parameters $\lambda_1,\lambda_2,\dots\lambda_n \in \mathbb{C}$. For every j, $1\leqslant j\leqslant n$, the functions

$$y_j(x_j,\lambda_1,\dots,\lambda_n) = \exp(\mathrm{i}(\alpha_{j1}\lambda_1 + \alpha_{j2}\lambda_2 + \cdots + \alpha_{jn}\lambda_n)x_j), \quad j = 1,\dots,n, \tag{2.4}$$

solve (2.2), where $\mathrm{i} := \sqrt{-1}$. Hence the eigenvalues of the system (2.2), (2.3) are determined by solving the linear system of equations

$$\left.\begin{aligned} \alpha_{11}\lambda_1 + \cdots + \alpha_{1n}\lambda_n &= k_1,\\ \alpha_{21}\lambda_1 + \cdots + \alpha_{2n}\lambda_n &= k_2,\\ &\vdots\\ \alpha_{n1}\lambda_1 + \cdots + \alpha_{nn}\lambda_n &= k_n, \end{aligned}\right\} \tag{2.5}$$

where $k_j \in \mathbb{Z}$. Let $K := (k_1,k_2,\dots,k_n) \in \mathbb{Z}^n$. Then the eigenvalues of (2.2), (2.3) are

$$\underline{\lambda}_K = (\lambda_{1,k_1},\dots,\lambda_{n,k_n}) = A^{-1}K, \quad K \in \mathbb{Z}^n, \tag{2.6}$$

where A^{-1} is the inverse of A. The corresponding set of eigenfunction is

$$\mathcal{O} = \biggl\{\psi_K(\underline{x}) = \prod_{j=1}^n \exp(\mathrm{i}k_jx_j): \ K = (k_1,k_2,\dots,k_n) \in \mathbb{Z}^n\biggr\}. \tag{2.7}$$

The Fourier system $\mathcal{O}$ is an orthogonal basis of $\mathfrak{H}$. The n-dimensional sampling theorem associated with the multiparameter problem (2.2), (2.3) is the following.

Theorem 2.1. *Let $f(\underline{\lambda})$, $\underline{\lambda} = (\lambda_1,\dots,\lambda_n) \in \mathbb{C}^n$, be the n-variable integral transform*

$$f(\underline{\lambda}) = \int_{\Omega_\pi} g(\underline{x})\Phi(\underline{x},\underline{\lambda})\,\mathrm{d}\underline{x}, \quad g(\cdot) \in \mathfrak{H}, \tag{2.8}$$

where

$$\Phi(\underline{x},\underline{\lambda}) := \prod_{j=1}^n \exp(\mathrm{i}(\alpha_{j1}\lambda_1 + \cdots + \alpha_{jn}\lambda_n)x_j). \tag{2.9}$$

Then $f(\underline{\lambda})$ admits the sampling representation

$$f(\underline{\lambda}) = \sum_{k_1,\dots,k_n=-\infty}^{\infty} f(A^{-1}K) \prod_{j=1}^{n} \frac{\sin \pi(\alpha_{j1}\lambda_1 + \cdots + \alpha_{jn}\lambda_n - k_j)}{\pi(\alpha_{j1}\lambda_1 + \cdots + \alpha_{jn}\lambda_n - k_j)} \quad \underline{\lambda} \in \mathbb{C}^n. \tag{2.10}$$

Moreover, $f(\underline{\lambda})$ is an entire function of exponential type

$$\sigma_\pi := \bigg(\sum_{k=1}^{n} |\alpha_{k1}|\pi, \dots, \sum_{k=1}^{n} |\alpha_{kn}|\pi \bigg).$$

Series (2.10) converges absolutely on $\mathbb{C}^n$ and uniformly on compact subsets of $\mathbb{C}^n$ and on the subset of all $\underline{\lambda} \in \mathbb{C}^n$ such that $A\underline{\lambda} \in \mathbb{R}^n$. In particular, if all entries of A are real, then (2.10) converges uniformly on $\mathbb{R}^n$.

Proof. Since $\mathcal{O}$ is an orthogonal basis of $\mathfrak{H}$, then, from Parseval's identity,

$$f(\underline{\lambda}) = \sum_{K \in \mathbb{Z}^n} \frac{\overline{\langle \bar{g}(\cdot), \Psi_K(\cdot) \rangle}_{\mathfrak{H}} \langle \Phi(\cdot, \underline{\lambda}), \Psi_K(\cdot) \rangle_{\mathfrak{H}}}{\|\Psi_K(\cdot)\|_{\mathfrak{H}}^2}, \quad \underline{\lambda} \in \mathbb{C}^n. \tag{2.11}$$

Thus, for $\underline{\lambda} \in \mathbb{C}^n$,

$$\begin{aligned} f(\underline{\lambda}) &= \sum_{k_1,\dots,k_n=-\infty}^{\infty} \frac{\int_{\Omega_\pi} g(\underline{x}) \prod_{\ell=1}^{n} \mathrm{e}^{\mathrm{i}k_\ell x_\ell} \, \mathrm{d}\underline{x} \cdot \int_{\Omega_\pi} \Phi(\underline{x}, \underline{\lambda}) \prod_{\ell=1}^{n} \mathrm{e}^{-\mathrm{i}k_\ell x_\ell} \, \mathrm{d}\underline{x}}{\int_{\Omega_\pi} \prod_{\ell=1}^{n} \mathrm{e}^{\mathrm{i}k_\ell x_\ell} \prod_{\ell=1}^{n} \mathrm{e}^{-\mathrm{i}k_\ell x_\ell} \, \mathrm{d}\underline{x}} \\ &= \sum_{k_1,\dots,k_n=-\infty}^{\infty} f(A^{-1}K) \frac{\int_{\Omega_\pi} \prod_{j=1}^{n} \mathrm{e}^{\mathrm{i}(\alpha_{j1}\lambda_1 + \cdots + \alpha_{jn}\lambda_n)x_j} \prod_{\ell=1}^{n} \mathrm{e}^{-\mathrm{i}k_\ell x_\ell} \, \mathrm{d}\underline{x}}{\int_{\Omega_\pi} \prod_{\ell=1}^{n} \mathrm{e}^{\mathrm{i}k_\ell x_\ell} \prod_{\ell=1}^{n} \mathrm{e}^{-\mathrm{i}k_\ell x_\ell} \, \mathrm{d}\underline{x}}. \end{aligned} \tag{2.12}$$

Simple calculations yield

$$\int_{\Omega_\pi} \prod_{j=1}^{n} \mathrm{e}^{\mathrm{i}(\alpha_{j1}\lambda_1 + \cdots + \alpha_{jn}\lambda_n)x_j} \prod_{\ell=1}^{n} \mathrm{e}^{-\mathrm{i}k_\ell x_\ell} \, \mathrm{d}\underline{x} = \prod_{j=1}^{n} \frac{2 \sin \pi(\alpha_{j1}\lambda_1 + \cdots + \alpha_{jn}\lambda_n - k_j)}{(\alpha_{j1}\lambda_1 + \cdots + \alpha_{jn}\lambda_n - k_j)}, \tag{2.13}$$

$$\int_{\Omega_\pi} \prod_{\ell=1}^{n} \mathrm{e}^{\mathrm{i}k_\ell x_\ell} \prod_{\ell=1}^{n} \mathrm{e}^{-\mathrm{i}k_\ell x_\ell} \, \mathrm{d}\underline{x} = (2\pi)^n, \quad \underline{\lambda} \in \mathbb{C}^n, \ j = 1, \dots, n. \tag{2.14}$$

Substituting from (2.13) and (2.14) in (2.12), we get the multivariate sampling expansion (2.10) with pointwise convergence on $\mathbb{C}^n$. Now we establish the other convergence properties. We start with the absolute convergence. Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ be fixed. Then, by the Cauchy–Schwarz inequality and Parseval's identity, we obtain

$$\begin{aligned} &\sum_{k_1,\dots,k_n=-\infty}^{\infty} \bigg| f(A^{-1}K) \prod_{j=1}^{n} \frac{\sin \pi(\alpha_{j1}\lambda_1 + \cdots + \alpha_{jn}\lambda_n - k_j)}{\pi(\alpha_{j1}\lambda_1 + \cdots + \alpha_{jn}\lambda_n - k_j)} \bigg| \\ &\quad = \sum_{K \in \mathbb{Z}^n}^{\infty} \bigg| \frac{\overline{\langle \bar{g}(\cdot), \Psi_K(\cdot) \rangle}_{\mathfrak{H}} \langle \Phi(\cdot, \underline{\lambda}), \Psi_K(\cdot) \rangle_{\mathfrak{H}}}{\|\Psi_K(\cdot)\|_{\mathfrak{H}}^2} \bigg| \\ &\quad \leqslant \bigg(\sum_{K \in \mathbb{Z}^n}^{\infty} \bigg| \frac{\langle \bar{g}(\cdot), \Psi_K(\cdot) \rangle_{\mathfrak{H}}}{\|\Psi_K(\cdot)\|_{\mathfrak{H}}} \bigg|^2 \bigg)^{1/2} \bigg(\sum_{K \in \mathbb{Z}^n}^{\infty} \bigg| \frac{\langle \Phi(\cdot, \underline{\lambda}), \Psi_K(\cdot) \rangle_{\mathfrak{H}}}{\|\Psi_K(\cdot)\|_{\mathfrak{H}}} \bigg|^2 \bigg)^{1/2} < \infty, \end{aligned} \tag{2.15}$$

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since $g(\cdot), \Phi(\cdot, \underline{\lambda}) \in \mathfrak{H}$. As for the uniform convergence let N be a positive integer and define the function

$$s_N(\underline{\lambda}) := \left| f(\underline{\lambda}) - \sum_{|K| \leqslant N} f(A^{-1}K) \prod_{j=1}^{n} \frac{\sin \pi(\alpha_{j1}\lambda_1 + \cdots + \alpha_{jn}\lambda_n - k_j)}{\pi(\alpha_{j1}\lambda_1 + \cdots + \alpha_{jn}\lambda_n - k_j)} \right|, \quad \underline{\lambda} \in \mathbb{C}^n, \tag{2.16}$$

where $|K| := \sqrt{k_1^2 + \cdots + k_n^2}$. To prove uniform convergence of (2.10) on a subset M, it is sufficient to prove that $s_N(\underline{\lambda})$ approaches zero as $N \to \infty$ without depending on $\underline{\lambda}$. Again, using the Cauchy–Schwarz and Bessel's inequalities, we obtain

$$\begin{aligned} s_N(\underline{\lambda}) &\leqslant \sum_{|K| > N} \left| f(A^{-1}K) \prod_{j=1}^{n} \frac{\sin \pi(\alpha_{j1}\lambda_1 + \cdots + \alpha_{jn}\lambda_n - k_j)}{\pi(\alpha_{j1}\lambda_1 + \cdots + \alpha_{jn}\lambda_n - k_j)} \right| \\ &\leqslant \left(\sum_{|K| > N} \left| \frac{\langle \bar{g}(\cdot), \Psi_K(\cdot) \rangle_{\mathfrak{H}}}{\|\Psi_K(\cdot)\|_{\mathfrak{H}}} \right|^2 \right)^{1/2} \|\Phi(\cdot, \underline{\lambda})\|_{\mathfrak{H}}, \quad \underline{\lambda} \in \mathbb{C}^n. \end{aligned} \tag{2.17}$$

To prove uniform convergence on a subset of $\mathbb{C}^n$, it suffices to show that $\|\Phi(\cdot, \underline{\lambda})\|_{\mathfrak{H}}$ is bounded on this subset. Indeed,

$$\|\Phi(\cdot, \underline{\lambda})\|_{\mathfrak{H}} = \prod_{j=1}^{n} \int_{-\pi}^{\pi} \exp(-2 \operatorname{Im} z_j x_j) \, \mathrm{d}x_j = \prod_{j=1}^{n} \frac{\sinh \pi \operatorname{Im} z_j}{\operatorname{Im} z_j}, \quad \underline{\lambda} \in \mathbb{C}^n, \tag{2.18}$$

where $z_j = \alpha_{j1}\lambda_1 + \cdots + \alpha_{jn}\lambda_n$ and $\operatorname{Im} z$ is the imaginary part of z. Then $\|\Phi(\cdot, \underline{\lambda})\|_{\mathfrak{H}}$ is bounded on compact subsets of $\mathbb{C}^n$, implying the uniform convergence of series (2.10) on compact subsets of $\mathbb{C}^n$. This also proves that $f(\underline{\lambda})$ is holomorphic on compact subsets of $\mathbb{C}^n$ in Weierstrass's sense. Hence f is entire. To prove that $f(\underline{\lambda})$ has exponential type σ_π we first apply the Cauchy–Schwarz inequality to the integral transform (2.8) to obtain

$$|f(\underline{\lambda})| \leqslant (2\pi)^{n/2} \max_{\underline{x} \in \Omega_\pi} |\phi(\underline{x}, \underline{\lambda})| \|g(\cdot)\|_{\mathfrak{H}}, \quad \underline{\lambda} \in \mathbb{C}^n. \tag{2.19}$$

Now we prove that

$$\max_{\underline{x} \in \Omega_\pi} |\phi(\underline{x}, \underline{\lambda})| \leqslant \exp\left\{ \sum_{k=1}^{n} \left(\sum_{j=1}^{n} |\alpha_{jk}|\pi \right) |\lambda_k| \right\}, \quad \underline{\lambda} \in \mathbb{C}^n. \tag{2.20}$$

Indeed, suppose that $\phi(\underline{x}, \underline{\lambda}) \neq 0$. Since $|\exp(z)| \leqslant \exp|z|$ for all $z \in \mathbb{C}$, then

$$|\phi(\underline{x}, \underline{\lambda})| \leqslant \prod_{j=1}^{n} \exp(|\alpha_{j1}||\lambda_1|\pi + \cdots + |\alpha_{jn}||\lambda_{jn}|\pi).$$

Taking the logarithm and collecting similar terms we obtain

$$\ln |\phi(\underline{x}, \underline{\lambda})| \leqslant \sum_{k=1}^{n} \left(\sum_{j=1}^{n} |\alpha_{jk}|\pi \right) |\lambda_k|,$$

which leads to inequality (2.20). The last inequality together with (2.19) proves that $f(\underline{\lambda})$ has exponential type σ_π. It remains to prove uniform convergence on the subset of $\mathbb{C}^n$ where $A\underline{\lambda} \in \mathbb{R}^n$. This is clear because in this case

$$\|\Phi(\cdot,\underline{\lambda})\|_{\mathfrak{H}}^2 = \prod_{j=1}^{n} \int_{-\pi}^{\pi} \exp(-2\operatorname{Im} z_j x_j)\,\mathrm{d}x_j = (2\pi)^n. \tag{2.21}$$

This completes the proof of Theorem 2.1. □

Remark 2.2. Under restrictive conditions, Prosser obtained a sampling formula of the type (2.10) [**32**, Equation (13)]. From the Paley–Wiener theorem, Theorem B above, the multivariate transform (2.8) can be written as the several-variable Fourier transform

$$f(\underline{\lambda}) = \frac{1}{(2\pi)^{n/2}} \int_{\Delta_{\sigma_\pi}} \psi(\underline{x}) \exp(\mathrm{i}\underline{x}\cdot\underline{\lambda})\,\mathrm{d}\underline{x}, \quad \psi(\underline{x}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\underline{\lambda}) \exp(-\mathrm{i}\underline{x}\cdot\underline{\lambda})\,\mathrm{d}\underline{\lambda}. \tag{2.22}$$

This leads to a sampling representation of $f(\underline{\lambda})$ of the form

$$f(\underline{\lambda}) = \sum_{K\in\mathbb{Z}} f\left(\frac{k_1\pi}{\sigma_1},\dots,\frac{k_n\pi}{\sigma_n}\right) \prod_{j=1}^{n} \frac{\sin(\sigma_j\lambda_j - k_j\pi)}{(\sigma_j\lambda_j - k_j\pi)}, \tag{2.23}$$

where $\sigma_j := \sum_{k=1}^{n} |\alpha_{kj}|\pi$, $j = 1,\dots,n$. As in the classical case, the role of the multi-parameter operators in deriving the previous sampling formulae is not seen here since we could check and compute everything explicitly. This is because of the exceptional situation of first-order problems. In higher-order problems, the derivation of the sampling theorems is impossible without the use of the theory of differential operators.

3. A two-parameter system

In this section we introduce a two-parameter system of second-order Sturm–Liouville problems. This system is studied by Faierman [**19**–**22**] and Sleeman [**35**, **36**] (see also [**6**, **33**]). We state the main results needed for the derivation of the sampling theorems, in particular, the discreteness of the eigenvalues and the completeness of the eigenfunctions. Consider the two-parameter system

$$-y''(x) + q(x)y(x) = (\lambda+\mu)y(x), \qquad 0 \leqslant x \leqslant \pi, \tag{3.1}$$

$$U_1(y) := y(0)\cos\alpha - y'(0)\sin\alpha = 0, \qquad 0 \leqslant \alpha < \pi, \tag{3.2}$$

$$U_2(y) := y(\pi)\cos\beta - y'(\pi)\sin\beta = 0, \qquad 0 \leqslant \beta < \pi, \tag{3.3}$$

and

$$-z''(t) + p(t)z(t) = (\lambda-\mu)z(t), \qquad 0 \leqslant t \leqslant \pi, \tag{3.4}$$

$$V_1(z) := z(0)\cos\gamma - z'(0)\sin\gamma = 0, \qquad 0 \leqslant \gamma < \pi, \tag{3.5}$$

$$V_2(z) := z(\pi)\cos\delta - z'(\pi)\sin\delta = 0, \qquad 0 \leqslant \delta < \pi. \tag{3.6}$$

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Here $\lambda, \mu \in \mathbb{C}$ are the eigenvalue parameters and $q(x)$, $p(t)$ are continuous real-valued functions on $[0, \pi]$. A complex pair $\Lambda := (\lambda, \mu)$ is called an eigenvalue of the system (3.1)–(3.6) if there are non-trivial solutions $y(x, \Lambda) := y(x, \lambda, \mu)$ of (3.1) which satisfy (3.2), (3.3) and $z(t, \Lambda) := z(t, \lambda, \mu)$ of (3.4) which satisfy (3.5), (3.6). In this case the product

$$\Psi(x, t, \Lambda) := \Psi(x, t, \lambda, \mu) = y(x, \Lambda) z(t, \Lambda), \tag{3.7}$$

is an eigenfunction of the two-parameter system (3.1)–(3.6) corresponding to the eigenvalue Λ. Let $\Omega_0 := [0, \pi] \times [0, \pi]$ and $\mathcal{H} := L_2(\Omega_0)$ denote the usual $L_2(\Omega_0)$ space of Lebesgue measurable functions on Ω_0 which are square integrable with the following inner product and norm:

$$\langle f, g \rangle_{\mathcal{H}} := \int_0^\pi \int_0^\pi f(x, t) \bar{g}(x, t) \, \mathrm{d}x \, \mathrm{d}t, \qquad \|f\|_{\mathcal{H}} := \left(\int_0^\pi \int_0^\pi |f(x, t)|^2 \, \mathrm{d}x \, \mathrm{d}t \right)^{1/2}. \tag{3.8}$$

Let $\varphi_i(x, \lambda, \mu) = \varphi_i(x, \lambda + \mu)$ and $\chi_i(t, \lambda, \mu) = \chi_i(t, \lambda - \mu)$, $i = 1, 2$, denote the solutions of (3.1) and (3.4) respectively which satisfy the initial conditions

$$\varphi_1(0, \lambda + \mu) = \sin \alpha, \qquad \varphi_1'(0, \lambda + \mu) = \cos \alpha; \tag{3.9}$$

$$\varphi_2(\pi, \lambda + \mu) = \sin \beta, \qquad \varphi_2'(\pi, \lambda + \mu) = \cos \beta; \tag{3.10}$$

$$\chi_1(0, \lambda - \mu) = \sin \gamma, \qquad \chi_1'(0, \lambda - \mu) = \cos \gamma; \tag{3.11}$$

$$\chi_2(\pi, \lambda - \mu) = \sin \delta, \qquad \chi_2'(\pi, \lambda - \mu) = \cos \delta. \tag{3.12}$$

Thus, $\varphi_i(x, \lambda + \mu)$ satisfy (3.1), $\chi_i(t, \lambda - \mu)$ satisfy (3.4) and

$$U_i(\varphi_i) = 0, \qquad V_i(\chi_i) = 0, \quad i = 1, 2, \quad \text{for all } (\lambda, \mu) \in \mathbb{C}^2. \tag{3.13}$$

To find the eigenvalues and the eigenfunctions of the system (3.1)–(3.6) we have four choices.

(i) The eigenvalues are the solutions of the system

$$U_2(\varphi_1) = 0, \qquad V_2(\chi_1) = 0, \tag{3.14}$$

and the corresponding eigenfunctions are

$$\Psi(x, t, \lambda, \mu) = \varphi_1(x, \lambda + \mu) \chi_1(t, \lambda - \mu). \tag{3.15}$$

(ii) The eigenvalues are the solutions of the system

$$U_2(\varphi_1) = 0, \qquad V_1(\chi_2) = 0, \tag{3.16}$$

and the corresponding eigenfunctions are

$$\Psi(x, t, \lambda, \mu) = \varphi_1(x, \lambda + \mu) \chi_2(t, \lambda - \mu). \tag{3.17}$$

(iii) The eigenvalues are the solutions of the system

$$U_1(\varphi_2) = 0, \qquad V_2(\chi_1) = 0, \tag{3.18}$$

and the corresponding eigenfunctions are

$$\Psi(x, t, \lambda, \mu) = \varphi_2(x, \lambda + \mu)\chi_1(t, \lambda - \mu). \tag{3.19}$$

(iv) The eigenvalues are the solutions of the system

$$U_1(\varphi_2) = 0, \qquad V_1(\chi_2) = 0, \tag{3.20}$$

and the corresponding eigenfunctions are

$$\Psi(x, t, \lambda, \mu) = \varphi_2(x, \lambda + \mu)\chi_2(t, \lambda - \mu). \tag{3.21}$$

It should be noted that the eigenvalues are the same in every case and the corresponding eigenfunctions are unique up to a multiplicative constant. Moreover, the following facts concerning the eigenvalues and the eigenfunctions of system (3.1)–(3.6) hold (cf. [**20**,**35**]).

Theorem C. *The eigenvalues of the system (3.1)–(3.6) form a denumerable set in* $\mathbb{R}^2$ *with no finite limit points. Eigenfunctions corresponding to different eigenvalues are orthogonal. The totality of all eigenfunctions is an orthogonal basis of* $\mathcal{H}$.

4. The sampling theorems

This section includes four different sampling formulae associated with the multiparameter system (3.1)–(3.6). Classifications will be according to whether there are eigenvalues of the two-parameter system (3.1)–(3.6) of the form $\Lambda = (\lambda, \mu)$, where $\lambda \neq \pm\mu$; $\lambda = \mu$ but $\lambda \neq -\mu$ (i.e. when there are eigenvalues of the form (μ, μ) but there are no eigenvalues of the form $(\lambda, -\lambda)$); $\lambda = -\mu$ but $\lambda \neq \mu$ and finally when $\lambda = \pm\mu$. Let $\{\Lambda_{nm}\}_{n,m=1}^{\infty}$ denote the sequence of all eigenvalues of the system (3.1)–(3.6) for which $\lambda \neq \pm\mu$. For convenience, let ω_{ij}, θ_{ij}, Ψ_{ij} be

$$\omega_{ij}(\lambda, \mu) = \omega_{ij}(\lambda + \mu) := U_i(\varphi_j), \qquad 1 \leqslant i \neq j \leqslant 2, \tag{4.1}$$

$$\theta_{ij}(\lambda, \mu) = \theta_{ij}(\lambda - \mu) := V_i(\chi_j), \qquad 1 \leqslant i \neq j \leqslant 2, \tag{4.2}$$

$$\Psi_{ij}(x, t, \lambda, \mu) := \phi_i(x, \lambda, \mu)\chi_j(t, \lambda, \mu), \qquad 1 \leqslant i, j \leqslant 2. \tag{4.3}$$

The first sampling theorem of this paper is the following.

Theorem 4.1. *Assume that system (3.1)–(3.6) has no eigenvalues of the form* $\Lambda = (\lambda, \mu), \lambda = \pm\mu$. *Let* $g(x, t) \in L_2(\Omega_0)$. *Let* $f_{11}(\lambda, \mu)$ *be the transform*

$$f_{11}(\lambda, \mu) = \int_0^{\pi}\int_0^{\pi} g(x, t)\Psi_{11}(x, t, \lambda, \mu)\,\mathrm{d}x\,\mathrm{d}t. \tag{4.4}$$

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Then $f_{11}(\lambda,\mu)$ is an entire function of order $\frac{1}{2}$ and type $\sigma_0 := (2\pi, 2\pi)$ that admits the sampling representation

$$f_{11}(\lambda,\mu) = \sum_{m,n=1}^{\infty} f_{11}(\lambda_{mn},\mu_{mn}) \frac{\omega_{21}(\lambda+\mu)}{(\lambda+\mu-(\lambda_{mn}+\mu_{mn}))\omega_{21}'(\lambda_{mn}+\mu_{mn})} \times \frac{\theta_{21}(\lambda-\mu)}{(\lambda-\mu-(\lambda_{mn}-\mu_{mn}))\theta_{21}'(\lambda_{mn}-\mu_{mn})}, \quad (4.5)$$

where the derivative of ω_{21} is with respect to $\lambda+\mu$ and that of θ_{21} is with respect to $\lambda-\mu$. The sampling series (4.5) converges absolutely on $\mathbb{C}^2$ and uniformly on compact subsets of $\mathbb{C}^2$.

Proof. Since $\{\Psi_{11}(x,t,\lambda_{mn},\mu_{mn})\}_{m,n=1}^{\infty}$ is an orthogonal basis of $\mathcal{H}$, then applying Parseval's identity to (4.4) implies

$$f_{11}(\lambda,\mu) = \sum_{m,n}^{\infty} \overline{\hat{\bar{g}}_{11}(m,n)} \frac{\hat{\Psi}_{11}(m,n)}{\|\Psi_{11}(x,t,\lambda_{mn},\mu_{mn})\|_{\mathcal{H}}^2}, \quad (4.6)$$

where

$$\hat{\bar{g}}_{11}(m,n) = \int_0^{\pi}\int_0^{\pi} \bar{g}(x,t)\overline{\Psi}_{11}(x,t,\lambda_{mn},\mu_{mn})\,\mathrm{d}x\,\mathrm{d}t = \bar{f}_{11}(\lambda_{mn},\mu_{mn}) \quad (4.7)$$

and

$$\hat{\Psi}_{11}(m,n) = \int_0^{\pi}\int_0^{\pi} \Psi_{11}(x,t,\lambda,\mu)\bar{\Psi}_{11}(x,t,\lambda_{mn},\mu_{mn})\,\mathrm{d}x\,\mathrm{d}t = \int_0^{\pi} \varphi_1(x,\lambda+\mu)\bar{\varphi}_1(x,\lambda_{mn}+\mu_{mn})\,\mathrm{d}x \cdot \int_0^{\pi} \chi_1(t,\lambda-\mu)\bar{\chi}_1(t,\lambda_{mn}-\mu_{mn})\,\mathrm{d}t. \quad (4.8)$$

Let $\Lambda = (\lambda,\mu) \in \mathbb{C}^2$ and $m,n \in \mathbb{Z}^+$ such that $\Lambda \neq \Lambda_{mn}$. Using integration by parts and the fact that $\varphi_1(x,\lambda+\mu)$, $\varphi_1(x,\lambda_{mn}+\mu_{mn})$ satisfy (3.1), we obtain

$$(\lambda+\mu-(\lambda_{mn}+\mu_{mn}))\int_0^{\pi} \varphi_1(x,\lambda+\mu)\bar{\varphi}_1(x,\lambda_{mn}+\mu_{mn})\,\mathrm{d}x = [\varphi_1(x,\lambda+\mu)\overline{\varphi'}_1(x,\lambda_{mn}+\mu_{mn}) - \varphi_1'(x,\lambda+\mu)\bar{\varphi}_1(x,\lambda_{mn}+\mu_{mn})]_0^{\pi}, \quad (4.9)$$

where the derivatives in the right-hand side are with respect to x. Substituting from (3.9) in (4.9) leads to

$$(\lambda+\mu-(\lambda_{mn}+\mu_{mn}))\int_0^{\pi} \varphi_1(x,\lambda+\mu)\bar{\varphi}_1(x,\lambda_{mn}+\mu_{mn})\,\mathrm{d}x = [\varphi_1(\pi,\lambda+\mu)\overline{\varphi'}_1(\pi,\lambda_{mn}+\mu_{mn}) - \varphi_1'(\pi,\lambda+\mu)\bar{\varphi}_1(\pi,\lambda_{mn}+\mu_{mn})]. \quad (4.10)$$

Since $\Psi_{11}(x,t,\lambda_{mn},\mu_{mn})$ is an eigenfunction of (3.1)–(3.6), then $\varphi_1(x,\lambda_{mn}+\mu_{mn})$ satisfies (3.3). We distinguish between two cases. First, if $\sin\beta \neq 0$, then substituting from (3.3)

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in (4.10), we obtain

$$(\lambda+\mu-(\lambda_{mn}+\mu_{mn}))\int_0^\pi \varphi_1(x,\lambda+\mu)\bar{\varphi}_1(x,\lambda_{mn}+\mu_{mn})\,\mathrm{d}x = \frac{\bar{\varphi}_1(\pi,\lambda_{mn}+\mu_{mn})}{\sin\beta}\omega_{21}(\lambda+\mu). \tag{4.11}$$

If $\sin\beta = 0$, then $\cos\beta = 1$ and substituting from (3.3) in (4.10) yields

$$(\lambda+\mu-(\lambda_{mn}+\mu_{mn}))\int_0^\pi \varphi_1(x,\lambda+\mu)\bar{\varphi}_1(x,\lambda_{mn}+\mu_{mn})\,\mathrm{d}x = -\overline{\varphi'}_1(\pi,\lambda_{mn}+\mu_{mn})\omega_{21}(\lambda+\mu). \tag{4.12}$$

Similar computations for $\chi(t,\lambda,\mu)$, $\chi(t,\lambda_{mn},\mu_{mn})$ lead to

$$(\lambda-\mu-(\lambda_{mn}-\mu_{mn}))\int_0^\pi \chi_1(t,\lambda-\mu)\bar{\chi}_1(t,\lambda_{mn}-\mu_{mn})\,\mathrm{d}t = \frac{\bar{\chi}_1(\pi,\lambda_{mn}-\mu_{mn})}{\sin\delta}\theta_{21}(\lambda-\mu), \tag{4.13}$$

if $\sin\delta \neq 0$, and

$$(\lambda+\mu-(\lambda_{mn}-\mu_{mn}))\int_0^\pi \chi_1(t,\lambda-\mu)\bar{\chi}_1(t,\lambda_{mn}-\mu_{mn})\,\mathrm{d}t = -\overline{\chi'}_1(\pi,\lambda_{mn}-\mu_{mn})\theta_{21}(\lambda-\mu), \tag{4.14}$$

otherwise. Now we prove that if $\sin\beta \neq 0$, then $\bar{\varphi}_1(\pi,\lambda_{mn}+\mu_{mn}) \neq 0$. Indeed, let $\sin\beta \neq 0$ and $\bar{\varphi}_1(\pi,\lambda_{mn}+\mu_{mn}) = 0$. Since $\Psi_{11}(x,t,\lambda_{mn},\mu_{mn})$ is an eigenfunction, then $\varphi_1(x,\lambda_{mn}+\mu_{mn})$ satisfies (3.3). Therefore, $\varphi_1'(\pi,\lambda_{mn}+\mu_{mn})\sin\beta = 0$. Thus $\varphi_1'(\pi,\lambda_{mn}+\mu_{mn}) = 0$, implying that $\varphi_1(x,\lambda_{mn}+\mu_{mn}) \equiv 0$ on $[0,\pi]$. Hence $\Psi_{11}(x,t,\Lambda_{mn}) \equiv 0$ on Ω_0, contradicting the fact that $\Psi_{11}(x,t,\Lambda_{mn})$ is an eigenfunction. Hence $\varphi_1(\pi,\lambda_{mn},\mu_{mn}) \neq 0$. Similarly $\varphi_1'(\pi,\lambda_{mn}+\mu_{mn}) \neq 0$ when $\sin\beta = 0$; $\chi_1(\pi,\lambda_{mn}-\mu_{mn}) \neq 0$ if $\sin\delta \neq 0$ and $\chi_1'(\pi,\lambda_{mn}-\mu_{mn}) \neq 0$ if $\sin\delta = 0$. To compute $\hat{\Psi}_{11}(m,n)/\|\Psi_{11}(x,t,\lambda_{mn},\mu_{mn})\|^2_{\mathcal{H}}$, we start with the case $\sin\beta \neq 0 \neq \sin\delta$. From (4.11) and (4.13), we obtain

$$\int_0^\pi |\varphi_1(x,\lambda_{mn}+\mu_{mn})|^2\,\mathrm{d}x = \frac{1}{\sin\beta}\bar{\varphi}_1(\pi,\lambda_{mn}+\mu_{mn})\omega_{21}'(\lambda_{mn}+\mu_{mn}), \tag{4.15}$$

$$\int_0^\pi |\chi_1(x,\lambda_{mn}-\mu_{mn})|^2\,\mathrm{d}t = \frac{1}{\sin\delta}\bar{\chi}_1(\pi,\lambda_{mn}-\mu_{mn})\theta_{21}'(\lambda_{mn}-\mu_{mn}), \tag{4.16}$$

where $\omega_{21}'(\lambda_{mn}+\mu_{mn})$ and $\theta_{21}'(\lambda_{mn}-\mu_{mn})$ are the derivatives of $\omega_{21}(\lambda+\mu)$ with respect to $\lambda+\mu$ at $\lambda_{mn}+\mu_{mn}$ and of $\theta_{21}(\lambda-\mu)$ with respect to $\lambda-\mu$ at $\lambda_{mn}-\mu_{mn}$. From (4.11), (4.13), (4.15) and (4.16), we obtain, when $\beta \neq 0 \neq \delta$,

$$\frac{\hat{\Psi}_{11}(m,n)}{\|\Psi_{11}(x,t,\lambda_{mn},\mu_{mn})\|^2_{\mathcal{H}}} = \frac{\omega_{21}(\lambda+\mu)}{((\lambda+\mu)-(\lambda_{mn}+\mu_{mn}))\omega_{21}'(\lambda_{mn}+\mu_{mn})} \times \frac{\theta_{21}(\lambda,\mu)}{((\lambda-\mu)-(\lambda_{mn}-\mu_{mn}))\theta_{21-}'(\lambda_{mn}-\mu_{mn})}. \tag{4.17}$$

Similarly (4.17) holds for the other choices of β, δ. Since $(\lambda,\mu) \in \mathbb{C}^2$ and m, n are arbitrary, provided that $(\lambda,\mu) \neq (\lambda_{mn},\mu_{mn})$, the combination (4.6), (4.7) and (4.17) leads to the desired sampling representation for $f_{11}(\lambda,\mu)$ when $(\lambda,\mu) \in \mathbb{C}^2$, $(\lambda,\mu) \neq (\lambda_{mn},\mu_{mn})$

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for all $m, n \in \mathbb{Z}^+$ and the convergence is pointwise. The proof in case $(\lambda, \mu) = (\lambda_{mn}, \mu_{mn})$ is trivial. The proof of the absolute convergence on $\mathbb{C}^2$ can be established as in § 2 above. As for the proof of the uniform convergence on compact subsets of $\mathbb{C}^2$, let $M \subset \mathbb{C}^2$ be compact and $N \in \mathbb{Z}^+$. Define $S_N(\Lambda) = S_N(\lambda, \mu)$ to be

$$S_N(\lambda,\mu) := \biggl| f_{11}(\lambda,\mu) - \sum_{m,n \leqslant N} f_{11}(\lambda_{mn},\mu_{mn}) \frac{\omega_{21}(\lambda+\mu)}{(\lambda+\mu-(\lambda_{mn}+\mu_{mn}))\omega_{21}'(\lambda_{mn}+\mu_{mn})} \times \frac{\theta_{21}(\lambda-\mu)}{(\lambda+\mu-(\lambda_{mn}-\mu_{mn}))\theta_{21}'(\lambda_{mn}-\mu_{mn})} \biggr|, \quad (\lambda,\mu) \in M. \tag{4.18}$$

Using the Cauchy–Schwarz and Bessel's inequalities we obtain

$$S_N(\lambda,\mu) \leqslant \|\Psi_{11}(x,t,\lambda,\mu)\|_{\mathcal{H}} \cdot \biggl(\sum_{m,n>N} \biggl| \frac{\hat{g}(n,m)}{\|\Psi_{11}(x,t,\lambda_{mn},\mu_{mn})\|_{\mathcal{H}}} \biggr|^2 \biggr)^{1/2}, \quad (\lambda,\mu) \in M. \tag{4.19}$$

To prove uniform convergence on M, it is sufficient to show that $\|\Psi_{11}(x,t,\lambda,\mu)\|_{\mathcal{H}}$ is bounded on M. Indeed,

$$\|\Psi_{11}(x,t,\lambda,\mu)\|_{\mathcal{H}}^2 = \|\varphi_1(x,\lambda+\mu)\|_{L_2(0,\pi)}^2 \|\chi_1(t,\lambda-\mu)\|_{L_2(0,\pi)}^2, \quad (\lambda,\mu) \in M. \tag{4.20}$$

Using a result of [**12**, p. 225] we can find positive constants $C_1(M)$ and $C_2(M)$ which depend only on M such that

$$\|\varphi_1(x,\lambda+\mu)\|_{L_2(0,\pi)}^2 \leqslant C_1(M), \quad \|\chi_1(t,\lambda-\mu)\|_{L_2(0,\pi)}^2 \leqslant C_2(M), \quad (\lambda,\mu) \in M. \tag{4.21}$$

The last inequalities complete the proof of uniform convergence on M. From the uniform convergence on compact subsets of $\mathbb{C}^2$, $f_{11}(\lambda, \mu)$ is entire. Now we prove that $f_{11}(\lambda, \mu)$ is of order $\frac{1}{2}$ and type σ_0. First, applying the Cauchy–Schwarz inequality to the integral transform (4.4) we obtain

$$|f_{11}(\lambda,\mu)| \leqslant \pi \max_{(x,t)\in\Omega_0} |\Psi_{11}(x,t,\lambda,\mu)| \|g(\cdot)\|_{\mathcal{H}}, \quad (\lambda,\mu) \in \mathbb{C}^2. \tag{4.22}$$

Using the method of variation of constants, we obtain

$$\varphi_1(x,\lambda+\mu) = \sin\alpha\cos(\sqrt{\lambda+\mu}x) + \cos\alpha \frac{\sin(\sqrt{\lambda+\mu}x)}{\sqrt{\lambda+\mu}} + \frac{1}{\sqrt{\lambda+\mu}} \int_0^x \sin(\sqrt{\lambda+\mu}(x-\xi))\varphi_1(\xi,\lambda+\mu)q(\xi)\,\mathrm{d}\xi. \tag{4.23}$$

Applying the same technique of [**13**, Chapter 5] (see also [**25**]), we have for large $|\lambda|$ and large $|\mu|$ the following asymptotic formula:

$$|\varphi_1(x,\lambda+\mu)| \leqslant |\sin\alpha||\cos(\sqrt{\lambda+\mu}x)| + |\cos\alpha| \biggl| \frac{\sin(\sqrt{\lambda+\mu}x)}{\sqrt{\lambda+\mu}} \biggr| + O\biggl(\frac{\exp(|\sqrt{\lambda+\mu}|x)}{|\sqrt{\lambda+\mu}|} \biggr) \tag{4.24}$$

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uniformly for $x \in [0, \pi]$. Let $R > 1$ be a sufficiently large positive number such (4.24) is satisfied for all $(\lambda, \mu) \in \mathbb{C}^2$, where $|\lambda| \geqslant R$, $|\mu| \geqslant R$. Using the inequalities

$$|\cos z| \leqslant \mathrm{e}^{|z|}, \quad |\sin z| \leqslant \mathrm{e}^{|z|}, \quad |\sqrt{\lambda + \mu}| \leqslant \sqrt{|\lambda|} + \sqrt{|\mu|}, \tag{4.25}$$

we can find a positive constant A_R such that

$$|\varphi_1(x, \lambda + \mu)| \leqslant A_R \exp((\sqrt{|\lambda|} + \sqrt{|\mu|})\pi), \quad |\lambda|, |\mu| \geqslant R. \tag{4.26}$$

Similarly, there is a positive constant B_R which is independent of t, λ, μ, for which

$$\max_{0 \leqslant t \leqslant \pi} |\chi_1(t, \lambda - \mu)| \leqslant B_R \exp((\sqrt{|\lambda|} + \sqrt{|\mu|})\pi), \quad |\lambda|, |\mu| \geqslant R. \tag{4.27}$$

Combining the last two inequalities together with (4.19), f_{11} eventually has order $\frac{1}{2}$ and type σ_0, which suffices to accomplish the proof. □

Similar results hold for the transforms

$$f_{ij}(\lambda, \mu) = \int_0^\pi \int_0^\pi g(x, t) \Psi_{ij}(x, t, \lambda, \mu) \,\mathrm{d}x \,\mathrm{d}t, \quad g(x, t) \in L_2(\Omega_0), \ 1 \leqslant i, j \leqslant 2, \ i + j > 2. \tag{4.28}$$

In the following we discuss the other three cases. We consider the boundary-value problems:

$$-y''(x) + q(x)y(x) = 0, \qquad U_1(y) = U_2(y) = 0; \tag{4.29}$$

$$-z''(t) + p(t)z(t) = 0, \qquad V_1(z) = V_2(z) = 0. \tag{4.30}$$

The remaining three cases are when (4.29) has a non-trivial solution but (4.30) does not, the converse situation and finally when both problems have non-trivial solutions. Let us consider the first case. Thus for $(\lambda, \mu) \in \mathbb{C}^2$, $\mu = -\lambda$ problem (3.1)–(3.3) has a non-trivial solution, $\varphi_0(x)$ say. Then, for such points, (3.5), (3.6) become

$$-z''(t) + p(t)z(t) = 2\lambda z(t), \qquad V_1(z) = V_2(z) = 0. \tag{4.31}$$

From Sturm–Liouville's theory (cf. [**13**, **25**]), problem (4.31) has a sequence of real eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$, which is bounded below and has no finite limit points. Moreover, by assumptions $\lambda_n \neq 0$ for all n. The sequence λ_n is the set of the solutions of

$$\theta_{21}(\lambda - \mu) = \theta_{21}(2\lambda) \quad \text{or} \quad \theta_{12}(\lambda - \mu) = \theta_{12}(2\lambda). \tag{4.32}$$

In this situation system (3.1)–(3.6) will have a sequence of eigenvalues $\Lambda_n = (\lambda_n, -\lambda_n)$ and the corresponding eigenfunctions are

$$\left.\begin{aligned} \Psi_{01}(x, t, \lambda_n, -\lambda_n) &:= \varphi_0(x)\chi_1(t, \lambda_n, -\lambda_n) \\ \text{or} \quad \Psi_{02}(x, t, \lambda_n, -\lambda_n) &:= \varphi_0(x)\chi_2(t, \lambda_n, -\lambda_n). \end{aligned}\right\} \tag{4.33}$$

In addition to this sequence we will have the sequence of eigenvalues $\Lambda_{mn} = (\lambda_{mn}, \mu_{mn})$ determined before as well as the corresponding eigenfunctions. The sampling result in this case will be as follows.

Theorem 4.2. *Suppose that problem (4.29) has a non-trivial solution and problem (4.30) has only the trivial solution. Then the two-variable integral transform*

$$f_{11}(\lambda,\mu)=\int_0^\pi\int_0^\pi g(x,t)\Psi_{11}(x,t,\lambda,\mu)\,\mathrm{d}x\,\mathrm{d}t,\quad g(x,t)\in L_2(\Omega_0). \tag{4.34}$$

is an entire function of order $\frac{1}{2}$ and type σ_0. It admits the two-variable sampling expansion

$$\begin{aligned}f_{11}(\lambda,\mu)=\frac{\omega_{21}(\lambda+\mu)}{(\lambda+\mu)\omega'_{21}(0)}\sum_{n=1}^{\infty}f_{11}(\lambda_n,-\lambda_n)\frac{\theta_{21}(\lambda-\mu)}{(\lambda+\mu-2\lambda_n)\theta'_{21}(2\lambda_n)}\\ +\sum_{m,n=1}^{\infty}f_{11}(\lambda_{mn},\mu_{mn})\frac{\omega_{21}(\lambda+\mu)}{(\lambda+\mu-(\lambda_{mn},\mu_{mn}))\omega'_{21}(\lambda_{mn}+\mu_{mn})}\\ \times\frac{\theta_{21}(\lambda-\mu)}{(\lambda-\mu-(\lambda_{mn}-\mu_{mn}))\theta'_{21}(\lambda_{mn}-\mu_{mn})}.\end{aligned} \tag{4.35}$$

The sampling series (4.35) converges absolutely on $\mathbb{C}^2$ and uniformly on compact subsets of $\mathbb{C}^2$. Similar results hold for transforms (4.28).

Proof. Since the only difference between (4.35) and (4.5) is the first single-variable sum of (4.35), we only indicate how to get this sum and the rest of the proof will be as that of Theorem 4.1 above. Indeed, applying Parseval's relation on (4.34), we get

$$f_{11}(\lambda,\mu)=\sum_{n=1}^{\infty}\overline{\hat{\bar{g}}(n,n)}\frac{\hat{\Psi}_{11}(n,n)}{\|\Psi_{11}(x,t,\lambda_n,-\lambda_n)\|^2_{\mathcal{H}}}+\sum_{m,n=1}^{\infty}\overline{\hat{\bar{g}}(m,n)}\frac{\hat{\Psi}_{11}(m,n)}{\|\Psi_{11}(x,t,\lambda_{mn},\mu_{mn})\|^2_{\mathcal{H}}}, \tag{4.36}$$

where, as in Theorem 4.1,

$$\hat{\bar{g}}(n,n)=\bar{f}_{11}(\lambda_n,-\lambda_n),\qquad \hat{\bar{g}}(n,m)=\bar{f}_{11}(\lambda_{mn},\mu_{mn}), \tag{4.37}$$

and

$$\hat{\Psi}_{11}(n,n)=\int_0^\pi\int_0^\pi \Psi_{11}(x,t,\lambda,\mu)\bar{\Psi}_{11}(x,t,\lambda_n,-\lambda_n)\,\mathrm{d}x\,\mathrm{d}t, \tag{4.38}$$

and $\hat{\Psi}_{11}(m,n)$ is given in (4.8) above. As we have indicated the double term of (4.36) is nothing but that of (4.35). It remains to compute the single-variable sum of (4.35). For $n\in\mathbb{Z}^+$, $(\lambda,\mu)\in\mathbb{C}^2$, we have

$$\begin{aligned}\hat{\Psi}_{11}(n,n)&=\int_0^\pi\int_0^\pi \Psi_{11}(x,t,\lambda,\mu)\bar{\Psi}_{11}(x,t,\lambda_n,-\lambda_n)\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_0^\pi \varphi_1(x,\lambda+\mu)\bar{\varphi}_1(x,0,0)\,\mathrm{d}x\int_0^\pi \chi_1(t,\lambda-\mu)\bar{\chi}_1(t,\lambda_n,-\lambda_n)\,\mathrm{d}t.\end{aligned} \tag{4.39}$$

and

$$\|\Psi_{11}(x,t,\lambda_n,-\lambda_n)\|^2_{\mathcal{H}}=\|\varphi_1(x,0,0)\|^2_{L_2(0,\pi)}\|\chi_1(t,\lambda_n,-\lambda_n)\|^2_{L_2(0,\pi)}. \tag{4.40}$$

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Using the same technique employed in proving Theorem 4.1, we obtain

$$\frac{\hat{\Psi}_{11}(n,n)}{\|\Psi_{11}(x,t,\lambda_n,-\lambda_n)\|^2_{\mathcal{H}}} = \frac{\omega_{21}(\lambda+\mu)}{(\lambda+\mu)\omega'_{21}(0)}\frac{\theta_{21}(\lambda-\mu)}{((\lambda-\mu)-(2\lambda_n))\theta'_{21}(2\lambda_n)}. \tag{4.41}$$

□

It should be noted that although expansion (4.35) contains two series it can be written as one sum. This might be done by rearranging the eigenvalues and the eigenfunctions in one sequence and then applying Parseval's relation with respect to all eigenfunctions. The same for expansions (4.44) and (4.46) below. The second case is similar to the first, but here problem (4.29) has only the trivial solution and problem (4.30) has a non-trivial solution $\chi_0(t)$. In this case for all $\Lambda = (\lambda,\mu) \in \mathbb{C}^2$, $\lambda = \mu$, problem (3.4)–(3.6) has a non-trivial solution, namely $\chi_0(t)$. So, if $\lambda = \mu$, (3.1)–(3.3) is the single-parameter Sturm–Liouville problem

$$-y'' + q(x)y = 2\mu y, \qquad U_1(y) = U_2(y) = 0. \tag{4.42}$$

From Sturm–Liouville's theory, (4.42) has a sequence of real eigenvalues $\{\mu_n\}_{n=1}^{\infty}$, where $\{\mu_n\}$ is bounded below with no finite limit points and $\mu_n \neq 0$. Hence $\Lambda_n = (\mu_n,\mu_n)$ is a sequence of eigenvalues of the two-parameter system (3.1)–(3.6) with the eigenfunctions

$$\Psi_{10}(x,t,\mu_n,\mu_n) := \varphi_1(x,\mu_n,\mu_n)\chi_0(t) \quad \text{or} \quad \Psi_{20}(x,t,\mu_n,\mu_n) := \varphi_2(x,\mu_n,\mu_n)\chi_0(t). \tag{4.43}$$

Theorem 4.3. *Assume that problem (4.29) has only the trivial solution and problem (4.30) has a non-trivial solution. Let $g(x,t) \in \mathcal{H}$ and*

$$f_{11}(\lambda,\mu) = \int_0^\pi\int_0^\pi g(x,t)\Psi_{11}(x,t,\lambda,\mu)\,\mathrm{d}x\,\mathrm{d}t. \tag{4.44}$$

Then $f_{11}(\lambda,\mu)$ is entire of order $\frac{1}{2}$ and type σ_0 and it can be recovered via the sampling series

$$\begin{aligned} f_{11}(\lambda,\mu) = {}& \frac{\theta_{21}(\lambda-\mu)}{(\lambda-\mu)\theta'_{21}(0)}\sum_{n=1}^{\infty} f_{11}(\mu_n,\mu_n)\frac{\omega_{21}(\lambda+\mu)}{(\lambda+\mu-2\mu_n)\omega'_{21}(2\mu_n)} \\ & + \sum_{m,n=1}^{\infty} f_{11}(\lambda_{mn},\mu_{mn})\frac{\omega_{21}(\lambda+\mu)}{(\lambda+\mu-(\lambda_{mn},\mu_{mn}))\omega'_{21}(\lambda_{mn}+\mu_{mn})} \\ & \times \frac{\theta_{21}(\lambda-\mu)}{(\lambda-\mu-(\lambda_{mn}-\mu_{mn}))\theta'_{21}(\lambda_{mn}-\mu_{mn})}. \end{aligned} \tag{4.45}$$

The sampling expansion (4.45) converges absolutely on $\mathbb{C}^2$ and uniformly on compact subsets of $\mathbb{C}^2$. Similar results hold for (4.28).

It remains to discuss the situation when both (4.29) and (4.30) have non-trivial solutions. In this case if λ_n, μ_n have the above meanings, the sampling theorem associated with the system (3.1)–(3.6) will be the following.

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Theorem 4.4. *Suppose that problem (4.29), (4.30) has non-trivial solutions. Then the transform*

$$f_{11}(\lambda,\mu)=\int_0^\pi\int_0^\pi g(x,t)\Psi_{11}(x,t,\lambda,\mu)\,\mathrm{d}x\,\mathrm{d}t,\quad g(x,t)\in\mathcal{H},\tag{4.46}$$

is entire in $\Lambda=(\lambda,\mu)$ of order $\frac{1}{2}$ and type σ_0. It can be reconstructed via the interpolation form

$$\begin{aligned}f_{11}(\lambda,\mu)&=f_{11}(0,0)\frac{\omega_{21}(\lambda+\mu)}{(\lambda+\mu)\omega_{21}'(0)}\frac{\theta_{21}(\lambda-\mu)}{(\lambda-\mu)\theta_{21}'(0)}\\&\quad+\frac{\omega_{21}(\lambda+\mu)}{(\lambda+\mu)\omega_{21}'(0)}\sum_{n=1}^{\infty}f_{11}(\lambda_n,-\lambda_n)\frac{\theta_{21}(\lambda-\mu)}{(\lambda-\mu-(2\lambda_n))\theta_{21}'(2\lambda_n)}\\&\quad+\frac{\theta_{21}(\lambda-\mu)}{(\lambda-\mu)\theta_{21}'(0)}\sum_{n=1}^{\infty}f_{11}(\mu_n,\mu_n)\frac{\omega_{21}(\lambda+\mu)}{(\lambda+\mu-2\mu_n)\omega_{21}'(2\mu_n)}\\&\quad+\sum_{m,n=1}^{\infty}f_{11}(\lambda_{mn},\mu_{mn})\frac{\omega_{21}(\lambda+\mu)}{(\lambda+\mu-(\lambda_{mn}+\mu_{mn}))\omega_{21}'(\lambda_{mn}+\mu_{mn})}\\&\qquad\qquad\times\frac{\theta_{21}(\lambda-\mu)}{(\lambda-\mu-(\lambda_{mn}-\mu_{mn}))\theta_{21}'(\lambda_{mn}-\mu_{mn})}.\end{aligned}\tag{4.47}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$ are the non-zero eigenvalues of problems (4.31) and (4.42), respectively. The sampling representation (4.47) converges absolutely on $\mathbb{C}^2$ and uniformly on compact subsets of $\mathbb{C}^2$. Similar results hold for the transforms (4.28).

5. Examples

In this section we introduce three examples exhibiting the sampling theorems established above. The first example illustrates Theorem 4.1, when zero is not an eigenvalue of the Sturm–Liouville problems (4.31) and (4.42), the second example is devoted to the case when zero is an eigenvalue of one problem only and the last example is when zero is an eigenvalue of both.

Example 5.1. Consider the system

$$-y''=(\lambda+\mu)y,\qquad y(0)=y(\pi)=0,\tag{5.1}$$

$$-z''=(\lambda-\mu)z,\qquad z(0)=z(\pi)=0.\tag{5.2}$$

In the notation of the above section $q(x)\equiv 0\equiv p(t)$, $\alpha=\beta=\gamma=\delta=0$. Therefore, in the above notation

$$\varphi_1(x,\lambda,\mu)=\frac{\sin\sqrt{\lambda+\mu}x}{\sqrt{\lambda+\mu}},\qquad \varphi_2(x,\lambda,\mu)=\frac{\sin\sqrt{\lambda+\mu}(x-\pi)}{\sqrt{\lambda+\mu}},\tag{5.3}$$

$(\lambda,\mu)\in\mathbb{C}^2$, $\lambda\neq-\mu$. If $\lambda=-\mu$, then

$$\varphi_1(x,\lambda,\mu)=x,\qquad \varphi_2(x,\lambda,\mu)=x-\pi.\tag{5.4}$$

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As for (5.2),

$$\chi_1(t,\lambda,\mu)=\frac{\sin\sqrt{\lambda-\mu}t}{\sqrt{\lambda-\mu}},\qquad \chi_2(t,\lambda,\mu)=\frac{\sin\sqrt{\lambda-\mu}(t-\pi)}{\sqrt{\lambda-\mu}} \tag{5.5}$$

$(\lambda,\mu)\in\mathbb{C}^2$, $\lambda\neq\mu$ and

$$\chi_1(t,\lambda,\mu)=t,\qquad \varphi_2(t,\lambda,\mu)=t-\pi \tag{5.6}$$

otherwise. We first notice that there are no eigenvalues of the form $\Lambda=(\lambda,\mu)$ such that $\lambda=\pm\mu$. Also, in the above notation,

$$\omega_{21}(\lambda+\mu)=-\omega_{12}(\lambda+\mu)=\frac{\sin\sqrt{\lambda+\mu}\pi}{\sqrt{\lambda+\mu}},\qquad \theta_{21}(\lambda-\mu)=-\theta_{12}(\lambda-\mu)=\frac{\sin\sqrt{\lambda-\mu}\pi}{\sqrt{\lambda-\mu}}. \tag{5.7}$$

Hence the eigenvalues are the solutions of the system

$$\lambda+\mu=n^2,\quad \lambda-\mu=m^2,\quad \lambda\neq\pm\mu,\quad n,m\in\mathbb{Z}^+. \tag{5.8}$$

Therefore, the eigenvalues of the system (5.1), (5.2) are

$$\Lambda_{mn}=(\lambda_{mn},\mu_{mn})=\left(\frac{n^2+m^2}{2},\frac{n^2-m^2}{2}\right),\quad n,m\in\mathbb{Z}^+, \tag{5.9}$$

and the corresponding sequence of eigenfunctions is $\{\sin nx/n\times\sin mx/m\}_{m,n=1}^{\infty}$. If we apply Theorem 4.1 to the transform

$$f(\lambda,\mu)=\int_0^{\pi}\int_0^{\pi}g(x,t)\frac{\sin\sqrt{\lambda+\mu}x}{\sqrt{\lambda+\mu}}\frac{\sin\sqrt{\lambda-\mu}t}{\sqrt{\lambda-\mu}}\,\mathrm{d}x\,\mathrm{d}t,\quad g(x,t)\in L_2(\Omega_0), \tag{5.10}$$

then we obtain the following sampling result, $(\lambda,\mu)\in\mathbb{C}^2$,

$$f(\lambda,\mu)=\sum_{n,m=1}^{\infty}f\left(\frac{n^2+m^2}{2},\frac{n^2-m^2}{2}\right)\frac{2n^2\sin\pi(\sqrt{\lambda+\mu}-n)}{\pi\sqrt{\lambda+\mu}(\lambda+\mu-n^2)}\frac{2m^2\sin\pi(\sqrt{\lambda-\mu}-m)}{\pi\sqrt{\lambda-\mu}(\lambda-\mu-m^2)}. \tag{5.11}$$

Example 5.2. Consider the system

$$-y''=(\lambda+\mu)y,\qquad y(0)=y(\pi)=0, \tag{5.12}$$

$$-z''=(\lambda-\mu)z,\qquad z'(0)=z'(\pi)=0. \tag{5.13}$$

In this example $q(x)\equiv 0\equiv p(t)$ on $[0,\pi]$ and $\alpha=\beta=0$, $\gamma=\delta=\pi/2$. In the notation of the previous section, $\varphi_i(x,\lambda+\mu)$ will be as in (5.3), (5.4) above and $\chi_i(t,\lambda-\mu)$ will be

$$\chi_1(t,\lambda-\mu)=\cos\sqrt{\lambda-\mu}t,\qquad \chi_1(t,\lambda-\mu)=\cos\sqrt{\lambda-\mu}(t-\pi), \tag{5.14}$$

if $\lambda\neq\mu$ and $\chi_1(t,\lambda-\mu)=\chi_2(t,\lambda-\mu)=1$ otherwise. Hence

$$\left.\begin{aligned}\omega_{21}(\lambda+\mu)=-\omega_{12}(\lambda+\mu)=\sin\sqrt{\lambda+\mu}\pi/\sqrt{\lambda+\mu},\quad \lambda\neq-\mu,\\ \theta_{21}(\lambda-\mu)=\theta_{21}(\lambda-\mu)=-\sqrt{\lambda-\mu}\sin\sqrt{\lambda-\mu}\pi,\quad \lambda\neq\mu.\end{aligned}\right\} \tag{5.15}$$

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We notice that for all $(\lambda, \mu) \in \mathbb{C}^2$, $\lambda = \mu$, (5.13) has a solution, namely $\chi_0(t) = 1$. In this case Theorem 4.3 is applicable. To compute the eigenvalues, we first need to compute the eigenvalues of the problem

$$-y'' = 2\lambda y, \qquad y(0) = y(\pi) = 0, \tag{5.16}$$

which are denoted by $\{\mu_n\}_{n=1}^{\infty}$ in Theorem 4.3 above. These eigenvalues are $\mu_n = n^2/2$, $n \in \mathbb{Z}^+$. The corresponding eigenfunctions are

$$\varphi_1(x, \mu_n, \mu_n) = \frac{\sin nx}{n}, \quad n \in \mathbb{Z}^+. \tag{5.17}$$

Therefore, $\Lambda_n = (\mu_n, \mu_n) = (n^2/2, n^2/2)$, $n \in \mathbb{Z}^+$, are eigenvalues of the system (5.12), (5.13) with the eigenfunctions

$$\frac{\sin nx}{n}\chi_0(t) = \varphi_1(t, \mu_n, \mu_n)\chi_1(t, \mu_n, \mu_n) = \Psi_{11}(x, t, \mu_n, \mu_n).$$

The rest of the eigenvalues can be determined from

$$\omega_{12}(\lambda + \mu) = \frac{\sin\sqrt{\lambda+\mu}\pi}{\sqrt{\lambda+\mu}} = 0, \quad \theta_{12}(\lambda - \mu) = \sqrt{\lambda-\mu}\sin\sqrt{\lambda-\mu}\pi = 0, \quad \lambda \neq \pm\mu. \tag{5.18}$$

Hence

$$\Lambda_{mn} = \left(\frac{n^2+m^2}{2}, \frac{n^2-m^2}{2}\right)$$

are the rest of the eigenvalues of (5.12), (5.13) with the eigenfunctions $\Psi_{11}(x, t, \lambda_{mn}, \mu_{mn})$, $m, n \in \mathbb{Z}^+$. The sampling result of this case will be following. Let $g(x, t) \in \mathcal{H}$ and

$$f(\lambda, \mu) = \int_0^\pi\int_0^\pi g(x, t)\frac{\sin\sqrt{\lambda+\mu}x}{\sqrt{\lambda+\mu}}\cos\sqrt{\lambda-\mu}t \,\mathrm{d}x\,\mathrm{d}t. \tag{5.19}$$

Then $f(\lambda, \mu)$ admits the sampling series

$$\begin{aligned} f(\lambda, \mu) = \frac{\sin\pi(\sqrt{\lambda-\mu})}{\pi\sqrt{\lambda-\mu}}\sum_{n,m=1}^{\infty} f\left(\frac{n^2}{2}, \frac{n^2}{2}\right)\frac{2n^2\sin\pi(\sqrt{\lambda+\mu}-n)}{\pi\sqrt{\lambda+\mu}(\lambda+\mu-n^2)} \\ + \sum_{n,m=1}^{\infty} f\left(\frac{n^2+m^2}{2}, \frac{n^2-m^2}{2}\right)\frac{2\sqrt{\lambda-\mu}\sin\pi(\sqrt{\lambda-\mu}-m)}{\pi(\lambda-\mu-m^2)} \\ \times\frac{2n^2\sin\pi(\sqrt{\lambda+\mu}-n)}{\pi\sqrt{\lambda+\mu}(\lambda+\mu-n^2)}. \end{aligned} \tag{5.20}$$

The following example exhibits the last sampling result of the previous section.

Example 5.3. Consider the two-parameter system

$$-y''(x) = (\lambda+\mu)y(x), \qquad y'(0) = y'(\pi) = 0, \tag{5.21}$$

$$-z''(t) = (\lambda-\mu)z(t), \qquad z'(0) = z'(\pi) = 0. \tag{5.22}$$

This case coincides with that considered in Theorem 4.4 above. In this example $\chi_1(t,\lambda-\mu)$, $\chi_2(t,\lambda-\mu)$, $\theta_{12}(\lambda-\mu)$ and $\theta_{21}(\lambda-\mu)$ will be as in the previous example, while

$$\varphi_1(x,\lambda+\mu)=\cos\sqrt{\lambda+\mu}x,\qquad \varphi_2(x,\lambda+\mu)=\cos\sqrt{\lambda+\mu}(x-\pi) \tag{5.23}$$

if $\lambda\neq-\mu$, $\varphi_1(x,\lambda+\mu)=\varphi_1(x,\lambda+\mu)=1$ otherwise and

$$\omega_{12}(\lambda+\mu)=\omega_{21}(\lambda+\mu)=-\sqrt{\lambda+\mu}\sin\sqrt{\lambda+\mu}\pi,\quad \lambda\neq-\mu. \tag{5.24}$$

The eigenvalues of system (5.21), (5.22) are the sequences

$$\Lambda_n=\left(\frac{n^2}{2},\frac{n^2}{2}\right),\quad \Lambda_m=\left(\frac{m^2}{2},\frac{-m^2}{2}\right),\quad \Lambda_{mn}=(\lambda_{mn},\mu_{mn})=\left(\frac{n^2+m^2}{2},\frac{n^2-m^2}{2}\right) \tag{5.25}$$

$n,m\in\mathbb{Z}^+$ and $\Lambda_0=(0,0)$. The corresponding sequences of eigenfunctions will be respectively

$$\varphi_1\left(x,\frac{n^2}{2},\frac{n^2}{2}\right),\quad \chi_1\left(t,\frac{m^2}{2},-\frac{m^2}{2}\right),\quad \varphi_1(t,\lambda_{mn},\mu_{mn})\chi_1(t,\lambda_{mn},\mu_{mn}), \tag{5.26}$$

in addition to the eigenfunction 1 corresponding to the eigenvalue $\Lambda_0=(0,0)$. In this case the sampling expansion of the transform

$$f(\lambda,\mu)=\int_0^\pi\int_0^\pi g(x,t)\cos\sqrt{\lambda+\mu}x\cos\sqrt{\lambda-\mu}t\,\mathrm{d}x\,\mathrm{d}t,\quad g(x,t)\in L_2(\Omega_0), \tag{5.27}$$

will be

$$\begin{aligned}f(\lambda,\mu)&=f(0,0)\frac{\sin\pi\sqrt{\lambda+\mu}\sin\pi\sqrt{\lambda-\mu}}{\pi^2\sqrt{\lambda^2-\mu^2}}\\&\quad+\frac{\sin\pi\sqrt{\lambda-\mu}}{\pi\sqrt{\lambda-\mu}}\sum_{n=1}^\infty f\left(\frac{n^2}{2},\frac{n^2}{2}\right)\frac{2\sqrt{\lambda+\mu}\sin\pi(\sqrt{\lambda+\mu}-n)}{\pi(\lambda+\mu-n^2)}\\&\quad+\frac{\sin\pi\sqrt{\lambda+\mu}}{\pi\sqrt{\lambda+\mu}}\sum_{n=1}^\infty f\left(\frac{n^2}{2},\frac{-n^2}{2}\right)\frac{2\sqrt{\lambda-\mu}\sin\pi(\sqrt{\lambda-\mu}-n)}{\pi(\lambda-\mu-n^2)}\\&\quad+\sum_{n,m=1}^\infty f\left(\frac{n^2+m^2}{2},\frac{n^2-m^2}{2}\right)\frac{2\sqrt{\lambda+\mu}\sin\pi(\sqrt{\lambda+\mu}-n)}{\pi(\lambda+\mu-n^2)}\\&\qquad\times\frac{2\sqrt{\lambda-\mu}\sin\pi(\sqrt{\lambda-\mu}-m)}{\pi(\lambda-\mu-m^2)}.\end{aligned} \tag{5.28}$$

In the above examples the eigenvalues—the sampling points—are determined explicitly. It is easy to derive sampling formulae where all eigenvalues cannot be computed explicitly. The results obtained in this article may be extended in several directions. First, when replacing the differential operator by either difference or integral ones as particular cases of the theory developed in [**6**]. In particular the use of Green's function in deriving the

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sampling forms. In another direction, the differential equations (3.1) and (3.4) might be replaced by the general ones of [**19**–**22**, **35**, **36**]. The main issue in this setting is to give concrete examples. Also, the problems when we have differential equations of distinct orders defined on different intervals are interesting. Finally, the derivation of the sampling theory associated with singular multiparameter eigenvalue problems is another possible extension.

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