On the Non-Euclidean Analogues of Tarry's Point.

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§ 1. *Introductory.* 1.1. The point known in elementary geometry by the name of Tarry * was first discussed by that writer as the point of concurrence of the perpendiculars respectively from the vertices of the base-triangle to the corresponding sides of Brocard's first triangle. Tarry's point is the point of the circumcircle diametrically opposite to Steiner's point, which is the fourth common point of the circumcircle and Steiner's circumellipse *

Tarry's point possesses the further property (due to Neuberg) that the three perpendiculars from that point to the sides of the base-triangle meet the three sides of that triangle at nine points which lie three by three in three further lines. Steiner's circumellipse is † the locus of a point which has the similar property that the lines through that point parallel to the sides of the base-triangle meet those sides at six points which lie three by three in two lines; the tangents to the ellipse at the vertices of the base-triangle are therefore respectively parallel to the opposite sides.

1.2. Berkhan states ‡ (without proof) that in non-euclidean geometry there are three points analogous to Tarry's point. It will be proved below that this theorem holds as regards Neuberg's property. For convenience the non-euclidean terminology will be used in the main; but it will be shewn that, as is frequently the case, || this expedient obscures the fact that the properties concerned belong to the general plane cubic, and usually occur in groups of three (*cf.* 3.4, below).

It is known (*v.* * § 17 ||) that in non-euclidean geometry the locus of a point (M) such that the feet (I, J, K) of the perpendiculars from M to the sides ($B_1 C_1$, $C_1 A_1$, $A_1 B_1$) of a given plane triangle are in line is an order-cubic (T_0) containing the

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* Neuberg, *Mathesis* (1) 6 (1886), pp. 5–7.

† Neuberg, *ibid.* (1) 1 (1881), p. 184; (1) 2 (1882), pp. 42–6.

‡ Berkhan, *Arch. d. Math. u. Phys.* (3) 11 (1907), p. 18.

|| *V.* Gabbatt, *Proc. Camb. Phil. Soc.* 21 (1922–3), pp. 297–362. References distinguished by an asterisk [thus : (* 4.1)] are to that paper.

points A_1, B_1, C_1 and the absolute poles (A_2, B_2, C_2) of the lines B_1C_1, C_1A_1, A_1B_1. In § 2 of the present paper are discussed some of the properties of two other order-cubics each containing the six points A_1, B_1, C_1, A_2, B_2, C_2. One of these cubics (U) is the locus of a point M such that the meets of the lines MI, MJ, MK with C_1A_1, A_1B_1, B_1C_1 respectively are in line; and the other (V) is the locus of a point M such that the meets of MI, MJ, MK with A_1B_1, B_1C_1, C_1A_1 respectively are in line. The mutual relations of the cubics U, V (the *by-pedal loci* for the triangle $A_1B_1C_1$) closely resemble the properties of the cubic T_0 (*Cf.* 2.1, 2.2).

In § 3 the analogues of Tarry's point are introduced as the common points (other than A_1, B_1, C_1, A_2, B_2, C_2) of the three cubics T_0, U, V. The place of an analogue of Steiner's circum-ellipse in the theory is also indicated.

§ 2. *The By-Pedal Loci.* 2.1. In non-euclidean geometry, if A_1, B_1, C_1 denote the vertices of any triangle and A_2, B_2, C_2 the absolute poles of the lines B_1C_1, C_1A_1, A_1B_1 respectively; then (* § 17) the locus of a point M_0 such that the lines A_2M_0, B_2M_0, C_2M_0 meet B_1C_1, C_1A_1, A_1B_1 respectively at three points in a line (m) is the order-cubic T_0 on which A_1, A_2; B_1, B_2; C_1, C_2 are pairs of correspondents of the same species. The cubic T_0 is also the locus of a point M_0 such that the lines A_1M_0, B_1M_0, C_1M_0 meet B_2C_2, C_2A_2, A_2B_2 respectively at three points in a line (m'). If m, m' be termed respectively the (1, 2)-and (2, 1)-pedal lines of M_0, and if M_0, M_3 denote correspondents (of the given species) on T_0; then m, m' are also respectively the (2, 1)-and (1, 2)-pedal lines of M_3.

The absolute poles of m, m' are a pair of correspondents (M_1, M_2) of the given species on T_0.

The envelope of the (1, 2)-or of the (2, 1)-pedal lines of the points M_0 of T_0 is a class-cubic, the absolute reciprocal of T_0. (2.11)

2.2. Now let P denote any point such that the lines A_2P, B_2P, C_2P meet C_1A_1, A_1B_1, B_1C_1 respectively at three points (Y, Z, X respectively) in a line (n). Then by the quadrilateral property $P(A_1, X; B_1, Y; C_1, Z)$ or what is the same thing $P(A_1, C_2; B_1, A_2; C_1, B_2)$ specifies conjugate pairs of rays in an involution pencil; thus* P is a point of the order-cubic (U) on which A_1, C_2; B_1, A_2; C_1, B_2 are pairs of correspondents of the same species.

Again, if P, P' denote correspondents (of the given species) on U, then* $A_1(B_1, A_2; C_1, B_2; P, P')$ specifies conjugate pairs of rays in an involution pencil; whence, if Z' denote the meet of the lines n, A_1P', then

$$P(A_1A_2B_2Z') \equiv P(A_1YZZ') \barwedge A_1(PYZZ')$$
$$\equiv A_1(PC_1B_1P') \barwedge A_1(P'B_2A_2P) \equiv A_1(Z'B_2A_2P),$$

and thus Z' is on A_2B_2; similarly the meets of n with the lines B_1P', C_1P' are on B_2C_2, C_2A_2 respectively; *i.e.*, the lines A_1P', B_1P', C_1P' meet A_2B_2, B_2C_2, C_2A_2 respectively at three points on the line n. Thus: *If A_1, B_1, C_1 denote the vertices of any triangle, and A_2, B_2, C_2 the absolute poles of the lines B_1C_1, C_1A_1, A_1B_1 respectively; then the locus of a point P such that the lines A_2P, B_2P, C_2P meet C_1A_1, A_1B_1, B_1C_1 respectively at three points in a line (n) is the order-cubic (U) on which A_1, C_2; B_1, A_2; C_1, B_2 are pairs of correspondents of the same species.*† *The cubic U is also the locus of a point P such that the lines A_1P, B_1P, C_1P meet A_2B_2, B_2C_2, C_2A_2 respectively at three points in a line (n') If n, n' be termed respectively the $(+1, 2)$-and $(-2, 1)$-by-pedal lines of P, and if P, P' denote correspondents (of the given species) on U; then n, n' are respectively the $(-2, 1)$-and $(+1, 2)$-by-pedal lines of P'.* (2.21)

Now if the lines A_2P, B_2P, C_2P meet C_1A_1, A_1B_1, B_1C_1 respectively at three points on n, and if N, p respectively denote the absolute pole and polar of n, P; then, by reciprocation q.‡ the Absolute, the lines A_2N, B_2N, C_2N meet A_1B_1, B_1C_1, C_1A_1 respectively at three points on p. Hence, as in (2.21), N is a point of the order-cubic (V) on which A_1, B_2; B_1, C_2; C_1, A_2 are pairs of correspondents of the same species. Again, if N' denote the absolute pole of n'; then similarly the lines A_1N', B_1N', C_1N' meet C_2A_2, A_2B_2, B_2C_2 at three points on p. The points N, N' are therefore correspondents (of the given species) on V, and there is complete reciprocity between the properties of the cubics U, V. Thus: *The locus of a point N such that the lines A_2N, B_2N, C_2N meet A_1B_1, B_1C_1, C_1A_1 respectively at three points in a line (p) is the order-cubic (V) on which A_1, B_2; B_1, C_2; C_1, A_2 are pairs of corre-*

* Cayley, *Liouville*, 9 (1844), p. 285 = *Papers*, 1, p. 183.

† A case of a theorem due to Grassmann and Clebsch; *v.* (* 17.1).

‡ The symbol q. signifies *with respect to.*

*spondents of the same species.** *The cubic V is also the locus of a point N such that the lines A_1N, B_1N, C_1N meet C_2A_2, A_2B_2, B_2C_2 respectively at three points in a line (p'). If p, p' be termed respectively the $(-1, 2)$-and $(+2, 1)$-by-pedal lines of N, and if N, N' denote correspondents (of the given species) on V; then p, p' are respectively the $(+2, 1)$-and $(-1, 2)$-by-pedal lines of N'.*

If P, P' denote any pair of correspondents (of the given species) on U; p, p' respectively the absolute polars of P, P'; n, n' respectively the $(+1, 2)$-and $(-2, 1)$-by-pedal lines of P; and N, N' respectively the absolute poles of n, n': then N, N' are correspondents (of the given species) on V, and p, p' are respectively the $(-1, 2)$-and $(+2, 1)$-by-pedal lines of N

The envelope of the $(+1, 2)$-or of the $(-2, 1)$-by-pedal lines of the points on U is a class-cubic, the absolute reciprocal of V; and reciprocally for V, U. (2.22)

2.3. The writer has shewn (* 17.11) that, if M_0 denote any point on the cubic T_0 (2.11), then the perpendiculars at A_1, B_1, C_1 to the lines A_1M_0, B_1M_0, C_1M_0 respectively meet at a point M_2; the perpendiculars at A_2, B_2, C_2 to the lines A_2M_0, B_2M_0, C_2M_0 meet at a point M_1; and M_1, M_2 are correspondents (of the same species as A_1, A_2) on T_0. These properties are analogous to the rectangular property of the euclidean circle. It may be shewn that the cubics U, V possess corresponding mutual relations.

For if P denote any point of U, then (2.22) the lines A_1P, A_2B_2 meet (at Z'', say) on the absolute polar of a point (N') on V. Z'' is thus the absolute pole of the line C_1N', which is therefore perpendicular to A_1P; similarly the lines A_1N', B_1N' are perpendicular to B_1P, C_1P respectively. Thus: *If P denote any point on the cubic U, then the lines through C_1, A_1, B_1 perpendicular to A_1P, B_1P, C_1P respectively meet at a point N'; the lines through B_2, C_2, A_2 perpendicular to A_2P, B_2P, C_2P respectively meet at a point N; and N, N' are correspondents (of the given species) on V. Similarly for V, U.* (2.31)

Again, it is well known that the lines A_1A_2, B_1B_2, C_1C_2 meet at a point (H), the common orthocentre of the triangles $A_1B_1C_1$, $A_2B_2C_2$; and that the lines B_1C_1, C_1A_1, A_1B_1 meet B_2C_2, C_2A_2, A_2B_2 respectively at three points (A_0, B_0, C_0 respectively) in a line,

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* **A case of a theorem due to Grassmann and Clebsch; *v.* (* 17.1).**

the common orthaxis of the two triangles. The writer has shewn (* 8.4, * 15) that the Absolute is the pole conic of the orthaxis q. a class-cubic (Θ_3) of which T_0 is the Cayleyan; Θ_3 being the class-cubic q. which the point-pairs A_1, A_2; B_1, B_2; C_1, C_2 are degenerate pole conics. It is easy to prove somewhat similar theorems for the cubics U, V.

For there is an order-cubic (U') q. which A_1, C_2; B_1, A_2; C_1, B_2 are pairs of conjugate poles; U being the Hessian of U'. Let the polar conic, q. U', of any point M be denoted by $S(M)$. Then $S(A_2)$ consists of a pair of lines meeting at B_1; and since B_2, C_1 are conjugate poles q. U', therefore the pair of lines B_1B_2, B_1C_1 are harmonic q. the pair of lines $S(A_2)$. Thus B_1C_1 is the polar of H q. $S(A_2)$, and is therefore the polar of A_2 q. $S(H)$; similarly the lines C_1A_1, A_1B_1 are the polars, q. $S(H)$, of B_2, C_2 respectively, and thus: *The Absolute is the polar conic of the orthocentre of the triangle $A_1B_1C_1$ q. either of two order-cubics (U', V') of which U, V are respectively the Hessians.* (2.32)

Now any pair of correspondents (of the given species) N, N' on V are conjugate points q. every polar conic q. V', and in particular q. the Absolute; whence (2.22): *The (+1, 2)-and (−2, 1)-by-pedal lines of any point P on U are perpendicular; and similarly for V.* (2.33)

§ 3. *The Analogues of Tarry's Point.* 3.1. The six points A_1, B_1, C_1, A_2, B_2, C_2 are common to the cubics T_0, U, V. If M denote any other common point of T_0 and U, and I, J, K the feet of the perpendiculars from M to B_1C_1, C_1A_1, A_1B_1 respectively; then (2.11) three of the nine points common to the three lines MI, MJ, MK and the three lines B_1C_1, C_1A_1, A_1B_1 are in a seventh line, and (2.21) three more of the nine points are in an eighth line. The remaining three of the nine points are therefore in a ninth line, and thus (2.22) M is a point of the cubic V. Hence: *The cubics T_0, U, V have three points (P_3, Q_3, R_3) other than A_1, B_1, C_1, A_2, B_2, C_2 common to all three. The points P_3, Q_3, R_3 and no other points are such that the three perpendiculars from any one of them to the sides of the triangle $A_1B_1C_1$ are definite and meet the three sides at nine distinct points which lie three by three in three further lines. P_3, Q_3, R_3 are similarly related to the triangle $A_2B_2C_2$.* (3.11)

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It follows from (2.31) that: *If and only if M denote one of the three points P_3, Q_3, R_3; then the three perpendiculars to the lines A_1M, B_1M, C_1M from A_1, B_1, C_1 respectively, from B_1, C_1, A_1 respectively, and from C_1, A_1, B_1 respectively are concurrent.* The three points of concurrence are on T_0, U, V respectively. Similarly for M; A_2, B_2, C_2; and T_0, V, U.* (3.12)

3.2. The points A_1, B_1, C_1 are respectively the correspondents of one species (on T_0), and the points A_2, B_2, C_2 respectively the correspondents of another species (on T_0) of the points A_0, B_0, C_0 (2.32). If the lines B_1C_2, C_1A_2, A_1B_2 meet B_2C_1, C_2A_1, A_2B_1 respectively at A_3, B_3, C_3; then (*5.1) the points A_3, B_3, C_3 are the correspondents of the third species (on T_0) of A_0, B_0, C_0 respectively, and the lines B_3C_3, C_3A_3, A_3B_3 contain A_0, B_0, C_0 respectively. More generally, if A_l, B_l, C_l denote respectively the correspondents (of a given species) of A_0, B_0, C_0; then A_m, B_m, C_m are respectively the correspondents (of that species) of A_n, B_n, C_n (l, m, $n = 1, 2, 3$). (3.21)

Further, if O, P, Q, R, S denote any five points on an order-cubic, and P', Q', R', S' respectively the correspondents of any (the same) species of P, Q, R, S; then †

$$O\,(PQRS) \barwedge O\,(P'Q'R'S') \qquad (3.22)$$

Thus, if M denote any one of the three points P_3, Q_3, R_3 (3.11), then (3.21) $A_3(A_0B_3C_3M) \barwedge M(A_0B_3C_3A_3)$

(3.21, 2)	$\barwedge M(A_1B_2C_2A_2)$
(2.22)	$\barwedge M(B_2A_1B_1C_1)$
(3.21, 2)	$\barwedge M(B_3A_0B_0C_0)$
(2.32)	$\barwedge B_3(MA_0B_0C_0)$
(3 21)	$\equiv B_3(MC_3B_0A_3)$,

whence the conic which contains M, B_3, C_3 and touches the line A_0A_3 at A_3 also touches the line B_0B_3 at B_3; similarly the conic touches the line C_0C_3 at C_3. Thus: *There is a conic which touches the lines A_0A_3, B_0B_3, C_0C_3 at A_3, B_3, C_3 respectively, and contains the points P_3, Q_3, R_3 specified in* (3.11). (3.23)

It follows that P_3, Q_3, R_3 are respectively correspondents (of

* The euclidean analogue is due to Neuberg, *Mathesis* (1) 5 (1885), p. 208.

† Cayley, *loc. cit.* (2.2).

the same species as the pair of correspondents A_1, A_2) of three points P_0, Q_0, R_0 in line on the cubic T_0. If T_3 be the order-cubic apolar (* 1.3) to Θ_3 (2.32); then the conic specified in (3.23) is the mixed poloconic (* 10.1) q. T_3 of the orthaxis $A_0B_0C_0$ and another line; it is also the mixed poloconic (* 10.6) q. Θ_3 of the orthocentre H and another point. (3.24)

3.3. Now let Q denote any point such that the lines A_0Q, B_0Q, C_0Q meet the lines C_3A_3, A_3B_3, B_3C_3 at three points (E, F, D respectively) in line; then A_0Q, B_0Q, C_0Q meet A_3B_3, B_3C_3, C_3A_3 respectively at three other points in line. Further, by the quadrilateral property $Q(A_3, A_0; B_3, B_0; C_3, C_0)$, and $Q(A_3, D; B_3, E; C_3, F)$ or what is the same thing $Q(A_3, C_0; B_3, A_0; C_3, B_0)$, specify conjugate pairs of rays in involution pencils. Thus $Q(A_0B_3C_3A_3) \barwedge Q(B_3A_0B_0C_0) \barwedge Q(B_0A_3B_3C_3)$, whence as in (3.23): *The locus of a point Q such that the lines A_0Q, B_0Q, C_0Q meet the lines B_3C_3, C_3A_3, A_3B_3 at six points (other than A_0, B_0, C_0) which lie three by three in two lines is the conic specified in* (3.23). (3.31)

3.4. Discarding the non-euclidean terminology, we may generalise the theorems (3.11, 3.23, 3.31) as follows: *Let A_0, B_0, C_0 denote any three collinear points on any order-cubic T_0, and A_n, B_n, C_n correspondents (of the same species) of A_0, B_0, C_0 respectively* ($n = 1, 2, 3$). *Then there are three and only three points $[G_l]$ such that the lines A_mG_l, B_mG_l, C_mG_l are definite and meet the lines B_nC_n, C_nA_n, A_nB_n at nine distinct points which lie three by three in three further lines. The lines A_nG_l, B_nG_l, C_nG_l also meet the lines B_mC_m, C_mA_m, A_mB_m at nine distinct points which lie three by three in three further lines. The three points $[G_l]$ are contained by the cubic T_0 and also by the conic (s_l) which touches the lines A_0A_l, B_0B_l, C_0C_l, at A_l, B_l, C_l respectively. The correspondents (associated with the suffix l) on T_0 of the three points $[G_l]$ are in line. The conic s_l is the locus of a point Q such that the lines A_0Q, B_0Q, C_0Q meet the lines B_lC_l, C_lA_l, A_lB_l at six points (other than A_0, B_0, C_0) which lie three by three in two lines* ($l, m, n = 1, 2, 3$).

(3.41)

It is to be observed that the lines of collinearity (g_1, g_2, g_3) of the (1, 2, 3)-correspondents respectively of the three groups of three points $[G_1]$, $[G_2]$, $[G_3]$ are not in general identical. It may be proved that the tangents to T_0 at A_n, B_n, C_n determine a triangle in perspective with the triangle $A_nB_nC_n$, and that g_n is the polar,

q. the triangle $A_n B_n C_n$, of the centre of perspective of the two triangles ($n = 1, 2, 3$). (3 42)

3.5. In the euclidean case (*cf.* * § 16), if $A_1 B_1 C_1$ remains an actual triangle (the *base-triangle*), then T_0 breaks up into the circumcircle of that triangle and the line at infinity, and each of the cubics U, V breaks up into a circumconic of the base-triangle * and the line at infinity. A_0, B_0, C_0 become the points at infinity of the lines $B_1 C_1$, $C_1 A_1$, $A_1 B_1$ respectively, and A_3, B_3, C_3 become the points of the circumcircle diametrically opposite to A_1, B_1, C_1 respectively. The conic specified in (3.23) becomes the Steiner circumellipse of the triangle $A_3 B_3 C_3$; † and, of the points P_3, Q_3, R_3 specified in (3.11), one becomes the Tarry point of the base-triangle, and the other two become the points at infinity on the ellipse just specified.

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* Neuberg, *Mathesis* (1) 3 (1883), p. 144 ; *ibid.* (1) 5 (1885), p. 208 ; *ibid.* (1) 6 (1886), p. 5. *Cf.* the remark *Encyk d. math. Wiss.* III. AB 10, p. 1266, ll. 2-7.

† The conic s_1 specified in (3.41) becomes the Steiner circumellipse of the base-triangle.

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