SOME EXPANSIONS OF HYPERGEOMETRIC FUNCTIONS IN SERIES OF HYPERGEOMETRIC FUNCTIONS†

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1. Introduction and main results. Throughout the present note we abbreviate the set of p parameters $a_1, \dots, a_p$ by (a_p), with similar interpretations for (b_q), etc. Also, by $[(a_p)]_m$ we mean the product $\Pi_{j=1}^{p}[a_j]_m$, where $[\lambda]_m = \Gamma(\lambda+m)/\Gamma(\lambda)$, and so on. One of the main results we give here is the expansion formula

$$\omega^{\mu}\,{}_{p+r}F_{q+s}\left[\begin{matrix}(a_p),(c_r);\\ \\ (b_q),(d_s);\end{matrix}\; z\omega\right] = \frac{h[(c_r)]_{-\mu}[(\alpha_t)]_{\mu}}{[(d_s)]_{-\mu}[(\beta_u)]_{\mu}} \tag{1}$$

$$\cdot \sum_{n=0}^{\infty} \frac{[-\mu]_n\Gamma(h-\alpha n+\mu)}{n!\Gamma(1-\alpha n+h)}\,{}_{p+t+2}F_{q+u+1}\left[\begin{matrix}\mu+1, h-\alpha n+\mu, (a_p), (\alpha_t)+\mu;\\ \\ \mu-n+1, (b_q), (\beta_u)+\mu;\end{matrix}\; z\right]$$

$$\cdot\,{}_{r+u+2}F_{s+t+2}\left[\begin{matrix}-n, 1+h/(1-\alpha), (c_r)-\mu, (\beta_u);\\ \\ h/(1-\alpha), 1-\alpha n+h, (d_s)-\mu, (\alpha_t);\end{matrix}\; \omega\right],$$

which is valid, by analytic continuation, when p, q, r, s, t and u are nonnegative integers such that $p+r < q+s+1$ (or $p+r = q+s+1$ and $|z\omega| < 1$), $p+t < q+u$ (or $p+t = q+u$ and $|z| < 1$), and the various parameters including μ are so restricted that each side of equation (1) has a meaning.

Here we tacitly assume that the parameter μ is not zero or a positive integer. However, if m is an arbitrary nonnegative integer, it is readily observed that a limiting case of the expansion formula (1) when $\mu \to m$ yields

$$\omega^{m}\,{}_{p+r}F_{q+s}\left[\begin{matrix}(a_p),(c_r);\\ \\ (b_q),(d_s);\end{matrix}\; z\omega\right] = \frac{h[(c_r)]_{-m}[(\alpha_t)]_{m}}{[(d_s)]_{-m}[(\beta_u)]_{m}} \tag{2}$$

$$\cdot \sum_{n=0}^{m-1} \frac{[-m]_n\Gamma(h-\alpha n+m)}{n!\Gamma(1-\alpha n+h)}\,{}_{p+t+2}F_{q+u+1}\left[\begin{matrix}m+1, h-\alpha n+m, (a_p), (\alpha_t)+m;\\ \\ m-n+1, (b_q), (\beta_u)+m;\end{matrix}\; z\right]$$

$$\cdot\,{}_{r+u+2}F_{s+t+2}\left[\begin{matrix}-n, 1+h/(1-\alpha), (c_r)-m, (\beta_u);\\ \\ h/(1-\alpha), 1-\alpha n+h, (d_s)-m, (\alpha_t);\end{matrix}\; \omega\right]+$$

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$$+\frac{(-1)^m h[(c_r)]_{-m}[(\alpha_t)]_m}{[(d_s)]_{-m}[(\beta_u)]_m}\sum_{n=0}^{\infty}\frac{[1-\alpha(m+n)+h]_{m+n-1}[(a_p)]_n[(\alpha_t)+m]_n(-z)^n}{n![(b_q)]_n[(\beta_u)+m]_n}$$

$$\cdot\,{}_{p+t+2}F_{q+u+1}\left[\begin{matrix}m+n+1,(1-\alpha)(m+n)+h,(a_p)+n,(\alpha_t)+m+n;\\ n+1,(b_q)+n,(\beta_u)+m+n;\end{matrix}\; z\right]$$

$$\cdot\,{}_{r+u+2}F_{s+t+2}\left[\begin{matrix}-m-n,1+h/(1-\alpha),(c_r)-m,(\beta_u);\\ h/(1-\alpha),1-\alpha(m+n)+h,(d_s)-m,(\alpha_t);\end{matrix}\; \omega\right],$$

which holds under essentially the same conditions as stated with (1).

For $m = 0$, this last expansion formula (2) would evidently reduce to the elegant form

(3)
$$_{p+r}F_{q+s}\left[\begin{matrix}(a_p),(c_r);\\ (b_q),(d_s);\end{matrix}\; z\omega\right]=h\sum_{n=0}^{\infty}\frac{[1-\alpha n+h]_{n-1}[(a_p)]_n[(\alpha_t)]_n(-z)^n}{n![(b_q)]_n[(\beta_u)]_n}$$

$$\cdot\,{}_{p+t+1}F_{q+u}\left[\begin{matrix}(1-\alpha)n+h,(a_p)+n,(\alpha_t)+n;\\ (b_q)+n,(\beta_u)+n;\end{matrix}\; z\right]$$

$$\cdot\,{}_{r+u+2}F_{s+t+2}\left[\begin{matrix}-n,1+h/(1-\alpha),(c_r),(\beta_u);\\ h/(1-\alpha),1-\alpha n+h,(d_s),(\alpha_t);\end{matrix}\; \omega\right],$$

provided that the relevant conditions of validity of (1) hold true.

2. Derivation of formula (1). In order to prove the expansion formula (1) we shall require the Mellin transform of the G-function given by the familiar formula (cf., e.g., [2], Vol. I, p. 157)

(4)
$$\int_0^{\infty}x^{\zeta-1}G_{p,q}^{m,n}\left(zx\,\middle|\,\begin{matrix}(a_p)\\ (b_q)\end{matrix}\right)dx$$

$$=\frac{\prod_{j=1}^{m}\Gamma(b_j+\zeta)\prod_{j=1}^{n}\Gamma(1-a_j-\zeta)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\zeta)\prod_{j=n+1}^{p}\Gamma(a_j+\zeta)}z^{-\zeta},$$

where

$$1\leqq n\leqq p<q,\ 1\leqq m\leqq q,\ \delta=m+n-\frac{1}{2}(p+q)>0,\ z\neq 0,\ |\arg(z)|<\delta\pi,$$

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and

$$(5) \qquad -\min_{1 \leqq j \leqq m} \{\mathrm{Re}(b_j)\} < \mathrm{Re}(\zeta) < 1 - \max_{1 \leqq j \leqq n} \{\mathrm{Re}(a_j)\}.$$

Making use of (4) if we take the Mellin transforms of the G-functions occurring on both sides of the known result† [5, p. 665, Eq. (1)], replace p by $r+u$ and set $a_j = \gamma_j$, $j = 1, \ldots, r$, $a_{r+j} = \beta_j$, $j = 1, \ldots, u$, and make a substantial change in the notations employed by the earlier writer, we shall obtain

$$(6) \qquad \omega^\mu = \frac{h\Gamma(\mu+1)[(\alpha_t)]_\mu[(\delta_s)]_\mu}{[(\beta_u)]_\mu[(\gamma_r)]_\mu} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(h-\alpha n+\mu)}{n!\Gamma(\mu-n+1)\Gamma(1-\alpha n+h)}$$

$$\cdot {}_{r+u+2}F_{s+t+2}\left[\begin{matrix} -n, 1+h/(1-\alpha), (\beta_u), (\gamma_r); \\ h/(1-\alpha), 1-\alpha n+h, (\alpha_t), (\delta_s); \end{matrix} \omega\right],$$

provided that the series on the right-hand side converges.

Now expand the first member of (1) in powers of ω, apply the formula (6) above, and then set $\gamma_j = c_j - \mu$, $j = 1, \ldots, r$, and $\delta_j = d_j - \mu$, $j = 1, \ldots, s$. A subsequent inversion of the order of the resulting double summation will finally yield (1) under the conditions stated already.

3. Extensions and particular cases. At the outset we remark that the method of the preceding section can be appropriately extended to derive various generalizations, for example, of (1) to hold for hypergeometric functions of two and more variables. As a matter of fact, one can easily apply (6) to obtain a result analogous to the Srivastava–Daoust expansion [4, p. 456, Eq. (4.3)] involving the generalized Lauricella function of several complex variables. The details are fairly straightforward and may well be omitted.

Next we observe that our expansion formulas (1), (2), (3) provide generalizations of several results given in the literature. For instance, these formulas would reduce, when $\alpha = 0$, to the known expansions due to Wimp and Luke ([6], p. 358, Eqs. (1.18) and (1.19); see also [2], Vol. II, pp. 9–11, Theorems 3 and 4, and [3], p. 355, Eq. (112)). On the other hand, a special case of (3) when $s = u = 0$ yields the expansion formula (1), p. 664 of Verma [5], while a special case of our expansion formula (1) when $r = s = 0$ gives us the recent result [1, p. 258, Eq. (18)]. Evidently, in view of this last special case, an alternative proof by induction on the integers r and s may be attributed to our expansion formula (1).

We should like to conclude by recording certain obvious further generalizations of our expansion formulas (2) and (3) given by the following

THEOREM. *Let $\{A_n\}$, $\{B_n\}$ and $\{C_n\}$ be sequences of arbitrary complex numbers such that $C_n \not\equiv 0$, $n \geqq 0$.*

Then, for every nonnegative integer m,

† Notice that this formula (1), p. 665 of A. Verma [5] appears with some obvious typographical errors.

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$$\omega^m \sum_{n=0}^{\infty} A_n \frac{[(c_r)]_n}{[(d_s)]_n} \frac{(z\omega)^n}{n!} = \sum_{n=0}^{m-1} \frac{[-m]_n}{n!} \sum_{k=0}^{n} \frac{[-n]_k}{k!} \left(\frac{h-\alpha k+k}{h-\alpha n+k}\right) \tag{7}$$

$$\cdot \frac{[(c_r)]_{k-m}}{[(d_s)]_{k-m}} C_k^{-1} \omega^k \sum_{l=0}^{\infty} \frac{[m+1]_l}{[m-n+1]_l} A_l C_{l+m} \frac{z^l}{l!}$$

$$+(-1)^m \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \sum_{k=0}^{m+n} \frac{[-m-n]_k}{k!} \left(\frac{h-\alpha k+k}{h-\alpha(m+n)+k}\right) \frac{[(c_r)]_{k-m}}{[(d_s)]_{k-m}} C_k^{-1} \omega^k$$

$$\cdot \sum_{l=0}^{\infty} \frac{[m+n+1]_l}{[n+1]_l} [h-\alpha(m+n)+k]_{l+m+n-k} A_{n+l} C_{l+m+n} \frac{z^l}{l!},$$

and

$$\sum_{n=0}^{\infty} A_n B_n \frac{(z\omega)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \sum_{k=0}^{n} \frac{[-n]_k}{k!} \left(\frac{h-\alpha k+k}{h-\alpha n+k}\right) B_k C_k^{-1} \omega^k \tag{8}$$

$$\cdot \sum_{l=0}^{\infty} [h-\alpha n+k]_{n+l-k} A_{n+l} C_{n+l} \frac{z^l}{l!},$$

provided that the series involved converge absolutely.

Equivalently, this last identity (8) may be rewritten in the form

$$\sum_{n=0}^{\infty} A_n B_n \frac{(z\omega)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \sum_{k=0}^{n} \frac{[-n]_k}{k!} \left(\frac{h-\alpha k+k}{h-\alpha n+k}\right) B_k \omega^k \tag{9}$$

$$\cdot \sum_{l=0}^{\infty} [h-\alpha n+k]_{n+l-k} A_{n+l} \frac{z^l}{l!},$$

whenever each side has a meaning.

Note added 16 July, 1975. The identity (9) above is substantially the same as the main result (3.1), p. 75 in the recent paper entitled "On generating functions of classical polynomials", *Proc. Amer. Math. Soc.* **46** (1974), 73–76, by A. Verma [see also *Math. Reviews* **49** (1975), # 9276 (by H. M. Srivastava)].

REFERENCES

1. R. F. A. Abiodun and B. L. Sharma, Summation of series involving generalized hypergeometric functions of two variables, *Glasnik Mat. Ser.* III **6** (**26**) (1971), 253–264. *MR* **46** # 7563.

2. Yudell L. Luke, *The Special Functions and their Approximations*, Vols. I and II (Academic Press, New York and London, 1969). *MR* **39** # 3039; *MR* **40** # 2909.

3. C. S. Meijer, Expansion theorems for the *G*-function. *V*, *Nederl. Akad. Wetensch. Proc. Ser.* A **56** = *Indag. Math.* **15** (1953), 349–357. *MR* **15**, 422.

4. H. M. Srivastava and Martha C. Daoust, Certain generalized Neumann expansions associated with the Kampé de Fériet function, *Nederl. Akad. Wetensch. Proc. Ser.* A **72** = *Indag. Math.* **31** (1969), 449–457. *MR* **40** # 5918.

https://doi.org/10.1017/S0017089500002664 Published online by Cambridge University Press

5. Arun Verma, A class of expansions of G-functions and the Laplace transform, *Math. Comp.* **19** (1965), 664–666.

6. Jet Wimp and Yudell L. Luke, Expansion formulas for generalized hypergeometric functions, *Rend. Circ. Mat. Palermo Ser.* II **11** (1962), 351–366. *MR* **29** ♯3681.

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