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ON THE MINIMAL CONVEX SHELL OF A CONVEX BODY

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ABSTRACT. For any convex body C in $\mathbb{R}^d$ we introduce the notion of the convex shell and we prove that there exists a unique "minimal" convex shell, extending the notion of the minimal spherical shell of C. Then we prove that a "typical" convex body touches the boundary of its minimal convex shell in precisely $d+2$ points.

1. **Introduction.** The study of the minimal spherical shell of a convex body has been motivated by its relation with the isoperimetric problem [2].

For a convex body C in $\mathbb{R}^d$, let R, r be the radii of two concentric spheres such that no point of C is outside the sphere with radius R and every point of the sphere with radius r is in C. When $R - r$ is minimal the corresponding spheres bound the minimal spherical shell of C.

The existence and uniqueness of the minimal spherical shell of a convex body C were established by T. Bonnesen [2] in dimension $d = 2$ and by N. Kritikos [5] in dimension $d = 3$; the general case was treated by I. Bárány [1].

Moreover, these authors showed that on each sphere bounding the minimal spherical shell of C there are at least two points of C.

A recent direction of research in convexity led to the discovery of many typical properties in the sense of Baire categories. For references on this topic see the monographs of P. M. Gruber [4] and T. Zamfirescu [8].

In this context, A. Zucco [9] proved that a typical convex body C touches the boundary of its minimal spherical shell in precisely $d+2$ points. By a well-known theorem of Radon each set of $d+2$ points in $\mathbb{R}^d$ is the union of two disjoint subsets whose convex hulls have a common point. In the same article A. Zucco showed that such subsets are determined by the points of C belonging to the same boundary component of the minimal spherical shell of C.

In the present paper we generalize these results as follows. First, we introduce the notion of the convex shell of a convex body by considering homothetic images of a fixed convex body. We prove that for every convex body C in $\mathbb{R}^d$ there exists a unique minimal convex shell $\mathcal{K}(C)$ of C and we give a characterization of $\mathcal{K}(C)$ involving the contact points of the boundaries of C and $\mathcal{K}(C)$.

Then, we prove that a typical convex body touches the boundary of its minimal convex shell in precisely $d+2$ points.

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It should be noted that the case $d = 2$ has already been treated in [6]. In that article the minimal convex shell has been studied by means of geometric methods, whereas here we use some tools from convex analysis, following the approach adopted by I. Bárány in [1].

2. **Preliminaries.** Let $\mathfrak{C}$ denote the space of all convex bodies in $\mathbb{R}^d$ equipped with the *Hausdorff metric*:

$$\delta(C,D) := \max\Big\{\sup_{x\in C}\inf_{y\in D} d(x,y), \sup_{y\in D}\inf_{x\in C} d(x,y)\Big\}, \text{ with } C, D \in \mathfrak{C},$$

where d denotes the Euclidean metric in $\mathbb{R}^d$.

$(\mathfrak{C},\delta)$ is a Baire space. By *typical* elements of a Baire space we shall mean all elements except those in a meager set.

Let S^{d-1} denote the unit sphere centred at the origin O of $\mathbb{R}^d$. For any $C \in \mathfrak{C}$ and $u \in S^{d-1}$, $h(C,u)$ will denote the support function of C.

Moreover, for any bounded subset X of $\mathbb{R}^d$ the symbols $\operatorname{bd} X$ and $\operatorname{conv} X$ will be used to indicate the boundary and the convex hull of X.

In the following, K will denote an arbitrary but fixed smooth and strictly convex body containing the origin O of $\mathbb{R}^d$ as an interior point. Moreover, for $x \in \mathbb{R}^d$ and $\rho > 0$ we put $K(x,\rho) = \rho K + x$.

The convex body K can be regarded as the gauge body of a Minkowski space by considering $\mathbb{R}^d$ equipped with the following *Minkowski length*:

$$\|x\| := \inf\{\rho \in \mathbb{R}_+ : x \in \rho K\}, \text{ for } x \in \mathbb{R}^d.$$

REMARK 1. Since K is a strictly convex body we have

$$\|x+y\| \le \|x\| + \|y\|$$

and equality holds if and only if x and y are linearly dependent.

For any $x_0 \in \mathbb{R}^d$ and $0 \le \rho_0 \le \sigma_0$, the set

$$\mathcal{K}_{x_0} := \{x \in \mathbb{R}^d : \rho_0 \le \|x - x_0\| \le \sigma_0\}$$

will be called the *convex shell of centre* x_0 *and radii* ρ_0, σ_0.

On the basis of the previous concept of length, we shall use the notion of the support function of a convex body C relative to K, say $h_K(C,u)$, defined by

$$h_K(C,u) := h(C,u)/h(K,u).$$

Moreover, for a given direction $u \in S^{d-1}$ we shall consider the vector u' defined by

$$u' := \big(1/h(K,u)\big)u. \tag{1}$$

We shall also need the following definitions and results from convex analysis.

Let $f\colon \mathbb{R}^d \longrightarrow \mathbb{R}$ be a convex function and let $x \in \mathbb{R}^d$. A vector x^* is said to be a *subgradient of f at x* if

$$\langle x^*, z-x\rangle \leq f(z) - f(x), \quad \forall z.$$

The set of all subgradients of f at x is called the *subdifferential of f at x* and it is denoted by $\partial f(x)$.

It is well-known that the subdifferential of a finite convex function is nonempty, convex and compact. Moreover we have

THEOREM A (SEE [7]). *Let $f\colon \mathbb{R}^d \longrightarrow \mathbb{R}$ be convex, $g\colon \mathbb{R}^d \longrightarrow \mathbb{R}$ concave functions, finite over the whole space. Then $f(x) - g(x)$ attains its minimum at x_0 if and only if $0 \in \partial f(x_0) + \partial(-g)(x_0)$.*

THEOREM B (SEE [3]). *Let Γ be a compact set and for $\gamma \in \Gamma$ let the map $\gamma \longrightarrow (x^*_\gamma, a_\gamma) \in \mathbb{R}^d \times \mathbb{R}$ be continuous. Let $f(x) = \sup\{\langle x^*_\gamma, x\rangle + a_\gamma : \gamma \in \Gamma\}$. Then $f\colon \mathbb{R}^d \longrightarrow \mathbb{R}$ is a finite convex function and $\partial f(x_0) = \operatorname{conv}\{x^*_\gamma : \gamma \in \Gamma$ and $\langle x^*_\gamma, x_0\rangle + a_\gamma = f(x_0)\}$.*

3. **The minimal convex shell of a convex body.** For any $C \in \mathfrak{C}$ and $x \in C$ we define

$$\begin{cases} R(x) := \min\{\rho \in \mathbb{R}_+ : C \subseteq K(x,\rho)\} \\ r(x) := \max\{\rho \in \mathbb{R}_+ : K(x,\rho) \subseteq C\}. \end{cases}$$

It is easy to see that the minimum and maximum above exist.

REMARK 2. For any $x \in C$, the functions $R(x)$ and $r(x)$ can be expressed by

$$R(x) = \max_{y \in \operatorname{bd} C} \|y-x\|, \qquad r(x) = \min_{y \in \operatorname{bd} C} \|y-x\|.$$

This means that there exist points $p, q \in \operatorname{bd} C$ such that $\|p-x\| = R(x)$ and $\|q-x\| = r(x)$. We shall say in this case that p *supports* $R(x)$ and q *supports* $r(x)$.

For any $C \in \mathfrak{C}$ and $x \in C$ the convex shell of centre x and radii $r(x)$, $R(x)$ will be called the *convex shell of C with centre x* and will be denoted by $\mathcal{K}_x(C)$.

LEMMA 1. *i)*

$$R\Big(\frac{x_1+x_2}{2}\Big) \leq \frac{1}{2}\big(R(x_1)+R(x_2)\big).$$

Moreover, if equality holds then there exists a unique point $y \in \operatorname{bd} C$ supporting $R\big((x_1+x_2)/2\big)$. This point lies on the straight line through x_1 and x_2 and it also supports both $R(x_1)$ and $R(x_2)$.

ii)

$$r\Big(\frac{x_1+x_2}{2}\Big) \geq \frac{1}{2}\big(r(x_1)+r(x_2)\big).$$

PROOF. i) Let $y \in \operatorname{bd} C$ be a point supporting $R\big((x_1+x_2)/2\big)$. By Remark 2 one has $\|y-x_i\| \leq R(x_i)$, for $i = 1,2$, so that the desired inequality follows from the triangle inequality relative to the length $\|\cdot\|$. Now, let us assume that the equality sign holds. Then

$$\|y-(x_1+x_2)/2\| = (1/2)(\|y-x_1\|+\|y-x_2\|).$$

By Remark 1, it happens if and only if y lies on the straight line through x_1 and x_2. Moreover, this implies the uniqueness of y.

ii) Clearly $\operatorname{conv}\Big(K\big(x_1, r(x_1)\big)\cup K\big(x_2, r(x_2)\big)\Big)\subseteq C$, so that it suffices to prove that

$$K\left(\frac{x_1+x_2}{2}, \frac{r(x_1)+r(x_2)}{2}\right)\subseteq \operatorname{conv}\Big(K\big(x_1, r(x_1)\big)\cup K\big(x_2, r(x_2)\big)\Big).$$

If $K\Big((x_1+x_2)/2, \big(r(x_1)+r(x_2)\big)/2\Big)$ contains a point y not belonging to $\operatorname{conv}\Big(K\big(x_1, r(x_1)\big)\cup K\big(x_2, r(x_2)\big)\Big)$ then, in the plane determined by x_1, x_2, y, there exists a straight line ℓ through y which does not meet the set $\operatorname{conv}\Big(K\big(x_1, r(x_1)\big)\cup K\big(x_2, r(x_2)\big)\Big)$.

Taking y_1, y_2 on ℓ so that the vectors y_i-x_i, with $i=1,2$, are parallel to $y-(x_1+x_2)/2$, we have

$$\|y-(x_1+x_2)/2\| = (1/2)(\|y_1-x_1\|+\|y_2-x_2\|),$$

with

$$\|y_1-x_1\|>r(x_1), \quad \|y_2-x_2\|>r(x_2).$$

Thus

$$\left\|y-\frac{x_1+x_2}{2}\right\|>\frac{r(x_1)+r(x_2)}{2},$$

contrary to the assumption that $y\in K\Big((x_1+x_2)/2, \big(r(x_1)+r(x_2)\big)/2\Big)$. ∎

This lemma enables us to state the following

THEOREM 1. *For every convex body C there exists a point $c\in C$ in which the function $R(x)-r(x)$ attains its minimal value. This point c is unique.*

We omit the proof as it does not differ essentially from that given by I. Bárány in [1].

The convex shell of C with centre c will be called the *minimal convex shell* of C and will be denoted by $\mathcal{K}(C)$.

In order to obtain a geometric characterization of $\mathcal{K}(C)$ we need the following definitions.

Let us consider a convex shell $\mathcal{K}_{x_0}$ with centre x_0 and radii ρ_0 and σ_0, with $0\le\rho_0\le\sigma_0$, and let us assume that $p_1,\dots,p_m\in \operatorname{bd}K(x_0,\sigma_0)$ and $q_1,\dots,q_n\in\operatorname{bd}K(x_0,\rho_0)$. For $i=1,\dots,m$, let u_i denote the outer normal to $\operatorname{bd}K(x_0,\sigma_0)$ at p_i and for $j=1,\dots,n$, let v_j denote the outer normal to $\operatorname{bd}K(x_0,\rho_0)$ at q_j. We then say that the sets

$$\{p_1,\dots,p_m\} \text{ and } \{q_1,\dots,q_n\}$$

cannot be separated if

$$\operatorname{conv}\{u_1',\dots,u_m'\}\cap\operatorname{conv}\{v_1',\dots,v_n'\}\neq\emptyset, \tag{2}$$

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where u_i', v_j' are the vectors associated with u_i, v_j by (1).

The set $\{p_1, \ldots, p_m, q_1, \ldots, q_n\}$ is said to be *irreducible* for $\mathcal{K}_{x_0}$ if, upon omitting any p_i or any q_j, the remaining set no longer satisfies the previous condition (2).

It may be noted that the points u_i', v_j' belong to the boundary of the polar reciprocal of K with respect to the unit sphere S^{d-1}. Therefore, if $\operatorname{bd} K$ is homothetic to S^{d-1} then our definitions coincide with the definitions given in [9].

Now, for $p \in \operatorname{bd} C$ let us consider the set of outer normals to $\operatorname{bd} C$ at p:

$$U(p) := \{u \in S^{d-1} : \langle u, p\rangle = \max_{y \in C} \langle u, y\rangle\}.$$

To $U(p)$ we make correspond the set $U'(p)$ defined by

$$U'(p) := \big\{u' \in \mathbb{R}^d : u' = \big(1/h(K,u)\big)u, u \in U(p)\big\}.$$

We also define

$$\Gamma := \{(p, u') \in \mathbb{R}^d \times \mathbb{R}^d : p \in \operatorname{bd} C, u' \in U'(p)\}.$$

Then we can state the following

LEMMA 2. *i)* $R(x) = \max\{\langle u', p - x\rangle, (p, u') \in \Gamma\}$,
ii) $r(x) = \min\{\langle u', p - x\rangle, (p, u') \in \Gamma\}$.

PROOF. i) For every $(p, u') \in \Gamma$ we have

$$\langle u', p - x\rangle = \big(1/h(k,u)\big)\langle u, p - x\rangle = h_K(C - x, u).$$

Since $C \subseteq K\big(x, R(x)\big)$ it follows

$$h_K(C - x, u) \leq h_K\Big(K\big(x, R(x)\big) - x, u\Big) = R(x),$$

so that

$$\langle u', p - x\rangle \leq R(x).$$

Moreover, if p supports $R(x)$ then the outer normal u to $\operatorname{bd} K\big(x, R(x)\big)$ at p is an outer normal to $\operatorname{bd} C$ as well. Thus, $u \in U(p)$, $(p, u') \in \Gamma$ and for this (p, u') we have

$$\langle u', p - x\rangle = h_K\Big(K\big(x, R(x)\big) - x, u\Big) = R(x).$$

Conversely, if the last equality holds, then p supports $R(x)$.

ii) The proof of the formula for $r(x)$ is similar. ■

THEOREM 2. *For every convex body C a point $c \in C$ is the centre of the minimal convex shell of C if and only if there exist two sets of points $p_1, \ldots, p_m \in \operatorname{bd} C$, supporting $R(c)$, and $q_1, \ldots, q_n \in \operatorname{bd} C$, supporting $r(c)$, which cannot be separated.*

PROOF. By Lemma 2 the function $r(x)$ can be extended over the whole space in such a way that it remains concave. Therefore we can extend the function $R(x) - r(x)$ to the whole space $\mathbb{R}^d$ in such a way that it attains its minimal value at the centre c of the minimal convex shell of C only.

Moreover, it follows from Theorem B that

$$\partial R(c) = \operatorname{conv}\{-u' : (p,u') \in \Gamma, \langle u', p - c\rangle = R(c)\}$$
$$\partial(-r)(c) = \operatorname{conv}\{u' : (p,u') \in \Gamma, \langle u', p - c\rangle = r(c)\}.$$

By Theorem A the function $R(x) - r(x)$ attains its minimal value at c if and only if there exists a point $y \in \mathbb{R}^d$ such that $y \in \partial R(c)$ and $-y \in \partial(-r)(c)$.

This means that there exist $\alpha_i \geq 0, \beta_j \geq 0, i = 1, \ldots, m, j = 1, \ldots, n$, with $\sum_{i=1}^m \alpha_i = 1$ and $\sum_{j=1}^n \beta_j = 1$, such that $y = -\sum_{i=1}^m \alpha_i u_i'$, $-y = \sum_{j=1}^n \beta_j v_j'$, where $\langle u_i', p_i - c\rangle = R(c)$ and $\langle v_j', q_j - c\rangle = r(c)$, for $(p_i, u_i'), (q_j, v_j') \in \Gamma$. This implies that the points $p_1, \ldots, p_m \in \operatorname{bd} C$ support $R(c)$, the points $q_1, \ldots, q_n \in \operatorname{bd} C$ support $r(c)$ and

$$\operatorname{conv}\{u_1', \ldots, u_m'\} \cap \operatorname{conv}\{v_1', \ldots, v_n'\} \neq \emptyset,$$

as required. ∎

4. **The case of a typical convex body.**

THEOREM 3. *A typical convex body C has exactly $d+2$ points in common with the boundary of its minimal convex shell $\mathcal{K}(C)$. These points form an irreducible set for $\mathcal{K}(C)$.*

PROOF. First, we shall show that the set $\mathcal{B} := \{C \in \mathfrak{C} : \operatorname{bd} C \cap \operatorname{bd} \mathcal{K}(C)$ is an irreducible set for $\mathcal{K}(C)$ consisting of exactly $d+2$ points$\}$ is dense in $\mathfrak{C}$.

Let D be a given element of $\mathfrak{C}$ and let P be a polytope approximating D. Moreover let $p \in P$ be the centre of the minimal convex shell of P. We set

$$\{p_1, \ldots, p_m\} = \operatorname{bd} P \cap \operatorname{bd} K\big(p, R(p)\big)$$
$$\{q_1, \ldots, q_n\} = \operatorname{bd} P \cap \operatorname{bd} K\big(p, r(p)\big).$$

By Theorem 2 these sets cannot be separated. Moreover, it can be shown (after reindexing if necessary) that there exist a subset $\{p_1, \ldots, p_k\}$ of $\{p_1, \ldots, p_m\}$ and a subset $\{q_1, \ldots, q_h\}$ of $\{q_1, \ldots, q_n\}$ such that $\{p_1, \ldots, p_k, q_1, \ldots, q_h\}$ is irreducible for $\mathcal{K}(P)$ and $4 \leq k + h \leq d + 2$.

If $k + h = d + 2$ then it follows from the same arguments used in [9] that there is a convex body sufficiently close to P which belongs to $\mathcal{B}$.

If $k + h < d + 2$ we proceed inductively by considering a convex body P', close to P, for which $\operatorname{bd} P' \cap \operatorname{bd} \mathcal{K}(P')$ is an irreducible set for $\mathcal{K}(P')$ consisting of $k + h + 1$ points.

As in [9] we can assume that p_1 does not belong to the hyperplanes tangent to $\operatorname{bd} K\big(p, r(p)\big)$ at the points $q_1, \ldots, q_h$.

Since the set

$$\{p_1, p_2, \ldots, p_k, q_1, q_2, \ldots q_h\}$$

is irreducible for $\mathcal{K}(P)$, we can choose two points r, s, near to p_1 such that

$$\{r, s, p_2, \ldots, p_k, q_1, q_2, \ldots, q_h\}$$

is also an irreducible set for $\mathcal{K}(P)$. This follows, for example, from the same construction used in [9] through replacement of S^{d-1} by the boundary of the polar reciprocal of K with respect S^{d-1}. We also note that the mapping (1) defines a homeomorphism from S^{d-1} to the boundary of the polar reciprocal of K with respect S^{d-1}.

Now, let $Q = \operatorname{conv}\{P \cup \{r\} \cup \{s\}\}$, where r and s are so close to p_1 that $\operatorname{bd} K\big(p, r(p)\big) \cap \operatorname{bd} Q$ contains $\{q_1, q_2, \ldots, q_h\}$. Moreover, we can choose a convex body P', sufficiently close to Q, such that $\operatorname{bd} P' \cap \operatorname{bd} \mathcal{K}(P')$ is an irreducible set for $\mathcal{K}(P')$ of $k + h + 1$ points. Since Q can be constructed sufficiently close to P, P' is the convex body we are looking for.

Since P can be chosen sufficiently close to D, it follows that $\mathfrak{B}$ is dense in $\mathfrak{C}$.

Finally, the same arguments used in [9] show that the complement of $\mathfrak{B}$ is a meager set in $\mathfrak{C}$, thus concluding the proof of the theorem. ■

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ADDED IN PROOF. Theorem 1 (but without the uniqueness) has also been proved in [4a].

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