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FINITARY GROUP COHOMOLOGY AND GROUP ACTIONS ON SPHERES

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Abstract We show that if G is an infinitely generated locally (polycyclic-by-finite) group with cohomology almost-everywhere finitary, then every finite subgroup of G acts freely and orthogonally on some sphere.

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1. Introduction

In [**3**] the question of which locally (polycyclic-by-finite) groups have cohomology almost-everywhere finitary was considered. Recall that a functor is *finitary* if it preserves filtered colimits (see [**7**, § 6.5] and also [**1**, § 3.18]). The nth cohomology of a group G is a functor $H^n(G, \cdot)$ from the category of $\mathbb{Z}G$-modules to the category of abelian groups. If G is a locally (polycyclic-by-finite) group, then [**5**, Theorem 2.1] shows that the *finitary set*

$$\mathcal{F}(G) := \{n \in \mathbb{N} : H^n(G, \cdot) \text{ is finitary}\}$$

is either cofinite or finite. If $\mathcal{F}(G)$ is cofinite, we say that G has *cohomology almost-everywhere finitary*, and if $\mathcal{F}(G)$ is finite, we say that G has *cohomology almost-everywhere infinitary.*

We proved the following results about locally (polycyclic-by-finite) groups with cohomology almost-everywhere finitary in [**3**].

Theorem 1.1. *Let G be a locally (polycyclic-by-finite) group. Then G has cohomology almost-everywhere finitary if and only if G has finite virtual cohomological dimension and the normalizer of every non-trivial finite subgroup is finitely generated.*

Corollary 1.2. *Let G be a locally (polycyclic-by-finite) group with cohomology almost-everywhere finitary. Then every subgroup of G also has cohomology almost-everywhere finitary.*

Recall (see, for example, [**10**]) that a finite group acts freely and orthogonally on some sphere if and only if every subgroup of order pq, where p and q are prime, is cyclic. In this paper, we prove the following result.

Theorem 1.3 (main theorem). *Let G be an infinitely generated locally (polycyclic-by-finite) group with cohomology almost-everywhere finitary. Then every finite subgroup of G acts freely and orthogonally on some sphere.*

Note that we cannot remove the 'infinitely generated' restriction, as, for example, every finite group is of type FP_∞ and so has nth cohomology functors finitary for all n, by a result of Brown (see the corollary to Theorem 1 in [**2**]).

2. Proof

The following proposition sets the scene for proving Theorem 1.3.

Proposition 2.1. *Let G be a locally (polycyclic-by-finite) group with cohomology almost-everywhere finitary. Then G has a characteristic subgroup S of finite index, such that S is torsion-free soluble of finite Hirsch length.*

Proof. By Theorem 2.1 in [**5**] we know that there is a finite-dimensional contractible G-CW-complex X on which G acts with finite isotropy groups, and that there is a bound on the orders of the finite subgroups of G.

We know that the rational cohomological dimension of G is bounded above by the dimension of X (see, for example, [**6**]), so G has finite rational cohomological dimension. Recall that the class of elementary amenable groups is the class generated from the finite groups and $\mathbb{Z}$ by the operations of extension and increasing union (see, for example, [**4**]), so G is elementary amenable. According to [**4**], the Hirsch length of an elementary amenable group is bounded above by its rational cohomological dimension, so G has finite Hirsch length.

Let $\tau(G)$ denote the join of the locally finite normal subgroups of G. As there is a bound on the orders of the finite subgroups of G, this implies that $\tau(G)$ is finite. Replacing G with $G/\tau(G)$, we may assume that $\tau(G) = 1$.

Now G is an elementary amenable group of finite Hirsch length, so it follows from a minor extension of a theorem by Mal'cev (see [**12**]) that $G/\tau(G) = G$ has a poly (torsion-free abelian) characteristic subgroup of finite index. $\square$

Before proving Theorem 1.3, we need four lemmas.

Lemma 2.2. *Let Q be a non-cyclic group of order pq, where p and q are prime, and let A be a $\mathbb{Z}$-torsion-free $\mathbb{Z}Q$-module such that the group $A \rtimes Q$ has cohomology almost-everywhere finitary. Then A is finitely generated.*

Proof. We write $G := A \rtimes Q$.

For any $K \leqslant Q$ we write $\hat{K}$ for the element of $\mathbb{Z}Q$ given by

$$\hat{K} := \sum_{k \in K} k.$$

Notice that $\hat{K}A$ is contained in the set of K-invariant elements A^K of A.

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There are two cases to consider.

(i) If Q is abelian, then $p = q$ and Q has $p + 1$ subgroups $E_0, \dots, E_p$ of order p. We have the following equation in $\mathbb{Z}Q$:

$$\sum_{i=0}^{p} \hat{E}_i = \hat{Q} + p,$$

so it follows that, for any $a \in A$,

$$pa = \sum_{i=0}^{p} \hat{E}_i a - \hat{Q} a \in \sum_{i=0}^{p} A^{E_i} + A^Q$$

and hence

$$pA \subseteq \sum_{i=0}^{p} A^{E_i} + A^Q.$$

If K is non-trivial, then it follows from Theorem 1.1 that $N_G(K)$ is finitely generated. Then, as $A^K \leqslant N_G(K)$, it follows that A^K is also finitely generated. Hence, we see that pA is finitely generated, and as A is torsion-free, we conclude that A is finitely generated.

(ii) If Q is non-abelian, then $p \neq q$, and without loss of generality we may assume that $p < q$. Then Q has one subgroup F of order q and q subgroups $H_0, \dots, H_{q-1}$ of order p. We have the following equation in $\mathbb{Z}Q$:

$$\sum_{i=0}^{q-1} \hat{H}_i + \hat{F} = \hat{Q} + q$$

and the proof continues as above. □

Recall (see, for example, [**8**, § 10.4]) that a group G is *upper-finite* if and only if every finitely generated homomorphic image of G is finite. The class of upper-finite groups is closed under extensions and homomorphic images. Also recall (see [**8**, § 10.4]) that the *upper-finite radical* of a group G is the subgroup generated by all of its upper-finite normal subgroups, and is itself upper-finite.

Lemma 2.3. *Let A and B be abelian groups. If A is upper-finite, then $A \otimes B$ is upper-finite.*

Proof. If $b \in B$, then $A \otimes b$ is a homomorphic image of A and hence is upper-finite. Then, as $A \otimes B$ is generated by all the $A \otimes b$, it is also upper-finite. □

Lemma 2.4. *Let G be an upper-finite nilpotent group. Then its derived subgroup G' is also upper-finite.*

Proof. As G is upper-finite, it follows that G/G' is also upper-finite.

As G is a nilpotent group, it has a finite lower central series

$$G = \gamma_1(G) \geqslant \gamma_2(G) \geqslant \cdots \geqslant \gamma_k(G) = 1,$$

where $\gamma_2(G) = G'$.

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For each i there is an epimorphism

$$\underbrace{G/G' \otimes \cdots \otimes G/G'}_{i} \twoheadrightarrow \gamma_i(G)/\gamma_{i+1}(G),$$

and as

$$\underbrace{G/G' \otimes \cdots \otimes G/G'}_{i}$$

is upper-finite, from Lemma 2.3, we see that each $\gamma_i(G)/\gamma_{i+1}(G)$ is upper-finite. Then, as the class of upper-finite groups is closed under extensions, we conclude that G' is also upper-finite. □

Lemma 2.5. *Let G be a torsion-free nilpotent group of finite Hirsch length. If the centre $\zeta(G)$ of G is finitely generated, then G is finitely generated.*

Proof. Let K be the upper-finite radical of G. As G is a torsion-free nilpotent group of finite Hirsch length, it is a special case of Lemma 10.45 in [**8**] that G/K is finitely generated. Suppose that $K \neq 1$.

Following an argument of Robinson [**8**, Lemma 10.44] we see that, for each $g \in G$, $[K, g]K'/K'$ is a homomorphic image of K, and so is upper-finite, so therefore $[K, G]/K'$ is upper-finite. Then, as K' is upper-finite, from Lemma 2.4, we see that $[K, G]$ is also upper-finite. Similarly, we see by induction that

$$[K, {}^m G] = [K, \underbrace{G, \dots, G}_{m}]$$

is upper-finite.

Choose the largest m such that $[K, {}^m G] \neq 1$. Then $[K, {}^m G] \subseteq \zeta(G)$, so $[K, {}^m G]$ is finitely generated, and hence finite. Then, as G is torsion-free, we see that $[K, {}^m G] = 1$, which is a contradiction. Therefore, $K = 1$, and so G is finitely generated. □

We can now prove Theorem 1.3.

Proof of Theorem 1.3. Let G be an infinitely generated locally (polycyclic-by-finite) group with cohomology almost-everywhere finitary. It follows from Proposition 2.1 that G has a characteristic subgroup S of finite index such that S is torsion-free soluble of finite Hirsch length.

Suppose that not every subgroup of G acts freely and orthogonally on some sphere, so there is a non-cyclic subgroup Q of order pq, where p and q are prime.

As S is a torsion-free soluble group of finite Hirsch length, it is linear over the rationals (see, for example, [**11**]), so by a result of Gruenberg (see [**11**, Theorem 8.2]) the Fitting subgroup $F := \mathrm{Fitt}(S)$ of S is nilpotent. Now the centre $\zeta(F)$ of F is a characteristic subgroup of G, so we can consider the group $\zeta(F)Q$. It then follows from Corollary 1.2 that $\zeta(F)Q$ has cohomology almost-everywhere finitary. Then, by Lemma 2.2, we see that $\zeta(F)$ is finitely generated. It then follows from Lemma 2.5 that F is finitely generated.

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Now, let K be the subgroup of S containing F such that $K/F = \tau(S/F)$. As S is linear over $\mathbb{Q}$, we see that S/F is also linear over $\mathbb{Q}$, and as locally finite $\mathbb{Q}$-linear groups are finite (see, for example, [**11**, Theorem 9.33]), we conclude that K/F is finite. An argument of Zassenhaus in [**9**, 15.1.2] shows that S/K is maximal abelian-by-finite, that is, crystallographic. Hence, S/F is finitely generated, so we conclude that S is finitely generated, which is a contradiction. □

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