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ON THE WEAK BASIS THEOREM IN F-SPACES

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Introduction. It is well-known that every weak basis in a Fréchet space is actually a basis. This result, called the *weak basis theorem* was first given for Banach spaces in 1932 by Banach [**1**, p. 238], and extended to Fréchet spaces by Bessaga and Pełczynski [**3**]. McArthur [**12**] proved an analogue for bases of subspaces in Fréchet spaces, and recently W. J. Stiles [**18**, Corollary 4.5, p. 413] showed that the theorem *fails* in the non-locally convex spaces l^p $(0 < p < 1)$. The purpose of this paper is to prove the following generalization of Stiles' result.

THEOREM 1. *The weak basis theorem fails in every locally bounded, non-locally convex F-space which has a weak basis.*

In other words, if a locally bounded F-space is not locally convex and has a weak basis, then it has a weak basis that is not a basis (of course, if the space does not have a weak basis, then the weak basis theorem holds vacuously). Theorem 1 resembles our earlier result [**13**, Theorem 1, p. 644]: *the Hahn Banach Extension Theorem fails in every non-locally convex F-space with a basis.* In fact, some of the elements of that proof, along with the Krein-Milman-Rutman theorem on perturbation of bases, provide the tools needed for the proof of Theorem 1. We present this proof in section 2, after setting out some preliminary material in section 1. In the third section we generalize Theorem 1 to F-spaces which admit continuous norms, and in the fourth section we apply Theorem 1 to the H^p spaces of functions analytic in the unit disc. We close by recording some open problems.

1. Preliminaries. A complete, metrizable topological vector space is called an *F-space*; and a locally convex F-space is a *Fréchet space*. A topological vector space is *locally bounded* if it has a bounded neighborhood of zero [**8**, section 6, p. 55, Problem M]. Every locally bounded space is pseudo-metrizable; and if it is locally convex, then by a theorem of Kolmogorov [**8**, Theorem 6.1, p. 44] it must be normable.

A sequence (e_n) in a topological vector space E is called a *basis* if for each f in E there exists a unique scalar sequence $(e_n'(f))$ such that the series $\sum e_n'(f)e_n$ converges to f in the topology of E. The linear functionals (e_n') are called the *coordinate functionals* of the basis: if E is an F-space then they must be continuous [**11**, IX.5, Theorem 2, p 126]. A basis for E taken in its

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weak topology is called a *weak basis*. We call a sequence in E *regular* if it lies entirely outside some neighborhood of zero.

The notion of *Mackey topology* will play a fundamental role in our work. If E is a linear space and F is a linear subspace of its algebraic dual, then there is a unique strongest locally convex topology τ on E such that F is the τ-dual of E [**8**, section 18]. This topology is called the *Mackey topology* of the pair (E, F), and we will denote it by $m(E, F)$. It is well-known that *the topology of a pseudo-metrizable locally convex space E is already the Mackey topology of the pair* (E, E') [**8**, Corollary 22.3, p. 210].

From these remarks it follows quickly that if E is a metrizable topological vector space with dual E', then a local base for $m(E, E')$ can be formed by taking the convex hulls of the members of a local base for the original topology of E [**13**, Proposition 3, p. 641]. Thus *the Mackey topology of a metrizable topological vector space is the unique locally convex pseudometrizable topology which is weaker than the original topology, and has the same continuous linear functionals*. For example, if $0 < p < 1$, then $L^p([0, 1])' = \{0\}$, and $(l^p)' = l^\infty$; so the Mackey topology of $L^p([0, 1])$ is the indiscrete topology, while the Mackey topology of l^p is the l^1-topology.

It is easy to see from our remark about local bases for $m(E, E')$ that if E is locally bounded in its original topology, then it is locally bounded in its Mackey topology. If in addition E has enough continuous linear functionals to separate points, then its weak topology is Hausdorff, so its Mackey topology is also Hausdorff, and therefore *normable* by the previously mentioned theorem of Kolmogorov.

2. Proof of Theorem 1. The outline of the proof is straightforward. Suppose E is a locally bounded, non-locally convex F-space with a weak basis (e_n). We will show that it is possible to make a perturbation of (e_n) that is so small in the Mackey topology that the new sequence is still a weak basis, yet so large in the original topology that it is not a basis. The next three propositions, contain the means for accomplishing this plan. The first is the Krein-Milman-Rutman theorem [9], which was used by Stiles [**18**] to show that l^p has a weak basis that is not a basis when $0 < p < 1$. The second is an observation about equicontinuity of the coordinate functionals of a basis, which occurs in the work of N. J. Kalton [7]; and the third is a standard criterion for a biorthogonal system to be a basis.

PROPOSITION A [**9**; **11**, IV. 3, Theorem 3, p. 63]. *Suppose E is a Banach space and (e_n) is a basis for E with coordinate functionals (e_n'). Suppose (f_n) is a sequence in E such that*

(3.1) $\sum ||e_n - f_n|| \, ||e_n'|| < 1.$

Then (f_n) is also a basis for E.

PROPOSITION B [**7**; Proposition 2.1, p. 92]. *Suppose (e_n) is a basis for a* *topological vector space E. If the coordinate functionals (e_n') are equicontinuous, then (e_n) is regular. Conversely, if E is an F-space and (e_n) is regular, then the coordinate functionals are equicontinuous.*

PROPOSITION C [**11**, III.2, Theorem 2 and Corollary 3, p. 31]. *Suppose (e_n, e_n') is a bi-orthogonal sequence in an F-space E in which (e_n) spans a dense subspace. Then (e_n) is a basis for E if and only if the partial sum operators (s_n) defined by*

$$s_nf = \sum_{k\leq n} e_k'(f)e_k \quad (f \text{ in } E) \tag{3.2}$$

form an equicontinuous family.

Although Proposition C is given in [**11**] for Banach spaces, the proof, suitably interpreted, works for general F-spaces, and we omit it.

Recall that we are assuming E is a locally bounded F-space with a weak basis (e_n), and we want to find a weak basis for E that is not a basis. We may as well assume that (e_n) is actually a basis: otherwise we are done. The coordinate functionals (e_n') are then continuous, so E has enough continuous linear functionals to separate points. It follows from this and the discussion in section 1 that the Mackey topology $m(E, E')$ is normable. Thus the Mackey completion $\hat{E}$ of E is a Banach space to which the coordinate functionals e_n' extend continuously and uniquely. We can therefore consider (e_n, e_n') to be a biorthogonal system in $\hat{E}$.

We claim that (e_n) is a basis for $\hat{E}$. Indeed, by Proposition C the partial sum operators (s_n) defined by equation (3.2) are equicontinuous on E; and hence $m(E, E')$-equicontinuous since the convex hulls of the E-neighborhoods of zero form a basis for the $m(E, E')$-neighborhoods. Now each s_n extends uniquely to $\hat{E}$, and its extension is still given by equation (3.2), where e_n' is now regarded as the continuous linear extension of the original coordinate functional to $\hat{E}$. It follows that the extended operators (s_n) are equicontinuous on $\hat{E}$, so again by Proposition C, (e_n) is a basis for $\hat{E}$.

Let the symbol $\|\cdot\|$ denote the norm on $\hat{E}$, and also the dual norm it induces on $E' = (\hat{E})'$. Since E is not locally convex, its Mackey topology is strictly weaker than its original topology, so there is a sequence (u_n) in E which is regular in the original topology, but Mackey-convergent to zero. By multiplying the elements of the basis (e_n) by appropriate non-zero scalars if necessary we can assume that $e_n \to 0$ in the original topology. Passing to a suitable subsequence of (u_n) we can therefore arrange that the sequence $(e_n + u_n)$ is regular in the original topology, and that $\sum \|u_n\|\|e_n'\| < 1$. Let $f_n = e_n + u_n$. Then (f_n) is a sequence in E that satisfies (3.1), so by Proposition A it is a basis for $\hat{E}$, and hence a weak basis for E. But (f_n) tends to zero in the Mackey topology, since (e_n) and (u_n) do, and is regular in the original topology. This implies that (f_n) cannot be a basis for E. For if it is a basis, then by Proposition B the coordinate functionals (f_n') are equicontinuous in the original topology;

and hence, as in the last paragraph, equicontinuous in the Mackey topology. So by Proposition B the sequence (f_n) is regular in the Mackey topology: a contradiction. This completes the proof of Theorem 1.

3. Generalization of Theorem 1. The proof of Theorem 1 can be modified to work for a more general class of F-spaces.

THEOREM 2. *Let E be an F-space which admits a continuous norm. If E is not locally convex, and has a weak basis, then E has a weak basis that is not a basis.*

The proof requires a generalization of the Krein-Milman-Rutman Theorem (Proposition A) to Fréchet spaces.

PROPOSITION D [7, Proposition 4.1, p. 97]. *Suppose E is a Fréchet space, and (e_n) is a basis for E with coordinate functionals (e_n'). Suppose further that p_0 is a continuous seminorm on E with $|e_n'(f)| \leqq p_0(f)$ for all n and all f in E. If (f_n) is a sequence in E with*

$$\sum p_0(e_n - f_n) < 1$$

and

$$\sum p(e_n - f_n) < \infty$$

for every continuous seminorm p on E, then (f_n) is also a basis for E.

Proof of Theorem 2. Let E be a non-locally convex F-space. As before we may assume that E has a basis $(e_n)_{n=1}^{\infty}$. We are assuming that E has a continuous *norm* p, hence p extends uniquely to a continuous norm on $\hat{E}$, the Mackey completion of E. As before, (e_n) is also a basis for $\hat{E}$. At this point the original proof requires some rearranging. By replacing e_n by $e_n/p(e_n)$ if necessary, we can insure that (e_n) is a regular sequence in $\hat{E}$, and hence that the coordinate functionals $(e_n)'$ are equicontinuous on $\hat{E}$, by Proposition B. Thus all the functionals e_n' are dominated by a single continuous seminorm p_0. Let $p_0 \leqq p_1 \leqq p_2 \ldots$ be a sequence of seminorms yielding the topology of $\hat{E}$. Choose scalars $\alpha_n \neq 0$ such that $\alpha_n e_n \to 0$ in E. Since $m(E, E')$ is strictly weaker than the original topology of E, we can choose a sequence (u_n) in E which is regular in the original topology yet for which

$$p_n(u_n/\alpha_n) < 1/2^n \qquad (n = 1, 2, \ldots).$$

Thus

$$\sum_{n \geqq 1} p_0(u_n/\alpha_n) < 1$$

while

$$\sum_{n \geqq k} p_k(u_n/\alpha_n) \leqq \sum_{n \leqq k} p_n(u_n/\alpha_n) < \sum_{n \geqq k} 2^{-n} < \infty$$

for $k = 1, 2, \ldots$; so the sequence $(e_n + u_n/\alpha_n)$ satisfies the hypotheses of

Proposiiton D, and is therefore a basis for $\hat{E}$. It follows that

$$f_n = \alpha_n e_n + u_n \qquad (n = 1, 2, \ldots)$$

defines a sequence in E which is a basis for $\hat{E}$, and hence a weak basis for E. Moreover $f_n \to 0$ in the Mackey topology of E, and is a regular sequence in the original topology (since $\alpha_n e_n \to 0$ in E; and (u_n) is regular, but Mackey convergent to zero). Thus (f_n) is not a basis for E by Proposition B, and the proof is complete.

We do not know if the requirement that E have a continuous norm in Theorem 2 can be removed. For example, let ω denote the space of all scalar sequences, with the topology of coordinatewise convergence. Then ω is an F-space which does not admit a continuous norm [**2**, Corollary 1, p. 375], so the same is true of the direct sum space $l^p \oplus \omega$. But if (e_n) is the standard basis for ω and (f_n) is a weak basis for l^p that is not a basis, then the set of pairs $\{(f_n, e_n)\}$ is a weak basis for $l^p \oplus \omega$ that is not a basis. Thus the weak basis theorem fails in $l^p \oplus \omega$, even though Theorem 2 does not apply.

However Theorem 2 still covers many spaces which arise naturally, and are not locally bounded. For example if $p_n \to 0$, then the space $l(p_n)$ consisting of sequences $f = (f(n))$ such that

$$\|f\| = \sum |f(n)|^{p_n},$$

falls into this class; as does the space $\bigcap_{p>p_0} l^p$ for $0 \leqq p_0 < 1$, taken in its natural least upper bound topology (cf. [**15**, section 2; **17**]). The l^1 norm is clearly a continuous norm on these spaces.

4. The weak basis theorem fails in H^p $(0 < p < 1)$. The *Hardy class* H^p $(0 < p < \infty)$ is the linear space of functions f analytic in the open unit disc $|z| < 1$ such that

$$||f||_p^p = \sup_{0 \leqq r < 1} \int_0^{2\pi} |f(re^{it})|^p dt < \infty.$$

For $0 < p < 1$ the functional $\|\cdot\|_p^p$ induces a translation invariant metric on H^p which turns it into a complete, locally bounded, non-locally convex space [**4**, section 3.2, p. 37] with a separating family of continuous linear functionals (evaluation at points of $|z| < 1$, for example). The Mackey completion of H^p $(0 < p < 1)$ has been identified by Duren, Romberg, and Shields [**5**, section 3] as the space B^p of functions f analytic in $|z| < 1$ with

$$||f||_{B^p} = \iint_{|z|<1} |f(z)| d\mu_p(z),$$

where

$$d\mu_p(z) = (1 - |z|)^{(1/p)-2}\, dxdy.$$

In [**16**] Shields and Williams show that B^p is a complemented subspace of $L^1(\mu_p)$, and Lindenstrauss and Pełczynski [**10**, p. 248] use this result to show that B^p is isomorphic to l^1. We use this last fact to prove:

THEOREM 3. *If* $0 < p < 1$, *then* H^p *has a weak basis that is not a basis.*

Proof. According to Theorem 1 we need only show that H^p has a weak basis. But the Mackey completion B^p of H^p, being isomorphic to l^1, has a basis; and since H^p is dense in B^p it follows from the Krein-Milman-Rutman Theorem (Proposition A, section 2) that there is a basis for B^p contained in H^p. This latter basis is therefore a weak basis for H^p, and the proof is complete.

5. Open problems. (a) It would be of interest to know if the perturbed sequence (f_n) which occurs in the proofs of Theorems 1 and 2 could be taken so that its closed linear span is not all of E. If so, then since the closed linear span of (f_n) is weakly dense in E ((f_n) is a weak basis), we would have a proof that every non-locally convex F-space which has a basis and admits a continuous norm has a proper, closed, weakly dense subspace. It is not known if this is true, although for $0 < p < 1$ the spaces l^p [**14**, p. 372] [**18**, p. 413], and H^p [**5**, Theorem 13, p. 53]; along with some other spaces of analytic functions [**14**] are known to contain such subspaces. From [**13**, Theorem 1, p. 644] and [**5**, Theorem 17, p. 59] it follows that every non-locally convex F-space with a basis has a closed subspace that is not weakly closed.

(b) Does Theorem 2 hold for all F-spaces with weak basis? If we omit the F-space hypothesis, then Theorem 2 fails. For it was shown in [**6**] that if E is a topological vector space whose weak and Mackey topologies differ (for example if E is an infinite dimensional Banach space), then there is a non-locally convex vector topology τ between them. So if E is a Fréchet space, then any weak basis for E is a basis (by the weak basis theorem), and hence a τ-basis. Thus the weak basis theorem holds for (E, τ); and if E has a basis, then it holds non-trivially.

(c) Although we have shown that H^p $(0 < p < 1)$ has a weak basis, we do not know if it has a basis.

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