



# On the sign changes of Dirichlet coefficients of triple product $L$ -functions

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*Abstract.* Let  $f$  and  $g$  be two distinct normalized primitive holomorphic cusp forms of even integral weight  $k_1$  and  $k_2$  for the full modular group  $SL(2, \mathbb{Z})$ , respectively. Suppose that  $\lambda_{f \times f \times f}(n)$  and  $\lambda_{g \times g \times g}(n)$  are the  $n$ -th Dirichlet coefficient of the triple product  $L$ -functions  $L(s, f \times f \times f)$  and  $L(s, g \times g \times g)$ . In this paper, we consider the sign changes of the sequence  $\{\lambda_{f \times f \times f}(n)\}_{n \geq 1}$  and  $\{\lambda_{f \times f \times f}(n)\lambda_{g \times g \times g}(n)\}_{n \geq 1}$  in short intervals and establish quantitative results for the number of sign changes for  $n \leq x$ , which improve the previous results.

## 1 Introduction

Let  $H_k$  be the set of normalized primitive holomorphic cusp forms of even integral weight  $k$  for the full modular group  $SL(2, \mathbb{Z})$ , which are eigenfunctions of all the Hecke operators  $T_n$ . Then  $f(z) \in H_k$  has a Fourier expansion at the cusp infinity

$$(1.1) \quad f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz) \quad (\Im z > 0),$$

where we normalize  $f(z)$  so that  $\lambda_f(1) = 1$ . From the theory of Hecke operators, the Fourier coefficient  $\lambda_f(n)$  is real and satisfies the multiplicative property

$$(1.2) \quad \lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right),$$

where  $m \geq 1$  and  $n \geq 1$  are any integers. In 1974, Deligne [1] proved the Ramanujan-Petersson conjecture: for all integers  $n \geq 1$ ,

$$(1.3) \quad |\lambda_f(n)| \leq d(n),$$

where  $d(n)$  is the number of positive divisors of  $n$ .

The sign changes of Fourier coefficients attached to automorphic forms is an important problem and has been studied extensively by several scholars. In [12], Knopp, Kohnen and Pribitkin showed  $\{\lambda_f(n)\}_{n \geq 1}$  has infinitely many sign changes. After that, Ram Murty [20] first considered the sign changes of the sequence of Fourier coefficients in short intervals. Later, Meher, Shankhadhar, and Viswanadham [19] established lower bounds for the number of sign changes of the sequence  $\{\lambda_f(n^j)\}_{n \geq 1}$  with  $j = 2, 3, 4$ .

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However, the analogous questions of simultaneous sign changes of two cusp forms have also been investigated by a number of mathematicians. Let  $f$  and  $g$  be two different cusp forms. In [13], Kumari and Ram Murty considered the simultaneous sign changes problem about  $\{\lambda_f(n)\lambda_g(n)\}_{n \geq 1}$ . Later, Gun, Kumar, and Paul [3] studied this problem about  $\{\lambda_f(n)\lambda_g(n^2)\}_{n \geq 1}$ . Recently, Lao and Luo [14] also considered more general cases and obtained better results about  $\{\lambda_f(n^i)\lambda_g(n^j)\}_{n \geq 1}$  for  $i \geq 1, j \geq 2$ . Further, Hua [5, 6] investigated the analogous problem over a certain integral binary quadratic form.

The triple product L-function  $L(s, f \times f \times f)$  satisfies analogous analytic properties such as those of the Hecke L-functions, and its coefficients also change signs. In this paper, we investigate the sign change about the sequence  $\{\lambda_{f \times f \times f}(n)\}_{n \geq 1}$  and  $\{\lambda_{f \times f \times f}(n)\lambda_{g \times g \times g}(n)\}_{n \geq 1}$  in short intervals and prove the following theorems.

**Theorem 1.1** *Let  $f \in H_k$  and  $\lambda_{f \times f \times f}(n)$  be the  $n$ -th normalized Dirichlet coefficient of the triple product L-function  $L(s, f \times f \times f)$ . Then for  $j \geq 2$  and any  $\delta$  with*

$$1 - \frac{315}{40\sqrt{30} + 8442} = 0.963\dots < \delta < 1,$$

*the sequence  $\{\lambda_{f \times f \times f}(n)\}_{n \geq 1}$  has at least one sign change for  $n \in (x, x + x^\delta]$  for sufficiently large  $x$ . Moreover, the number of sign changes of the above sequence for  $n \leq x$  is  $\gg x^{1-\delta}$ .*

**Remark 1.2** By comparison, in Theorem 1.1, our results about the number of sign changes for  $n \leq x$  improve the results of Hua [7, Theorem 1.1].

**Theorem 1.3** *Let  $f \in H_{k_1}, g \in H_{k_2}$  be two different forms. Also let  $\lambda_{f \times f \times f}(n)$  and  $\lambda_{g \times g \times g}(n)$  be the  $n$ -th normalized Dirichlet coefficient of the triple product L-function  $L(s, f \times f \times f)$  and  $L(s, g \times g \times g)$ , respectively. Then for any  $\delta$  with*

$$1 - \frac{882}{400\sqrt{21} + 1771497} = 0.99950\dots < \delta < 1,$$

*the sequence  $\{\lambda_{f \times f \times f}(n)\lambda_{g \times g \times g}(n)\}_{n \geq 1}$  has at least one sign change for  $n \in (x, x + x^\delta]$  for sufficiently large  $x$ . Moreover, the number of sign changes of the above sequence for  $n \leq x$  is  $\gg x^{1-\delta}$ .*

In Section 2, we give some preliminary lemmas. In Section 3, we prove three propositions which play an important part in proving Theorem 1.1 and Theorem 1.3. In Section 4 and Section 5, we complete the proofs of Theorem 1.1 and Theorem 1.3, respectively. And, throughout the paper, we denote by  $\varepsilon$  a sufficiently small positive constant, whose value may not be necessarily the same in all occurrences.

## 2 Preliminary and some lemmas

In this section, we will establish and recall some preliminary results for the proofs of Theorem 1.1 and Theorem 1.3. We first recall the definitions about some specific L-functions.

The Hecke  $L$ -function attached to  $f \in H_k$  is given by

$$(2.1) \quad L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1},$$

which converges absolutely for  $\Re(s) > 1$ . The local parameters  $\alpha_f(p)$  and  $\beta_f(p)$  satisfy

$$(2.2) \quad \alpha_f(p) + \beta_f(p) = \lambda_f(p) \quad \text{and} \quad |\alpha_f(p)| = |\beta_f(p)| = 1.$$

The  $j$ -th symmetric power  $L$ -function attached to  $f \in H_k$  is defined as

$$(2.3) \quad L(s, \text{sym}^j f) := \prod_p \prod_{m=0}^j (1 - \alpha_f(p)^{j-m} \beta_f(p)^m p^{-s})^{-1},$$

for  $\Re(s) > 1$ . Then,  $L(s, \text{sym}^j f)$  can be expressed as the Dirichlet series

$$(2.4) \quad L(s, \text{sym}^j f) = \sum_{n=1}^{\infty} \frac{\lambda_{\text{sym}^j f}(n)}{n^s} = \prod_p \left(1 + \sum_{k \geq 1} \frac{\lambda_{\text{sym}^j f}(p^k)}{p^{ks}}\right),$$

where  $\lambda_{\text{sym}^j f}(n)$  is a real multiplicative function, and

$$(2.5) \quad L(s, \text{sym}^0 f) = \zeta(s), \quad L(s, \text{sym}^1 f) = L(s, f).$$

According to (2.1) and (2.3), we obtain

$$(2.6) \quad \lambda_{\text{sym}^j f}(p) = \sum_{m=0}^j \alpha_f^{j-m}(p) \beta_f^m(p) = \lambda_f(p^j).$$

**Remark 2.1** The result of Newton-Thorne [21] implies that  $\text{sym}^j f$  ( $j \geq 1$ ) is an automorphic cuspidal representation of  $GL(j+1)$ . This means that  $L(s, \text{sym}^j f)$  has an analytic continuation as an entire function in the whole complex plane  $\mathbb{C}$  and satisfies a certain functional equation of Riemann zeta-type of degree  $j+1$ .

The Rankin–Selberg  $L$ -function associated with  $\text{sym}^i f$  and  $\text{sym}^j g$  is defined by

$$(2.7) \quad \begin{aligned} L(s, \text{sym}^i f \times \text{sym}^j g) &= \prod_p \prod_{u=0}^i \prod_{v=0}^j \left(1 - \frac{\alpha_f(p)^{i-u} \beta_f(p)^u \alpha_g(p)^{j-v} \beta_g(p)^v}{p^s}\right)^{-1} \\ &= \sum_{n=1}^{\infty} \frac{\lambda_{\text{sym}^i f \times \text{sym}^j g}(n)}{n^s}, \quad \Re(s) > 1, \end{aligned}$$

where  $\lambda_{\text{sym}^i f \times \text{sym}^j g}(n)$  is a real multiplicative function, and

$$(2.8) \quad \lambda_{\text{sym}^i f \times \text{sym}^j g}(p) = \sum_{u=0}^i \sum_{v=0}^j \alpha_f(p)^{i-2u} \alpha_g(p)^{j-2v} = \lambda_{\text{sym}^i f}(p) \lambda_{\text{sym}^j g}(p).$$

In particular, we have

$$(2.9) \quad L(s, \text{sym}^1 f \times \text{sym}^1 g) = L(s, f \times g), \quad L(s, \text{sym}^2 \times \text{sym}^1 g) = L(s, \text{sym}^2 f \times g).$$

**Remark 2.2** Due to the works of Jacquet and Shalika [10] [11], Shahidi [26] [27], Rudnick and Sarnak [25], Lau and Wu [15] and Newton-Thorne [21], the Rankin-Selberg function  $L(s, \text{sym}^i f \times \text{sym}^j g)$  ( $f \in H_{k_1}, g \in H_{k_2}$  are two different forms) has an analytic continuation as an entire function in the whole complex plane  $\mathbb{C}$  and satisfies a certain functional equation of Riemann zeta-type of degree  $(i + 1)(j + 1)$ .

The triple product  $L$ -function associated with  $f$  is defined by

$$\begin{aligned}
 L(s, f \times f \times f) &= \prod_p \left(1 - \frac{\alpha_f(p)^3}{p^s}\right)^{-1} \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-3} \left(1 - \frac{\beta_f(p)^3}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-3} \\
 (2.10) \quad &:= \sum_{n=1}^{\infty} \frac{\lambda_{f \times f \times f}(n)}{n^s}, \quad \Re(s) > 1,
 \end{aligned}$$

where  $\lambda_{f \times f \times f}$  is real and multiplicative.

**Remark 2.3** Recalling that the triple product  $L$ -functions  $L(s, f \times f \times f)$  are automorphic  $L$ -functions has been showed by Garrett [2], Piatetski-Shapiro and Rallis [24], etc. Furthermore, we learn that the  $L$ -function  $L(f \times f \times f, s)$  has an analytic continuation as an entire function in the whole complex plane  $\mathbb{C}$  and satisfies certain Riemann zeta-type functional equations of degree 8.

Thus for  $i, j \geq 1$ ,  $L(s, \text{sym}^i f)$  and  $L(s, \text{sym}^i f \times \text{sym}^j g)$  are also general  $L$ -functions in the sense of Perelli [23]. For general  $L$ -functions, we have the following result.

**Lemma 2.4** Suppose that  $\mathfrak{L}(s)$  is a general  $L$ -function of degree  $m$ . Then, for any  $\varepsilon > 0$ , we have

$$(2.11) \quad \mathfrak{L}(\sigma + it) \ll (1 + |t|)^{\max\{\frac{m(1-\sigma)}{2}, 0\} + \varepsilon},$$

uniformly for  $1/2 \leq \sigma \leq 1 + \varepsilon$  and  $|t| \geq 1$ . And

$$(2.12) \quad \int_1^T |\mathfrak{L}(\sigma + it)|^2 dt \ll T^{\max\{m(1-\sigma), 1\} + \varepsilon},$$

uniformly for  $\frac{1}{2} \leq \sigma \leq 2$  and  $T \geq 1$ .

**Proof** See [23]. ■

**Lemma 2.5** Let  $k = \frac{8}{63}\sqrt{15} = 0.4918\dots$ . Then for any  $\varepsilon > 0$ , we have

$$(2.13) \quad \zeta(\sigma + it) \ll t^{k(1-\sigma)^{3/2} + \varepsilon}$$

uniformly for  $|t| \geq 1$  and  $1/2 \leq \sigma \leq 1$ .

**Proof** The bound is proved by Heath-Brown in [4, Theorem 5]. ■

**Lemma 2.6** For any  $\varepsilon > 0$ , we have

$$(2.14) \quad L(\sigma + it, \text{sym}^2 f) \ll (1 + |t|)^{\max\{\frac{6}{5}(1-\sigma), 0\} + \varepsilon},$$

uniformly for  $\frac{1}{2} \leq \sigma \leq 2$  and  $|t| \geq 1$ .

**Proof** See [16, Corollary 1.2]. ■

Suppose that  $\pi$  is a unitary cuspidal automorphic representation of  $GL_r(\mathbb{A}_{\mathbb{Q}})$  and  $L(s, \pi)$  is the automorphic  $L$ -function related to  $\pi$ . For  $1/2 < \sigma < 1$ , let  $m(\sigma) \geq 2$  be the supremum of all numbers  $m$  such that

$$(2.15) \quad \int_1^T |L(s, \pi)|^m dt \ll T^{1+\varepsilon}.$$

**Lemma 2.7** Let  $m(\sigma)$  be defined by (2.15). Then for each  $1 - 1/r < \sigma < 1$  with  $r \geq 4$ , we have

$$(2.16) \quad m(\sigma) \geq \frac{2}{r(1-\sigma)}.$$

**Proof** See [8, Theorem 1.1]. ■

Newton and Thorne [21, 22] proved that  $\text{sym}^j f$  corresponds to a cuspidal automorphic representation of  $GL_r(\mathbb{A}_{\mathbb{Q}})$  for all  $j \geq 1$  with  $f \in H_k$ . As a result, we obtain the following lemma.

**Lemma 2.8** For  $f \in H_k$  and any  $\varepsilon > 0$ , we have

$$(2.17) \quad \int_1^T |L(23/25 + it, \text{sym}^4 f)|^5 dt \ll T^{1+\varepsilon},$$

uniformly for  $T \geq 1$ .

**Proof** According to Lemma 2.7, for  $r = 5$ , we take  $\sigma = 23/25$ . ■

**Lemma 2.9** For  $f \in H_k$ , we have

$$(2.18) \quad L(s, f \times f \times f) = L(s, f)^2 L(s, \text{sym}^3 f).$$

**Proof** See [18, Lemma 2.1]. ■

**Lemma 2.10** For  $f \in H_k$  and  $\Re(s) > 1$ , let

$$(2.19) \quad L(s) = \sum_{n=1}^{\infty} \frac{\lambda_{f \times f \times f}^2(n)}{n^s}.$$

Then we have

$$(2.20) \quad L(s) = \zeta^5(s)L^9(s, \text{sym}^2 f)L^5(s, \text{sym}^4 f)L(s, \text{sym}^6 f)U(s),$$

where the function  $U(s)$  is a Dirichlet series absolutely convergent in  $\Re(s) > 1/2$  and  $U(s) \neq 0$  for  $\Re(s) = 1$ .

**Proof** See [17, Lemma 5]. ■

**Lemma 2.11** Let  $f \in H_{k_1}, g \in H_{k_2}$  be two different forms. For  $\Re(s) > 1$ , let

$$(2.21) \quad G(s) = \sum_{n=1}^{\infty} \frac{\lambda_{f \times f \times f}(n)\lambda_{g \times g \times g}(n)}{n^s}.$$

Then we have

$$(2.22) \quad G(s) = L(s, \text{sym}^3 f \times \text{sym}^3 g)L^2(s, f \times \text{sym}^3 g)L^2(s, \text{sym}^3 f \times g)L^4(s, f \times g)V(s),$$

where the function  $V(s)$  is a Dirichlet series absolutely convergent in  $\Re(s) > 1/2$  and  $V(s) \neq 0$  for  $\Re(s) = 1$ .

**Proof** Noting that  $\lambda_{f \times f \times f}(n)\lambda_{g \times g \times g}(n)$  is multiplicative and satisfies the upper bound  $O(n^\epsilon)$  due to (1.3), we obtain for  $\Re(s) > 1$ ,

$$(2.23) \quad G(s) = \prod_p \left( 1 + \frac{\lambda_{f \times f \times f}(p)\lambda_{g \times g \times g}(p)}{p^s} + \frac{\lambda_{f \times f \times f}(p^2)\lambda_{g \times g \times g}(p^2)}{p^{2s}} + \dots \right).$$

From Lemma 2.9, we have

$$(2.24) \quad \lambda_{f \times f \times f}(p) = \lambda_{\text{sym}^3 f}(p) + 2\lambda_f(p).$$

Then,

$$(2.25) \quad \begin{aligned} \lambda_{f \times f \times f}(p)\lambda_{g \times g \times g}(p) &= (\lambda_{\text{sym}^3 f}(p) + 2\lambda_f(p))(\lambda_{\text{sym}^3 g}(p) + 2\lambda_g(p)) \\ &= \lambda_{\text{sym}^3 f}(p)\lambda_{\text{sym}^3 g}(p) + 2\lambda_f(p)\lambda_{\text{sym}^3 g}(p) + 2\lambda_{\text{sym}^3 f}(p)\lambda_g(p) + 4\lambda_f(p)\lambda_g(p) \\ &:= b(p). \end{aligned}$$

Define

$$(2.26) \quad G_1(s) = L(s, \text{sym}^3 f \times \text{sym}^3 g)L^2(s, f \times \text{sym}^3 g)L^2(s, \text{sym}^3 f \times g)L^4(s, f \times g).$$

Then it can be written as

$$(2.27) \quad G_1(s) = \prod_p \left( 1 + \sum_{k \geq 1} \frac{b(p^k)}{p^{ks}} \right).$$

As a result,

$$\begin{aligned}
 G(s) &= G_1(s) \times \prod_p \left( 1 + \frac{\lambda_{f \times f \times f}(p^2) \lambda_{g \times g \times g}(p^2) - b(p^2)}{p^{2s}} + \dots \right) \\
 &:= L(s, \text{sym}^3 f \times \text{sym}^3 g) L^2(s, f \times \text{sym}^3 g) L^2(s, \text{sym}^3 f \times g) L^4(s, f \times g) V(s),
 \end{aligned}
 \tag{2.28}$$

where  $V(s)$  converges absolutely and uniformly in the half-plane  $\Re(s) > 1/2$ . ■

**Lemma 2.12** *Let  $f \in H_{k_1}$ ,  $g \in H_{k_2}$  be two different forms. For  $\Re(s) > 1$ , let*

$$H(s) = \sum_{n=1}^{\infty} \frac{\lambda_{f \times f \times f}^2(n) \lambda_{g \times g \times g}^2(n)}{n^s}.
 \tag{2.29}$$

Then we have

$$H(s) = H_1(s) W(s),
 \tag{2.30}$$

where

$$\begin{aligned}
 H_1(s) &= \zeta^{25}(s) L^{45}(s, \text{sym}^2 f) L^{45}(s, \text{sym}^2 g) L^{25}(s, \text{sym}^4 f) L^{25}(s, \text{sym}^4 g) L^5(s, \text{sym}^6 f) \\
 &\quad L^5(s, \text{sym}^6 g) L(s, \text{sym}^6 f \times \text{sym}^6 g) L^5(s, \text{sym}^6 f \times \text{sym}^4 g) L^9(s, \text{sym}^6 f \times \text{sym}^2 g) \\
 &\quad L^5(s, \text{sym}^4 f \times \text{sym}^6 g) L^{25}(s, \text{sym}^4 f \times \text{sym}^4 g) L^{45}(s, \text{sym}^4 f \times \text{sym}^2 g) \\
 &\quad L^9(s, \text{sym}^2 f \times \text{sym}^6 g) L^{45}(s, \text{sym}^2 f \times \text{sym}^4 g) L^{81}(s, \text{sym}^2 f \times \text{sym}^2 g),
 \end{aligned}$$

and the function  $W(s)$  is a Dirichlet series absolutely convergent in  $\Re(s) > 1/2$  and  $W(s) \neq 0$  for  $\Re(s) = 1$ .

**Proof** From (2.6) and Lemma 2.9, we have

$$\lambda_{f \times f \times f}(p) = \lambda_{\text{sym}^3 f}(p) + 2\lambda_f(p) \quad \text{and} \quad \lambda_{\text{sym}^j f}(p) = \lambda_f(p^j).
 \tag{2.31}$$

According to (1.2), we have

$$\begin{aligned}
 \lambda_{f \times f \times f}^2(p) &= (\lambda_f(p^3) + 2\lambda_f(p))^2 = \lambda_f^2(p^3) + 4\lambda_f^2(p) + 4\lambda_f(p^3)\lambda_f(p) \\
 &= \lambda_f(p^6) + 5\lambda_f(p^4) + 9\lambda_f(p^2) + 5 = \lambda_{\text{sym}^6 f}(p) + 5\lambda_{\text{sym}^4 f}(p) + 9\lambda_{\text{sym}^2 f}(p) + 5.
 \end{aligned}$$

From (2.8),

$$\begin{aligned}
 &\lambda_{f \times f \times f}^2(p) \lambda_{g \times g \times g}^2(p) \\
 &= \lambda_{\text{sym}^6 f \times \text{sym}^6 g}(p) + 5\lambda_{\text{sym}^6 f \times \text{sym}^4 g}(p) + 9\lambda_{\text{sym}^6 f \times \text{sym}^2 g}(p) + 5\lambda_{\text{sym}^6 f}(p) \\
 &\quad + 5\lambda_{\text{sym}^4 f \times \text{sym}^6 g}(p) + 25\lambda_{\text{sym}^4 f \times \text{sym}^4 g}(p) + 45\lambda_{\text{sym}^4 f \times \text{sym}^2 g}(p) + 25\lambda_{\text{sym}^4 f}(p) \\
 &\quad + 9\lambda_{\text{sym}^2 f \times \text{sym}^6 g}(p) + 45\lambda_{\text{sym}^2 f \times \text{sym}^4 g}(p) + 81\lambda_{\text{sym}^2 f \times \text{sym}^2 g}(p) + 45\lambda_{\text{sym}^2 f}(p) \\
 &\quad + 5\lambda_{\text{sym}^6 g}(p) + 25\lambda_{\text{sym}^4 g}(p) + 45\lambda_{\text{sym}^2 g}(p) + 25.
 \end{aligned}$$

Now the lemma follows by standard argument like Lemma 2.11, so we omit it here. ■

**Lemma 2.13** For  $f \in H_k$  and any  $\varepsilon > 0$ , we have

$$(2.32) \quad \sum_{n \leq x} \lambda_{f \times f \times f}(n) \ll x^{7/10+\varepsilon}.$$

**Proof** See [18, Theorem 1.1]. ■

### 3 The main proposition

In this section, we shall establish asymptotic formula of the sum

$$(3.1) \quad \sum_{n \leq x} \lambda_{f \times f \times f}^2(n), \quad \sum_{n \leq x} \lambda_{f \times f \times f}(n) \lambda_{g \times g \times g}(n) \quad \text{and} \quad \sum_{n \leq x} \lambda_{f \times f \times f}^2(n) \lambda_{g \times g \times g}^2(n),$$

respectively (see Propositions 3.1, 3.2, 3.3). These asymptotic formulas are the key to prove Theorem 1.1 and Theorem 1.3.

#### 3.1 Proof of Proposition 3.1

**Proposition 3.1** For  $f \in H_k$  and any  $\varepsilon > 0$ , we have

$$(3.2) \quad \sum_{n \leq x} \lambda_{f \times f \times f}^2(n) = xP(\log x) + O_{f,\varepsilon}(x^{1-\frac{315}{40\sqrt{30+8442}+\varepsilon}}),$$

where  $P(t)$  is a polynomial of degree 4.

**Proof** Recalling Lemma 2.10 and applying Perron’s formula (see [9, Proposition 5.54]), we have

$$(3.3) \quad \sum_{n \leq x} \lambda_{f \times f \times f}^2(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} L(s) \frac{x^s}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T}\right),$$

where  $b = 1 + \varepsilon$  and  $3 \leq T \leq x$  is a parameter to be chosen later.

Then, we move the line of integration to the parallel segment with  $\Re s = 23/25$ . By Cauchy’s residue theorem, we obtain

$$(3.4) \quad \sum_{n \leq x} \lambda_{f \times f \times f}^2(n) = \operatorname{Res}_{s=1} \left\{ L(s) \frac{x^s}{s} \right\} + \frac{1}{2\pi i} \int_{\mathfrak{L}} L(s) \frac{x^s}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T}\right),$$

where  $\mathfrak{L}$  is the contour joining  $1 + \varepsilon - iT$ ,  $23/25 - iT$ ,  $23/25 + iT$ ,  $1 + \varepsilon + iT$  with straight lines. The residue at  $s = 1$  is equal to  $xP(\log x)$ ,  $P(t)$  is a polynomial of degree 4. We also have  $U(s) \ll 1$  in that the absolutely convergence of  $U(s)$  for  $\Re(s) \geq 1/2 + \varepsilon$ . Consequently, formula (3.4) can be written as

$$(3.5) \quad \sum_{n \leq x} \lambda_{f \times f \times f}^2(n) = xP(\log x) + O\left(\mathfrak{J}_1^h + \mathfrak{J}_1^v + \frac{x^{1+\varepsilon}}{T}\right),$$

where

$$(3.6) \quad \mathfrak{J}_1^h := \frac{1}{T} \int_{23/25}^{1+\varepsilon} |L(\sigma + iT)| x^\sigma d\sigma \ll \sup_{23/25 \leq \sigma \leq 1+\varepsilon} x^\sigma T^{-1} |L(\sigma + iT)|,$$



and

$$(3.7) \quad \mathcal{J}_1^v := x^{23/25} \int_1^T |L(23/25 + it)| \frac{dt}{t} \ll x^{23/25+\varepsilon} \sup_{3 \leq T_1 \leq T} T_1^{-1} \int_{T_1}^{2T_1} |L(23/25 + it)| dt.$$

By Lemma 2.5,  $k = \frac{8}{63}\sqrt{15} = 0.4918\dots$ , for any  $\varepsilon > 0$ , we have

$$(3.8) \quad \zeta(23/25 + it) \ll t^{\frac{\sqrt{2}k}{5} \times \frac{2}{25} + \varepsilon}.$$

Following from the well-known Phragmen-Lindelof principle, we obtain

$$(3.9) \quad \zeta(\sigma + it) \ll t^{\max\{\frac{\sqrt{2}k}{5}(1-\sigma), 0\} + \varepsilon},$$

uniformly for  $23/25 \leq \sigma \leq 2$  and  $|t| \geq 3$ . According to Lemma 2.4, Lemma 2.6, and (3.9), we deduce that for  $23/25 \leq \sigma \leq 1 + \varepsilon$ ,

$$|L(\sigma + iT)| \ll T^{\{5 \cdot \frac{\sqrt{2}k}{5} + 9 \cdot \frac{6}{5} + 5 \cdot \frac{5}{2} + \frac{7}{2}\}(1-\sigma) + \varepsilon} = T^{\{\sqrt{2}k + \frac{134}{5}\}(1-\sigma) + \varepsilon}.$$

Then it follows that

$$(3.10) \quad \mathcal{J}_1^h \ll T^{\sqrt{2}k + \frac{129}{5} + \varepsilon} \sup_{23/25 \leq \sigma \leq 1 + \varepsilon} \left( \frac{x}{T^{\sqrt{2}k + \frac{134}{5}}} \right)^\sigma \ll x^{23/25 + \varepsilon} T^{\frac{80\sqrt{30} + 9009}{7875}} + \frac{x^{1+\varepsilon}}{T}.$$

For  $\mathcal{J}_1^v$ , we have

$$(3.11) \quad \mathcal{J}_1^v \ll x^{23/25 + \varepsilon} \sup_{3 \leq T_1 \leq T} \frac{I_{1,1}(T_1)}{T_1} \int_{T_1}^{2T_1} |L(23/25 + it, \text{sym}^4 f)|^5 dt,$$

where

$$I_{1,1}(T_1) = \max_{T_1 \leq t \leq 2T_1} \zeta^5(23/25 + it) L^9(23/25 + it, \text{sym}^2 f) L(23/25 + it, \text{sym}^6 f).$$

According to Lemma 2.4, Lemma 2.6, Lemma 2.8, and (3.8), we have

$$(3.12) \quad I_{1,1}(T_1) \ll T_1^{5 \times \frac{8\sqrt{15}}{63} \times (\frac{2}{25})^{3/2} + 9 \times \frac{6}{5} \times \frac{2}{25} + \frac{7}{2} \times \frac{2}{25} + \varepsilon} = T_1^{\frac{80\sqrt{30} + 9009}{7875} + \varepsilon},$$

and

$$(3.13) \quad \int_{T_1}^{2T_1} |L(23/25 + it, \text{sym}^4 f)|^5 dt \ll T_1^{1+\varepsilon}.$$

Consequently,

$$(3.14) \quad \mathcal{J}_1^v \ll x^{23/25 + \varepsilon} T^{\frac{80\sqrt{30} + 9009}{7875}}.$$

Inserting (3.10) and (3.14) into (3.5), and taking  $T = x^{\frac{315}{40\sqrt{30} + 8442}}$ , we obtain

$$(3.15) \quad \sum_{n \leq x} \lambda_{f \times f \times f}^2(n) = xP(\log x) + O(x^{1 - \frac{315}{40\sqrt{30} + 8442} + \varepsilon}). \quad \blacksquare$$

### 3.2 Proof of Proposition 3.2

**Proposition 3.2** Let  $f \in H_{k_1}$ ,  $g \in H_{k_2}$  be two different forms. For any  $\varepsilon > 0$ , we have

$$(3.16) \quad \sum_{n \leq x} \lambda_{f \times f \times f}(n) \lambda_{g \times g \times g}(n) \ll x^{31/32+\varepsilon}.$$

**Proof** According to Lemma 2.11 and applying Perron’s formula (see [9, Proposition 5.54]), we have

$$(3.17) \quad \sum_{n \leq x} \lambda_{f \times f \times f}(n) \lambda_{g \times g \times g}(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} G(s) \frac{x^s}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T}\right),$$

where  $b = 1 + \varepsilon$  and  $3 \leq T \leq x$  is a parameter to be chosen later.

Then, we move the line of integration to the parallel segment with  $\Re s = 1/2$ . By Cauchy’s residue theorem, we deduce that

$$(3.18) \quad \sum_{n \leq x} \lambda_{f \times f \times f}(n) \lambda_{g \times g \times g}(n) = \frac{1}{2\pi i} \int_{\mathcal{L}} G(s) \frac{x^s}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T}\right),$$

where  $\mathcal{L}$  is the contour joining  $1 + \varepsilon - iT$ ,  $1/2 - iT$ ,  $1/2 + iT$ ,  $1 + \varepsilon + iT$  with straight lines.  $G(s)$  has no poles in the half-plane  $\Re(s) > 1/2$  by using the analytic properties of Rankin-Selberg L-functions. We also have  $V(s) \ll 1$  in that the absolutely convergence of  $V(s)$  for  $\Re(s) \geq 1/2 + \varepsilon$ . Consequently, formula (3.18) can be written as

$$(3.19) \quad \sum_{n \leq x} \lambda_{f \times f \times f}(n) \lambda_{g \times g \times g}(n) = O\left(\mathcal{J}_1^h + \mathcal{J}_1^v + \frac{x^{1+\varepsilon}}{T}\right),$$

where

$$(3.20) \quad \mathcal{J}_1^h := \frac{1}{T} \int_{1/2}^{1+\varepsilon} |G(\sigma + iT)| x^\sigma d\sigma \ll \sup_{1/2 \leq \sigma \leq 1+\varepsilon} x^\sigma T^{-1} |G(\sigma + iT)|,$$

and

$$(3.21) \quad \mathcal{J}_1^v := x^{1/2} \int_1^T |G(1/2 + it)| \frac{dt}{t} \ll x^{1/2+\varepsilon} \sup_{3 \leq T_1 \leq T} T_1^{-1} \int_{T_1}^{2T_1} |G(1/2 + it)| dt.$$

According to Lemma 2.4, we obtain for  $1/2 \leq \sigma \leq 1 + \varepsilon$

$$(3.22) \quad |G(\sigma + iT)| \ll T^{\frac{16+16+16}{2}(1-\sigma)+\varepsilon} = T^{32(1-\sigma)+\varepsilon}.$$

Therefore,

$$(3.23) \quad \mathcal{J}_1^h \ll T^{31+\varepsilon} \sup_{1/2 \leq \sigma \leq 1+\varepsilon} \left(\frac{x}{T^{32}}\right)^\sigma \ll x^{1/2+\varepsilon} T^{15} + \frac{x^{1+\varepsilon}}{T}.$$

For  $\mathcal{J}_1^v$ , we get

$$(3.24) \quad \mathcal{J}_1^v \ll x^{1/2+\varepsilon} \sup_{1 \leq T_1 \leq T} \frac{I_{2,1}(T_1)}{T_1} \int_{T_1}^{2T_1} |L(1/2 + it, f \times \text{sym}^3 g)|^2 dt,$$

where

$$(3.25) \quad I_{2,1}(T_1) = \max_{\substack{T_1 \leq t \leq 2T_1, \\ s_0 = 1/2 + it}} L(s_0, \text{sym}^3 f \times \text{sym}^3 g) L^2(s_0, \text{sym}^3 f \times g) L^4(s_0, f \times g).$$

By Lemma 2.4, we have

$$(3.26) \quad I_{2,1}(T_1) \ll T_1^{12+\varepsilon} \quad \text{and} \quad \int_{T_1}^{2T_1} |L(1/2 + it, f \times \text{sym}^3 g)|^2 dt \ll T_1^{4+\varepsilon}.$$

As a result,

$$(3.27) \quad \mathcal{J}_1^v \ll x^{1/2+\varepsilon} T^{15+\varepsilon}.$$

Inserting (3.23) and (3.27) into (3.19), and taking  $T = x^{1/32}$ , we obtain

$$(3.28) \quad \sum_{n \leq x} \lambda_{f \times f \times f}(n) \lambda_{g \times g \times g}(n) \ll x^{31/32+\varepsilon}. \quad \blacksquare$$

### 3.3 Proof of Proposition 3.3

**Proposition 3.3** *Let  $f \in H_{k_1}$ ,  $g \in H_{k_2}$  be two different forms. For any  $\varepsilon > 0$ , we have*

$$(3.29) \quad \sum_{n \leq x} \lambda_{f \times f \times f}^2(n) \lambda_{g \times g \times g}^2(n) = xQ(\log x) + O\left(x^{1 - \frac{882}{400\sqrt{21} + 1771497} + \varepsilon}\right),$$

where  $Q(t)$  is a polynomial of degree 24.

**Proof** Recalling Lemma 2.12, we obtain

$$(3.30) \quad H_1(s) := \zeta^{25}(s) L^{45}(s, \text{sym}^2 f) L^{45}(s, \text{sym}^2 g) L^5(s, \text{sym}^4 f \times \text{sym}^6 g) H_2(s),$$

where

$$\begin{aligned} H_2(s) = & L^{25}(s, \text{sym}^4 f) L^{25}(s, \text{sym}^4 g) L^5(s, \text{sym}^6 f) L^5(s, \text{sym}^6 g) L(s, \text{sym}^6 f \times \text{sym}^6 g) \\ & L^5(s, \text{sym}^6 f \times \text{sym}^4 g) L^9(s, \text{sym}^6 f \times \text{sym}^2 g) L^{25}(s, \text{sym}^4 f \times \text{sym}^4 g) \\ & L^{45}(s, \text{sym}^4 f \times \text{sym}^2 g) L^9(s, \text{sym}^2 f \times \text{sym}^6 g) L^{45}(s, \text{sym}^2 f \times \text{sym}^4 g) \\ & L^{81}(s, \text{sym}^2 f \times \text{sym}^2 g) \end{aligned}$$

is a general  $L$ -function of degree 3626.

Applying Perron’s formula (see [9, Proposition 5.54]), we have

$$(3.31) \quad \sum_{n \leq x} \lambda_{f \times f \times f}^2(n) \lambda_{g \times g \times g}^2(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} H(s) \frac{x^s}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T}\right),$$

where  $b = 1 + \varepsilon$  and  $3 \leq T \leq x$  is a parameter to be chosen later.

Then, we move the line of integration to the parallel segment with  $\Re s = 34/35$ . By Cauchy’s residue theorem, we obtain

$$(3.32) \quad \sum_{n \leq x} \lambda_{f \times f \times f}^2(n) \lambda_{g \times g \times g}^2(n) = \text{Res}_{s=1} \left\{ H(s) \frac{x^s}{s} \right\} + \frac{1}{2\pi i} \int_{\mathcal{L}} H(s) \frac{x^s}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T}\right),$$

where  $\mathcal{L}$  is the contour joining  $1 + \varepsilon - iT$ ,  $34/35 - iT$ ,  $34/35 + iT$ ,  $1 + \varepsilon + iT$  with straight lines. The residue at  $s = 1$  is equal to  $xQ(\log x)$ ,  $Q(t)$  is a polynomial of degree 24. We also have  $W(s) \ll 1$  in that the absolutely convergence of  $W(s)$  for  $\Re(s) \geq 1/2 + \varepsilon$ . Consequently, formula (3.32) can be written as

$$(3.33) \quad \sum_{n \leq x} \lambda_{f \times f \times f}^2(n) \lambda_{g \times g \times g}^2(n) = xQ(\log x) + O\left(\mathcal{J}_3^h + \mathcal{J}_3^v + \frac{x^{1+\varepsilon}}{T}\right),$$

where

$$(3.34) \quad \mathcal{J}_3^h := \frac{1}{T} \int_{34/35}^{1+\varepsilon} |H(\sigma + iT)| x^\sigma d\sigma \ll \sup_{34/35 \leq \sigma \leq 1+\varepsilon} x^\sigma T^{-1} |H(\sigma + iT)|,$$

and

$$(3.35) \quad \mathcal{J}_3^v := x^{34/35} \int_1^T |H(34/35 + it)| \frac{dt}{t} \ll x^{34/35+\varepsilon} \sup_{3 \leq T_1 \leq T} T_1^{-1} \int_{T_1}^{2T_1} |H(34/35 + it)| dt.$$

By Lemma 2.5,  $k = \frac{8}{63} \sqrt{15} = 0.4918\dots$ , for any  $\varepsilon > 0$ , we have

$$(3.36) \quad \zeta(34/35 + it) \ll t^{\frac{k}{\sqrt{35}} \times \frac{1}{35} + \varepsilon}.$$

Following from the well-known Phragmen-Lindelof principle, we obtain

$$(3.37) \quad \zeta(\sigma + it) \ll t^{\max\{\frac{k}{\sqrt{35}}(1-\sigma), 0\} + \varepsilon},$$

uniformly for  $34/35 \leq \sigma \leq 2$  and  $|t| \geq 3$ . According to Lemma 2.4, Lemma 2.6, and (3.37), we obtain for  $34/35 \leq \sigma \leq 1 + \varepsilon$

$$|H(\sigma + iT)| \ll T^{\{25 \cdot \frac{k}{\sqrt{35}} + 45 \cdot \frac{6}{5} + 45 \cdot \frac{6}{5} + 5 \cdot \frac{35}{2} + \frac{3626}{2}\}(1-\sigma) + \varepsilon} = T^{\{\frac{25k}{\sqrt{35}} + \frac{4017}{2}\}(1-\sigma) + \varepsilon}.$$

Therefore,

$$(3.38) \quad \begin{aligned} \mathcal{J}_3^h &\ll T^{\frac{25k}{\sqrt{35}} + \frac{4015}{2} + \varepsilon} \sup_{34/35 \leq \sigma \leq 1+\varepsilon} \left( \frac{x}{T^{\frac{25k}{\sqrt{35}} + \frac{4017}{2}}} \right)^\sigma \\ &\ll x^{34/35+\varepsilon} T^{\{\frac{25k}{\sqrt{35}} + \frac{3947}{2}\} \cdot \frac{1}{35} + \frac{x^{1+\varepsilon}}{T}}. \end{aligned}$$

According to (3.30), we deduce that

$$(3.39) \quad \mathcal{J}_3^v \ll x^{34/35+\varepsilon} \sup_{3 \leq T_1 \leq T} \frac{I_{3,1}(T_1)}{T_1} \int_{T_1}^{2T_1} |L(34/35 + it, \text{sym}^4 f \times \text{sym}^6 g)|^2 dt,$$

where

$$I_{3,1}(T_1) = \max_{\substack{T_1 \leq t \leq 2T_1, \\ s_0 = 34/35 + tT_1}} \zeta^{25}(s_0)L^{45}(s_0, \text{sym}^2 f)L^{45}(s_0, \text{sym}^2 g)L^3(s_0, \text{sym}^4 f \times \text{sym}^6 g)H_2(s_0).$$

From Lemma 2.4, Lemma 2.6, and (3.36), we have

$$(3.40) \quad I_{3,1}(T_1) \ll T_1^{\{25 \cdot \frac{k}{\sqrt{35}} + 45 \cdot \frac{6}{5} + 45 \cdot \frac{6}{5} + 3 \cdot \frac{35}{2} + \frac{3626}{2}\}/35 + \epsilon} = T^{\{\frac{25k}{\sqrt{35}} + \frac{3947}{2}\} \cdot \frac{1}{35} + \epsilon},$$

and

$$(3.41) \quad \int_{T_1}^{2T_1} |L(34/35 + it, \text{sym}^4 f \times \text{sym}^6 g)|^2 dt \ll T_1^{1+\epsilon}.$$

As a result,

$$(3.42) \quad \mathcal{J}_3^v \ll x^{34/35 + \epsilon} T^{\{\frac{25k}{\sqrt{35}} + \frac{3947}{2}\} \cdot \frac{1}{35}}.$$

Inserting (3.38) and (3.42) into (3.33), and taking  $T = x^{\frac{882}{400\sqrt{21} + 1771497}}$ , we obtain

$$(3.43) \quad \sum_{n \leq x} \lambda_{f \times f \times f}^2(n) \lambda_{g \times g \times g}^2(n) = xQ(\log x) + O\left(x^{1 - \frac{882}{400\sqrt{21} + 1771497} + \epsilon}\right). \quad \blacksquare$$

### 4 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by the argument of contradiction. Let

$$(4.1) \quad 1 - \frac{315}{40\sqrt{30} + 8442} = 0.963\dots < \delta < 1.$$

Suppose that the sequence  $\{\lambda_{f \times f \times f}(n)\}_{n \geq 1}$  has the same sign in the interval  $(x, x + x^\delta]$ . Without loss of generality, let the sign is positive. Then by Lemma 2.13 and Deligne’s bound (1.3), we obtain

$$(4.2) \quad \sum_{x \leq n \leq x + x^\delta} \lambda_{f \times f \times f}^2(n) \ll x^\epsilon \sum_{x \leq n \leq x + x^\delta} \lambda_{f \times f \times f}(n) \ll x^{7/10 + \epsilon}.$$

According to Proposition 3.1, we deduce that

$$(4.3) \quad \begin{aligned} \sum_{x \leq n \leq x + x^\delta} \lambda_{f \times f \times f}^2(n) &= (x + x^\delta)P(\log(x + x^\delta)) - xP(\log x) + O_{f,\epsilon}(x^{1 - \frac{315}{40\sqrt{30} + 8442} + \epsilon}) \\ &\geq (x + x^\delta)P(\log x) - xP(\log x) + O_{f,\epsilon}(x^{1 - \frac{315}{40\sqrt{30} + 8442} + \epsilon}) \\ &= x^\delta P(\log x) + O_{f,\epsilon}(x^{1 - \frac{315}{40\sqrt{30} + 8442} + \epsilon}) \gg x^\delta. \end{aligned}$$

From (4.2) and (4.3), we get the contradiction. As a result, the sequence  $\{\lambda_{f \times f \times f}(n)\}_{n \geq 1}$  has at least one sign change in the interval  $(x, x + x^\delta]$  with  $0.963\dots < \delta < 1$ . Therefore, the sequence  $\{\lambda_{f \times f \times f}(n)\}_{n \geq 1}$  has at least  $\gg x^{1-\delta}$  sign change in the interval  $(x, x + x^\delta]$  for sufficiently large  $x$ .

## 5 Proof of Theorem 1.2

In this section, we prove Theorem 1.3 by the argument of contradiction. Let

$$(5.1) \quad 1 - \frac{882}{400\sqrt{21} + 1771497} = 0.99950\dots < \delta < 1.$$

Suppose that the sequence  $\{\lambda_{f \times f \times f}(n)\lambda_{g \times g \times g}(n)\}_{n \geq 1}$  has the same sign in the interval  $(x, x + x^\delta]$ . Without loss of generality, let the sign is positive. Then by Proposition 3.2 and Deligne’s bound (1.3), we obtain

$$(5.2) \quad \sum_{x \leq n \leq x + x^\delta} \lambda_{f \times f \times f}^2(n)\lambda_{g \times g \times g}^2(n) \ll x^\epsilon \sum_{x \leq n \leq x + x^\delta} \lambda_{f \times f \times f}(n)\lambda_{g \times g \times g}(n) \ll x^{31/32 + \epsilon}.$$

According to Proposition 3.3, we have

$$(5.3) \quad \begin{aligned} & \sum_{x \leq n \leq x + x^\delta} \lambda_{f \times f \times f}^2(n)\lambda_{g \times g \times g}^2(n) \\ &= (x + x^\delta)P(\log(x + x^\delta)) - xP(\log x) + O_{f,\epsilon}(x^{1 - \frac{882}{400\sqrt{21} + 1771497} + \epsilon}) \\ &\geq (x + x^\delta)P(\log x) - xP(\log x) + O_{f,\epsilon}(x^{1 - \frac{882}{400\sqrt{21} + 1771497} + \epsilon}) \\ &= x^\delta P(\log x) + O_{f,\epsilon}(x^{1 - \frac{882}{400\sqrt{21} + 1771497} + \epsilon}) \gg x^\delta. \end{aligned}$$

From (5.2), (5.3), and  $\frac{31}{32} = 0.96\dots$ , we get the contradiction. As a result, the sequence  $\{\lambda_{f \times f \times f}(n)\lambda_{g \times g \times g}(n)\}_{n \geq 1}$  has at least one sign change in the interval  $(x, x + x^\delta]$  with  $0.99950\dots < \delta < 1$ . Therefore, the sequence  $\{\lambda_{f \times f \times f}(n)\lambda_{g \times g \times g}(n)\}_{n \geq 1}$  has at least  $\gg x^{1-\delta}$  sign change in the interval  $(x, x + x^\delta]$  for sufficiently large  $x$ .

## References

- [1] P. Deligne, *La conjecture de Weil. I*. Inst. Hautes Études Sci. Publ. Math. 43(1974), no. 43, 273–307. MR 340258
- [2] P. B. Garrett, *Decomposition of Eisenstein series: Rankin triple products*. Ann. of Math. (2) 125(1987), no. 2, 209–235. MR 881269
- [3] S. Gun, B. Kumar and B. Paul, *The first simultaneous sign change and non-vanishing of Hecke eigenvalues of newforms*. J. Number Theory 200(2019), 161–184. MR 3944435
- [4] D. R. Heath-Brown, *A new  $k$ th derivative estimate for a trigonometric sum via Vinogradov’s integral*. Tr. Mat. Inst. Steklova 296(2017), 95–110, English version published in Proc. Steklov Inst. Math. 296(2017), no. 1, 88–103.
- [5] G. Hua, *On the simultaneous sign changes of Hecke eigenvalues over an integral binary quadratic form*. Acta Math. Hungar. 167(2022), no. 2, 476–491. MR 4487620
- [6] G. Hua, *On the simultaneous sign changes of coefficients of Rankin-Selberg L-functions over a certain integral binary quadratic form*. Math. Pannon. (2023), no. 2, 244–257.
- [7] G. Hua, *On the sign changes of the coefficients attached to triple product L-functions*. Int. J. Number Theory 20(2024), no. 08, 2069–2081.
- [8] J. Huang, *Higher moments of automorphic L-functions of  $GL(r)$  and its applications*. Acta Math. Hungar. 169(2023), no. 2, 489–502. MR 4594312
- [9] H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, Vol. 53, American Mathematical Society, Providence, RI, 2004.
- [10] H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic representations. I*. Amer. J. Math. 103(1981), no. 3, 499–558. MR 618323
- [11] H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic representations. I*. Amer. J. Math. 103(1981), no. 3, 499–558. MR 618323

- [12] M. Knopp, W. Kohnen and W. Pribitkin, *On the signs of fourier coefficients of cusp forms*. Ramanujan J. 7(2003), no. 1, 269–277.
- [13] M. Kumari and M. Ram Murty, *Simultaneous non-vanishing and sign changes of Fourier coefficients of modular forms*. Int. J. Number Theory 14(2018), no. 8, 2291–2301. MR 3846406
- [14] H. Lao and S. Luo, *Sign changes and nonvanishing of Fourier coefficients of holomorphic cusp forms*. Rocky Mountain J. Math. 51(2021), no. 5, 1701–1714. MR 4382993
- [15] Y.-K. Lau and J. Wu, *A density theorem on automorphic L-functions and some applications*. Trans. Amer. Math. Soc. 358(2006), no. 1, 441–472. MR 2171241
- [16] Y. Lin, R. Nunes and Z. Qi, *Strong subconvexity for self-dual  $GL(3)$  L-functions*. Int. Math. Res. Not. IMRN 2023(2023), no. 13, 11453–11470.
- [17] H. Liu, *The second moment of the Fourier coefficients of triple product L-functions*. Proc. Indian Acad. Sci. Math. Sci. 133(2023), no. 1, Paper No. 8, 12. MR 4572184
- [18] G. Lü and A. Sankaranarayanan, *On the coefficients of triple product L-functions*. Rocky Mountain J. Math. 47(2017), no. 2, 553–570. MR 3635374
- [19] J. Meher, K. D. Shankhadhar and G. K. Viswanadham, *A short note on sign changes*. Proc. Indian Acad. Sci. Math. Sci. 123(2013), no. 3, 315–320. MR 3102375
- [20] M. Ram Murty, *Oscillations of Fourier coefficients of modular forms*. Math. Ann. 262(1983), no. 4, 431–446. MR 696516
- [21] J. Newton and J. A. Thorne, *Symmetric power functoriality for holomorphic modular forms*. Publ. Math. Inst. Hautes Études Sci. 134(2021), 1–116. MR 4349240
- [22] J. Newton and J. A. Thorne, *Symmetric power functoriality for holomorphic modular forms, II*. Publ. Math. Inst. Hautes Études Sci. 134(2021), 117–152. MR 4349241
- [23] A. Perelli, *General L-functions*. Ann. Mat. Pura Appl. 130(1982), 287–306.
- [24] I. Piatetski-Shapiro and S. Rallis, *Rankin triple L functions*. Compos. Math. 64(1987), no. 1, 31–115.
- [25] Z. Rudnick and P. Sarnak, *Zeros of principal L-functions and random matrix theory*. Duke Math. J. 81(1996), no. 2, 269–322 (English).
- [26] F. Shahidi, *On certain L-functions*. Amer. J. Math. 103(1981), no. 2, 297–355. MR 610479
- [27] F. Shahidi, *Third symmetric power L-functions for  $GL(2)$* . Compos. Math. 70(1989), no. 3, 245–273. MR 1002045

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