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ON THE ALEKSANDROV–RASSIAS PROBLEM OF DISTANCE PRESERVING MAPPINGS

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Abstract

In this paper, we introduce the concept of a semi-parallelogram and obtain some results for the Aleksandrov–Rassias problem using this concept. In particular, we resolve an important case of this problem for mappings preserving two distances with a nonintegral ratio.

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1. Introduction

Let X and Y be normed spaces. A mapping $T : X \to Y$ is called an isometry if T satisfies

$$\|T(x) - T(y)\| = \|x - y\|$$

for all $x, y \in X$. A distance $r > 0$ is said to be contractive by $T : X \to Y$ if $\|x - y\| = r$ always implies $\|T(x) - T(y)\| \le r$. Similarly, a distance $r > 0$ is said to be extensive by T if the inequality $\|T(x) - T(y)\| \ge r$ is true for all $x, y \in X$ with $\|x - y\| = r$. We say that r is conservative (or preserved) by T if r is contractive and extensive by T simultaneously. Obviously, T is an isometry if and only if every distance $r > 0$ is conservative by T.

In 1970, Aleksandrov [1] posed a question now known as the Aleksandrov problem by asking whether a mapping $T : X \to X$ with a single conservative distance is an isometry. The Aleksandrov problem, not only for the mappings of a space into itself but also for the general mappings $T : X \to Y$ from one space into another, has been studied considerably (see [2, 5–10]). Note that we may assume without loss of generality that $r = 1$ when X and Y are normed spaces (see [7]).

In 1953, Beckman and Quarles [2] had already given a positive answer to the Aleksandrov problem for $T : E^n \to E^n$ $(2 \le n < \infty)$, where E^n is an n-dimensional real

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Euclidean space. Moreover, they also presented counterexamples for the cases $n = 1$ and $n = \infty$. In order to extend the result to infinite-dimensional spaces, Schröder [10] introduced a sufficient condition such that if E is a real inner product space with $\dim E \geq 2$, $T : E \to E$ is surjective and

$$\|T(x) - T(y)\| = r \Leftrightarrow \|x - y\| = r$$

for all $x, y \in E$ and for some positive number $r > 0$, then T is an isometry of E. In addition, in [10], he showed that with the same assumptions on the space E, if $T : E \to E$ preserves both r and $2r$, then T is an isometry of E.

In 1985, Benz [3] generalised the latter result of Schröder to real normed spaces under an additional condition. Two years later, Benz and Berens [4] showed that the condition attached to the domain space was redundant.

THEOREM 1.1 [4]. *Let X and Y be real normed spaces such that $\dim X \geq 2$ and Y is strictly convex. Suppose that $T : X \to Y$ is a mapping and $N \geq 2$ is a fixed positive integer. If a distance r is contractive and Nr is extensive by T, then T is a linear isometry up to translation.*

By the triangle inequality, it is easy to verify that the condition in Theorem 1.1 that a distance r is contractive and Nr is extensive by T is equivalent to the property that T preserves the two distances r and Nr.

In this connection, Rassias [6] asked whether a mapping $T : X \to Y$ preserving two distances with a nonintegral ratio is an isometry. This is now called the Aleksandrov–Rassias problem. Some results on this problem can be found in [8, 11].

Xiang [11] obtained several impressive results when $T : X \to Y$ preserves two or three distances with a nonintegral ratio and X and Y are real Hilbert spaces.

THEOREM 1.2 [11]. *Let X and Y be real Hilbert spaces with $\dim X \geq 2$. Suppose that $T : X \to Y$ preserves the two distances 1 and $\sqrt{3}$. Then T is a linear isometry up to translation.*

THEOREM 1.3 [11]. *Let X and Y be real Hilbert spaces with $\dim X \geq 2$. Suppose that $T : X \to Y$ preserves the two distances 1 and $n\sqrt{3}$ for some positive integer n. Then T is a linear isometry up to translation.*

THEOREM 1.4 [11]. *Let X and Y be real Hilbert spaces with $\dim X \geq 2$. Suppose that $T : X \to Y$ preserves the three distances 1, a $(0 \leq a \leq 2)$ and $n\sqrt{4 - a^2}$ for some nonnegative constant a and for some positive integer $n \geq 2$. Then T is a linear isometry up to translation.*

Obviously, Theorem 1.2 is a particular case $(n = 1)$ in Theorem 1.3. In fact, in [11], Theorem 1.2 is one of the main theorems, while Theorem 1.3 is just a corollary of it in view of Theorem 1.1. The reason why we list Theorem 1.2 here is that it will be used to generalise Theorem 1.3 in Section 3 (see Theorem 3.3). Moreover, Theorem 1.4 will also be generalised to Theorem 3.2 in Section 3.

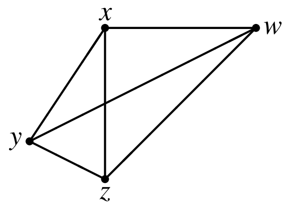

FIGURE 1. Illustration of the semi-parallelogram condition (Definition 2.2).

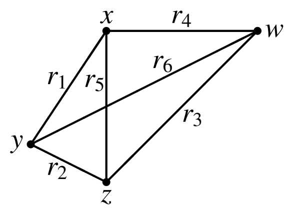

FIGURE 2. Illustration of the semi-parallelogram condition (Definition 2.3).

In this paper, we introduce the concept of a semi-parallelogram and obtain some results for the Aleksandrov–Rassias problem using this concept. In particular, we resolve an important case of this problem for mappings preserving two distances with a nonintegral ratio (see Theorem 3.3).

2. Some definitions and lemmas

All the above theorems from Theorem 1.2 to Theorem 1.4 take the parallelogram for their geometric interpretation (see [11]). In this paper, we work in a real inner product space X with $\dim X \geq 2$ and consider, more generally, planar convex quadrilaterals one of whose two diagonals is divided equally by the other (see Figure 1).

DEFINITION 2.1. A planar convex quadrilateral in X, one of whose two diagonals is divided equally by the other, is called a *semi-parallelogram*.

Since a planar convex quadrilateral in X is a parallelogram if and only if its two diagonals are divided equally by each other, all parallelograms are semi-parallelograms, but not *vice versa*.

From now on, '(SPC)' is short for 'the semi-parallelogram condition'.

DEFINITION 2.2. Let x, y, z and w be four elements in X. We say that a 4-tuple $\{x, y, z, w\}$ satisfies (SPC) if x, y, z and w, as four vertices in turn, form a semi-parallelogram, where the line segment with end points y and w passes through the mid point of the line segment with end points x and z (see Figure 1).

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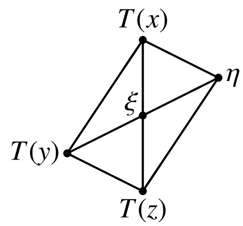

FIGURE 3. A parallelogram occurring in the proof of Lemma 2.5.

DEFINITION 2.3. Let r_1, r_2, r_3, r_4, r_5 and r_6 be six positive numbers. We say that a 6-tuple $\{r_1, r_2, r_3, r_4, r_5, r_6\}$ satisfies (SPC) if there exists a 4-tuple $\{x, y, z, w\}$ satisfying (SPC) such that r_1, r_2, r_3, r_4, r_5 and r_6 are the lengths of x–y, y–z, z–w, w–x, x–z and y–w, respectively (see Figure 2).

By the parallelogram law, one can easily verify the following result.

REMARK 2.4. A 6-tuple $\{r_1, r_2, r_3, r_4, r_5, r_6\}$ satisfies (SPC) if and only if both

$$\max\{|r_1 - r_2|, |r_3 - r_4|\} < r_5 < \min\{r_1 + r_2, r_3 + r_4\}$$

and

$$r_6 = \tfrac{1}{2}\Big(\sqrt{2r_1^2 + 2r_2^2 - r_5^2} + \sqrt{2r_3^2 + 2r_4^2 - r_5^2}\Big)$$

hold.

LEMMA 2.5. *Let X and Y be real inner product spaces with* $\dim X \geq 2$*, $T : X \to Y$ a mapping and $\{r_1, r_2, r_3, r_4, r_5, r_6\}$ a* 6*-tuple satisfying (SPC). Suppose that r_1, r_2, r_3, r_4 are contractive and r_5, r_6 extensive by T. Then $r_1, r_2, r_3, r_4, r_5, r_6$ are conservative by T.*

PROOF. By the hypothesis, there exists a 4-tuple $\{x, y, z, w\}$ satisfying (SPC) in X, where r_1, r_2, r_3, r_4, r_5 and r_6 are the lengths of x–y, y–z, z–w, w–x, x–z and y–w, respectively. Thus, x, y, z and w form a semi-parallelogram in X, as shown in Figure 2.

Set

$$\xi = \tfrac{1}{2}(T(x) + T(z)), \quad \eta = 2\xi - T(y).$$

Then $T(x), T(y), T(z)$ and η form a parallelogram in Y, as shown in Figure 3.

By the parallelogram law and the assumptions on r_1, r_2 and r_5,

$$\begin{aligned} \|T(y) - \xi\| &= \tfrac{1}{2}\|T(y) - \eta\| \\ &= \tfrac{1}{2}\sqrt{2\|T(x) - T(y)\|^2 + 2\|T(y) - T(z)\|^2 - \|T(x) - T(z)\|^2} \\ &\leq \tfrac{1}{2}\sqrt{2\|x - y\|^2 + 2\|y - z\|^2 - \|x - z\|^2} \\ &= \tfrac{1}{2}\sqrt{2r_1^2 + 2r_2^2 - r_5^2}. \end{aligned}$$

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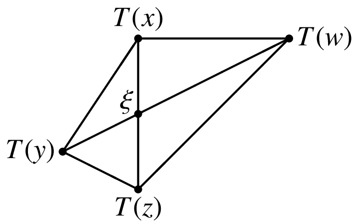

FIGURE 4. A semi-parallelogram arising from the proof of Lemma 2.5.

Similarly,

$$\|T(w) - \xi\| \le \tfrac{1}{2}\sqrt{2r_3^2 + 2r_4^2 - r_5^2}.$$

Hence,

$$\begin{aligned}\|T(y) - T(w)\| &\le \|T(y) - \xi\| + \|\xi - T(w)\| \\ &\le \tfrac{1}{2}\Big(\sqrt{2r_1^2 + 2r_2^2 - r_5^2} + \sqrt{2r_3^2 + 2r_4^2 - r_5^2}\Big).\end{aligned}$$

By Remark 2.4 and the assumption on r_6,

$$\|T(y) - T(w)\| = \tfrac{1}{2}\Big(\sqrt{2r_1^2 + 2r_2^2 - r_5^2} + \sqrt{2r_3^2 + 2r_4^2 - r_5^2}\Big).$$

Thus, all the '$\le$' signs in the above inequalities can be replaced with '='. This completes the proof. □

From the above proof, we can draw a further conclusion that $T(x), T(y), T(z)$ and $T(w)$ form a semi-parallelogram in Y as shown in Figure 4, which is the same as that formed by x, y, z and w in X in Figure 1. To show this, we need only to prove that $T(y), \xi$ and $T(w)$ are collinear in Y. However, this is obviously true because of the strict convexity of Y and the equality $\|T(y) - T(w)\| = \|T(y) - \xi\| + \|\xi - T(w)\|$.

Furthermore, the proof of Lemma 2.5 implies the following result.

REMARK 2.6. Let X and Y be real inner product spaces with $\dim X \ge 2$, $T : X \to Y$ a mapping and $\{r_1, r_2, r_3, r_4, r_5, r_6\}$ a 6-tuple satisfying (SPC). Suppose that r_1, r_2, r_3, r_4 are contractive and r_5 extensive by T. Then r_6 is contractive by T.

LEMMA 2.7. *Let X and Y be normed spaces, $T : X \to Y$ a mapping and N a fixed positive integer. Suppose that a distance r is contractive by T. Then Nr is also contractive by T.*

PROOF. It is easy to verify this lemma by the triangle inequality. □

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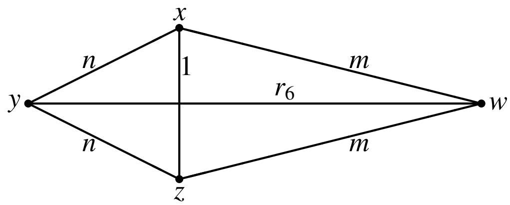

FIGURE 5. Geometric interpretation of Theorem 3.3.

3. Main results

THEOREM 3.1. *Let X and Y be real inner product spaces with* $\dim X \geq 2$, $T : X \to Y$ *a mapping and* $\{r_1, r_2, r_3, r_4, r_5, r_6\}$ *a 6-tuple satisfying (SPC), where there exist two different numbers r_i and r_j $(1 \leq i, j \leq 6)$ such that $r_i = Nr_j$ for some positive integer $N \geq 2$. Suppose that r_1, r_2, r_3, r_4 are contractive and r_5, r_6 extensive by T. Then T is a linear isometry up to translation.*

PROOF. The proof follows at once from Lemma 2.5 and Theorem 1.1. □

THEOREM 3.2. *Let X and Y be real inner product spaces with* $\dim X \geq 2$, $T : X \to Y$ *a mapping and* $\{r_1, r_2, r_3, r_4, r_5, r_6\}$ *a 6-tuple satisfying (SPC). Suppose that r_1, r_2, r_3, r_4 are contractive and r_5, Nr_6 extensive by T for some positive integer $N \geq 2$. Then T is a linear isometry up to translation.*

PROOF. The proof follows from Remark 2.6 and Theorem 1.1. □

The case $a = 0$ or $a = 2$ in Theorem 1.4 follows from Theorem 1.1 and the case $0 < a < 2$ can be seen as the case $\{r_1, r_2, r_3, r_4, r_5, r_6\} = \{1, 1, 1, 1, a, \sqrt{4 - a^2}\}$ in Theorem 3.2. Thus, to some extent, Theorem 3.2 is a generalisation of Theorem 1.4.

THEOREM 3.3. *Let X and Y be real inner product spaces with* $\dim X \geq 2$ *and* $T : X \to Y$ *a mapping. Suppose that 1 is conservative and $k(\sqrt{4n^2 - 1} + \sqrt{4m^2 - 1})/2$ extensive by T for some positive integers k, n and m. Then T is a linear isometry up to translation.*

PROOF. The case $k = n = m = 1$ follows from Theorem 1.2. By Lemma 2.7, the case $k = 1$, $\max\{n, m\} > 1$ can be seen as the case $\{r_1, r_2, r_3, r_4, r_5, r_6\} = \{n, n, m, m, 1, (\sqrt{4n^2 - 1} + \sqrt{4m^2 - 1})/2\}$ in Theorem 3.1, and the case $k > 1$ is a corollary of Theorem 3.2. □

Theorem 1.3 can be seen as the particular case $n = m = 1$ in Theorem 3.3. Thus, to some extent, Theorem 3.3 is a generalisation of Theorem 1.3. In contrast to Theorem 1.3, whose geometric interpretation is based on a rhombus (which is also a parallelogram), Theorem 3.3 takes its geometric interpretation from a kite quadrilateral (which is not necessarily a parallelogram), as shown in Figure 5, where $r_6 = (\sqrt{4n^2 - 1} + \sqrt{4m^2 - 1})/2$.

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