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EXTENSIONS OF SEMI-HEREDITARY RINGS

M. W. EVANS

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Abstract

Hattori (1960) defined a right R-module A to be torsion-free if for all $a \in A$ and $x \in R$, $ax = 0$ implies that there exist elements $\{x_1, x_2, \cdots, x_n\} \subseteq R$ with $x_i x = 0$ for all $1 \leqq i \leqq n$ and $\{a_1, a_2, \cdots, a_n\} \subseteq A$ such that $a = \sum_{i=1}^{n} a_i x_i$. Left torsion-free is defined similarly. It is shown that for a ring R, these torsion-free modules are the torsion-free class of a hereditary torsion theory, corresponding to a perfect topology, if and only if the left flat epimorphic hull of R is a regular ring which is both left and right torsion-free. A class of right semi-hereditary rings for which the torsion-free modules of Hattori satisfy the above property are found and this class of rings is discussed.

1. Introduction

In Hattori (1960) a right R-module A was defined to be torsion-free if for all $a \in A$ and $x \in R$, $ax = 0$ implies that there exist elements $\{x_1, x_2, \cdots, x_n\} \subseteq R$ and $\{a_1, a_2, \cdots, a_n\} \subseteq A$, with $x_i x = 0$ for all $1 \leqq i \leqq n$, such that $a = \sum_{i=1}^{n} a_i x_i$. We call such modules H-torsion-free and will denote the class of all such R-modules by $\mathscr{S}_R$. Necessary and sufficient conditions are found for $\mathscr{S}_R$ to be the torsion-free class of a hereditary torsion theory corresponding to a perfect topology.

It is shown in Section 4 that a right semi-hereditary ring R has a left flat epimorphic hull which is also a right flat R-module and is regular (von Neumann) if and only if Q_R, the maximal right quotient ring of R, is flat as a right R-module. This is shown to be a sufficient condition for the torsion-free R-modules of Hattori to be the torsion-free class of a hereditary torsion theory which corresponds to a perfect topology. The ring of quotients with respect to this topology may be identified with the left flat epimorphic hull of R.

Throughout this paper R is an associative ring with identity and all modules are unitary. Let $\mathscr{M}_R$ denote the category of right R-modules. If $A \in \mathscr{M}_R$, $E_R(A)$ is the right injective hull of A. A ring will be said to be right (left) P.P. if every principal right (left) ideal is projective. Given a subset X of a right R-module A_R we set $r_R(X) = \{y \in R : Xy = 0\}$. $l_R(X)$ is defined similarily when X is a subset of a left R module. For ring and homological notation the reader is referred to Cartan and Eilenberg (1956) and Lambek (1966).

For $(\mathcal{T}, \mathcal{F})$ a torsion theory, $\mathcal{T}$ denotes the torsion class and $\mathcal{F}$ a torsion-free class. *Throughout this paper we assume* $R_R \in \mathcal{F}$. A hereditary torsion theory with corresponding perfect topology will be said to be a perfect torsion theory. We will denote the left flat epimorphic hull of a ring by $M(R)$. For all details of torsion theories we refer the reader to Stenström (1971).

2. Flat epimorphic extensions

Following Stenström (1971) we call the torsion theory co-generated by $E(R)$ the *Lambek torsion theory*. A right ideal I of R will be said to be *dense* if $\mathrm{Hom}_R(R/I, E(R)) = 0$. It is easily shown that a right ideal is dense if and only if $q \in E(R)$, $qI = 0$ implies $q = 0$. We will denote the class of right ideals of R which contain a finitely generated dense right ideal by G_R.

LEMMA 2.1. *Let S be an overring of R such that $R \subseteq S \subseteq Q$ and $I_1, I_2, \cdots, I_n$ right ideals of R. If $I_iS = S$ for all $1 \leqq i \leqq n$ then*

(i) *$I_i \in G_R$ for all $1 \leqq i \leqq n$.*

(ii) *If S is left flat as an R-module, $\bigcap_{i=1}^n I_i \in G_R$.*

PROOF. The proof of (i) is routine and will be omitted. Part (ii) follows from Theorem 4 of Hinohara (1960).

COROLLARY 2.2. *If $\{s_1, s_2, \cdots, s_n\} \subseteq S$ where S is a left flat epimorphic extension of R, $\bigcap_{i=1}^n (R : s_i) \in G_R$.*

PROOF. This is immediate from Lemma 2.1 and Theorem 13.10 of Stenström (1971).

We will say a ring R is *right special saturated* if R contains no finitely generated dense right ideals. It is clear that every regular ring is right special saturated.

LEMMA 2.3. *If S is a left flat epimorphic extension of R and S is a right special saturated ring then $S \cong M(R)$ over R.*

PROOF. S may be considered as a subring of $M(R)$. Let $m \in M(R)$. Then Corollary 2.2 gives that $(R : m) \in G_R$, so $(R : m)S \in G_S$ and thus $(R : m)\ S = S$. Therefore $m = m \,.\, 1_S \in m\,(R :_R m)S \subseteq RS \subseteq S$.

LEMMA 2.4. *Let S be a left flat epimorphic extension of R and $I \in G_S$. Then there exists $K \in G_R$ such that $K \subseteq I \cap R$.*

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PROOF. Let $\{s_1, s_2, \cdots, s_n\}$ be a generating set of I. Corollary 2.2 gives that there exists a finitely generated dense right ideal J of R such that $J \subseteq \bigcap_{i=1}^{n}(R : s_i)$. If X generates J, the right ideal of R generated by $\{s_i x \mid 1 \leqq i \leqq n, x \in X\}$ may easily be shown to be a dense right ideal of R.

From previous results of this section and Theorem 13.1 and 13.10 of Stenström (1970) the following result may be obtained.

LEMMA 2.5. *A ring R with left flat epimorphic hull $M(R)$ which is a right special saturated ring satisfies the following conditions:*

(i) $\{I : IM(R) = M(R)$, *where I is a right ideal of $R\} = G$.*

(ii) *G is a perfect topology*

(iii) *$R_G \cong M(R)$ over R.*

Conversely if G is a perfect topology then $R_G \cong M(R)$ is a right special saturated ring.

REMARKS 1. (a) Quentel (1971) shows that if R is a commutative non-singular ring with $M(R)$ regular then G is a topology with $R_G \cong M(R)$.

(b) An example of a right special saturated ring which is not regular is presented in Wiegand (1971).

EXAMPLE 1 (Storrer 1971). The ring Q of Storrer in Example 7.2 is right rationally complete. Hence $Q = M(Q)$ and the ideal D' is a finitely generated dense right ideal of Q. Thus Q is not a right special saturated ring but $Q = M(Q)$.

3. H-torsion-free R-modules

A ring R is said to be a P.F. ring if every principal right ideal of R is flat. In Jøndrup (1971) it is shown that R is a P.F. ring if and only if every principal left ideal of R is also flat.

LEMMA 3.1. *If R is a P.F. ring, S a regular ring with $R \subseteq S$ and S a left H-torsion-free R-module, then R is a right P.P. ring.*

PROOF. Proposition 1 of Hattori (1960) gives that for each $x \in R$ $\mathrm{Tor}_1(R/xR, S) = 0$. Thus we have the short exact sequence $0 \to xR \otimes S \to S \to R/xR \otimes S \to 0$. Hence $xR \otimes_R S$ may be considered as a finitely generated submodule of S a regular ring which implies $xR \otimes_R S$ is projective. Theorem 3.1 of Jøndrup (1970) gives that xR is projective.

THEOREM 3.2. *The following are equivalent for a ring R.*

(i) *The right H-torsion-free R-modules are the torsion-free class of a perfect torsion theory for $\mathcal{M}_R$.*

(ii) *R is a right P.P. ring, $M(R)$ is regular and $M(R)$ is right H-torsion free when considered as an R-module.*

PROOF. (i) $\Rightarrow$ (ii) Let S be the quotient ring of R corresponding to the torsion theory for $\mathcal{M}_R$ of which $\mathcal{S}_R$ is the torsion-free class. S is a left flat epimorphic extension of R. If $x \in A$ an S-module, then $xR \in \mathcal{S}_R$ and hence is flat as an R-module. We thus have $xS \cong xR \otimes_R S$ is a flat right S-module and hence A is H-torsion-free as an S-module. Proposition 4 of Hattori (1960) gives that S is regular and by Lemma 2.3, $S \cong M(R)$. Propositiop 13 of Hattori gives that R is a P.F. ring and since $M(R)$ is regular, R is a right P.P. ring (Lemma 3.1).

(ii) $\Rightarrow$ (i) $M(R)$ is a right special saturated ring. Let G be the perfect topology in Lemma 2.5. We show $\mathcal{S}_R = \mathcal{F}$ the torsion-free class corresponding to G. Let $x \in A \in \mathcal{S}_R$. Since xR is right flat we have the exact sequence $0 \to xR \to xR \otimes_R R_G$ and as $xR \otimes R_G \in \mathcal{F}$ this implies $xR \in \mathcal{F}$. Hence $A_R \in \mathcal{F}$.

Conversely assume $A \in \mathcal{F}$. Since R_G is regular $A \otimes R_G$ is an H-torsion-free R_G-module and the transitivity of H-torsion-freeness gives that $A \otimes_R R_G \in \mathcal{S}_R$. But R is a P.F. ring which implies $\mathcal{S}_R$ is closed under submodules and hence $A \in \mathcal{S}_R$.

REMARKS 2 (a). It is shown in the proof of (ii) $\Rightarrow$ (i) that $\mathcal{S}_R$ coincides with the torsion-free class of the torsion theory corresponding to G.

(b). If a ring R satisfies the equivalent condition of the theorem then R is also a left P.P. ring. Thus not every right P.P. ring satisfies the condition of the theorem for there exist right P.P. rings which are not left P.P. (see e.g. Small (1966)).

4. Regular flat epimorphic hulls

In this section necessary and sufficient conditions are found for a right semi-hereditary ring to have a regular left flat epimorphic hull which is also the right flat epimorphic hull.

LEMMA 4.1. *If $Z(R_R) = 0$ and Q is left flat as an R-module then G_R is a topology.*

PROOF. Let I be a right ideal of R and assume there exists $j \in G_r$ such that $(I :_R a) \in G_R$ for every $a \in J$. Since Q is left flat, Theorem 4 of Hinohara (1960) gives $(I :_R a)Q = (IQ :_Q a)$. Thus there exists a finitely generated dense right ideal $K \subseteq (I :_R a)$ with $KQ \subseteq (IQ :_Q a)$. But Q is regular and hence $Q = KQ \subseteq (IQ :_Q a)$. Thus $J \subseteq IQ$ and as $JQ = Q$, $IQ = Q$ and $I \in G_R$.

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By again using Theorem 4 of Hinohara (1960) we may show that if $I \in G_R$, then $(I:_R x) \in G_R$ for each $x \in R$.

THEOREM 4.2. *If R is a right semi-hereditary ring then,*
(i) *G is a perfect topology.*
(ii) *$R_G \cong M(R)$ over R.*
(iii) *R_G is a right special saturated ring.*

PROOF. Q_R is left flat (Sandomierski (1968)). Lemma 4.1 gives that R is a topology. Theorems 4.3 and 4.5 of Goldman (1969) may be used to show G_R is perfect and Lemma 2.5 completes the result.

If a ring R satisfies the property that the left annihilator of a finitely generated proper right ideal is always non-zero, then R is a right special saturated ring. However it is unknown whether the converse holds.

LEMMA 4.3. *If $Z(R_R) = 0$ and Q_R is right flat then every finitely generated right ideal of R with zero left annihilator is a dense right ideal.*

PROOF. Let I be a finitely generated right ideal of R such that $l_R(I) = 0$. Let $\{x_1, x_2, \cdots, x_n\}$ be a generating set for I. Then $\bigcap_{i=1}^n l_R(x_i) = 0$ and by using Theorem 4 of Hinohara (1960) we find that $\bigcap_{i=1}^n l_Q(x_i) = \bigcap_{i=1}^n Ql_R(x_i) = 0$. Since Q is regular this implies $IQ = Q$ and hence I is a dense right ideal.

THEOREM 4.4. *The following are equivalent for a ring R:*
(i) *R is right semi-hereditary and Q_R is right flat,*
(ii) *$w.gl \dim R \leqq 1$, $M(R)$ is a regular ring and is right flat when considered as an R-module.*

PROOF. (i) $\Rightarrow$ (ii) From Theorem 2.10 of Sandomierski (1968), if R is a right semi-hereditary ring $w.gl.\dim R \leqq 1$. Since Q_R is right flat, $M(R)$ is also flat as a right R-module. From Theorem 4.2 we know that $M(R)$ is a right special saturated ring. Now using Lemma 4.3 we see that $M(R)$ contains no finitely generated right ideals with zero left annihilator. Hence $M(R)$ satisfies the equivalent conditions of Theorem 5.4, Bass (1960).

If I is a finitely generated left ideal of $M(R)$, then $Q \otimes_{M(R)} I \cong QI$ is a projective left Q-module. By Proposition 14.6 Stenström (1971) $w.gl.\dim M(R) \leqq 1$. Hence I is a flat $M(R)$ module and Theorem 3.1 of Jøndrup (1970) now gives that I is a projective $M(R)$ module. Thus I is a direct summand of $M(R)$ and $M(R)$ is a regular ring.

(ii) $\Rightarrow$ (i) This is clear from Theorem 2.1 of Sandomierski (1968).

Knight (1970) has shown that for every ring R there exists a maximal both right and left flat epimorphic extension of R. He called this extension the flat injective closure of R and denoted it Epi (R). If R satisfies the equivalent conditions of Theorem 4.4, Epi $R = M(R)$.

COROLLARY 4.5. *If R is a right semi-hereditary ring then the following are equivalent*:

(i) Q_R *is right flat.*

(ii) Epi (R) *is regular.*

REMARKS 3. If R is a right and left semi-hereditary ring with two sided maximal quotient ring then R satisfies the equivalent conditions of the Theorem 4.4. Not every semi-hereditary ring satisfies the equivalent conditions of the theorem. For a counter-example choose R to be a right semi-hereditary ring which is not left semi-hereditary (Small (1966)).

When R is a commutative P.P. ring $M(R)$ is the classical quotient ring of R and every idempotent of $M(R)$ is a member of R (see e.g. Evans (1972)). This is not true generally for the non commutative case as the following example shows.

EXAMPLE 2. *A ring satisfying the equivalent conditions of Theorem* 6.4 *for which $M(R)$ is not the right classical quotient ring of R.* Let D be a principal ideal domain, K the quotient field of D. Let R be the ring of all 2×2 matrices $\begin{bmatrix} d_1 & q \\ 0 & d_2 \end{bmatrix}$ where $d_1, d_2 \in R$, $q \in K$. Then R is both left and right semi-hereditary and S, the ring of upper triangular matrices over K is the two sided classical quotient ring of R. $M_2(K)$, the ring of 2×2 matrices over K, is the semi-simple maximal quotient ring of R. Clearly $M_2(K) \neq S$. Since $M_2(K)$ is semi-simple it is the left flat epimorphic hull of R (Findlay (1971)). Note also that not every idempotent of $M_2(K)$ is a member of R (for example $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$).

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84 Glencairn Ave.,
East Brighton, 3187,
Australia.

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