# 7

# Spontaneous symmetry breaking and restoration

In the standard model of particle physics, which has been thoroughly tested to energies above 100 GeV, a central role is played by scalar fields introduced in the Lagrangian with a negative mass-squared. These fields are introduced to spontaneously break a gauge symmetry and so yield the massive vector mesons $W$ and $Z$, as observed in nature, in the framework of a renormalizable field theory. This is the Higgs mechanism, to be discussed in Section 7.4, and more specifically in the Glashow–Weinberg–Salam model of electroweak interactions in Chapter 15. Spontaneous symmetry breaking is more general, and arises in the strong interactions too as is elucidated in later chapters. We now turn our attention to a simple model to illustrate the phenomenon. This will be followed by a general statement of Goldstone's theorem, and a consideration of loop corrections and of the Higgs model.

## 7.1 Charged scalar field with negative mass-squared

Consider a complex scalar field $\Phi$ with Lagrangian

$$\mathcal{L} = \partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi - \lambda (\Phi^* \Phi)^2 \tag{7.1}$$

This Lagrangian has a global U(1) symmetry $\Phi \to \Phi \mathrm{e}^{-i\alpha}$, as discussed in Section 2.4. What happens if $m^2 = -c^2 < 0$? First suppose that $\lambda = 0$. Then in frequency–momentum space the action is

$$S_0 = -\tfrac{1}{2}\beta^2 \sum_n \sum_{\mathbf{p}} \left(\omega_n^2 + \mathbf{p}^2 - c^2\right) \times \left[\phi_{1;n}(\mathbf{p})\phi_{1;-n}(-\mathbf{p}) + \phi_{2;n}(\mathbf{p})\phi_{2;-n}(-\mathbf{p})\right] \tag{7.2}$$

where $\Phi = \phi_1 + i\phi_2$ in the usual notation. This action is not negative definite and therefore the functional integral is not convergent. Another



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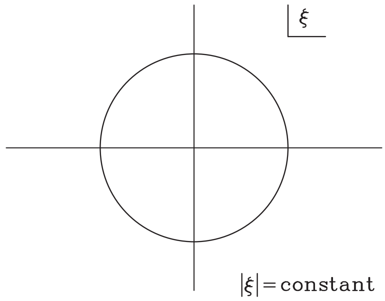


Fig. 7.1.

way to see this is to recall the expression for the partition function, (2.40), and simply replace $m^2$ with $-c^2$:

$$\begin{aligned} \ln Z_0 &= 2V \int \frac{d^3p}{(2\pi)^3} \left[ -\tfrac{1}{2}\beta\omega - \ln\left(1 - \mathrm{e}^{-\beta\omega}\right) \right] \\ \omega &= \sqrt{\mathbf{p}^2 - c^2} \end{aligned} \tag{7.3}$$

The dispersion relation indicates an instability when $|\mathbf{p}| < c$.

The basic problem is that the potential is unbounded from below when $\lambda = 0$. To stabilize the system we require $\lambda > 0$, which means a repulsive interaction between the particles. The aforementioned instability occurs at small momenta. This suggests that the bosons condense, or accumulate, in the zero-momentum mode. Therefore, following the discussion of Bose–Einstein condensation, we separate out explicitly the static infrared part of the field as

$$\begin{aligned} \Phi &= \xi + \chi \\ \Phi^* &= \xi^* + \chi^* \end{aligned} \tag{7.4}$$

Here $\xi$ is a constant and $\chi_{n=0}(\mathbf{p} = 0) = 0$; that is, the thermal average $\langle\Phi\rangle = \xi$. Owing to the global U(1) symmetry, $\mathcal{L}$ depends only on the magnitude of $\xi$ and not on its phase, as illustrated in Figure 7.1. For convenience we shall choose $\xi$ real.

In terms of the shifted field, the Lagrangian is given by

$$\mathcal{L} = -\mathrm{U}(\xi) + \mathcal{L}_0 + \mathcal{L}_\mathrm{I} \tag{7.5}$$

where

$$\begin{aligned} \mathrm{U}(\xi) &= -c^2\xi^2 + \lambda\xi^4 \\ \mathcal{L}_0 &= \tfrac{1}{2}\partial_\mu\chi_1\partial^\mu\chi_1 - \tfrac{1}{2}\left(6\lambda\xi^2 - c^2\right)\chi_1^2 \\ &\quad + \tfrac{1}{2}\partial_\mu\chi_2\partial^\mu\chi_2 - \tfrac{1}{2}\left(2\lambda\xi^2 - c^2\right)\chi_2^2 \\ \mathcal{L}_\mathrm{I} &= -\sqrt{2}\lambda\xi\left(\chi_1^2 + \chi_2^2\right)\chi_1 - \tfrac{1}{4}\lambda\left(\chi_1^2 + \chi_2^2\right)^2 \end{aligned}$$

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In addition, $\mathcal{L}$ contains terms linear in $\chi_1$ and $\chi_2$, but these contribute nothing and may be dropped. (Using the Fourier expansion (2.30), we see that these terms contribute to the action an amount proportional to $\int_0^\beta d\tau \int d^3x\, \chi(\mathbf{x},\tau) \propto \chi_{n=0}(\mathbf{p}=\mathbf{0})$.) The procedure of shifting the field in this way and regarding $\chi_1$ and $\chi_2$ as the elementary excitations instead of $\phi_1$ and $\phi_2$ is called the mean field expansion. The mean field potential energy density is $\mathrm{U}(\xi)$, as we show below. The mean field masses can be read off from $\mathcal{L}_0$ as

$$\begin{aligned} \bar{m}_1^2 &= 6\lambda\xi^2 - c^2 \\ \bar{m}_2^2 &= 2\lambda\xi^2 - c^2 \end{aligned} \tag{7.6}$$

Finally, notice that a cubic interaction is induced if $\xi \neq 0$.

The mean field approximation is obtained by calculating $\ln Z$ with the neglect of $\mathcal{L}_\mathrm{I}$. One might expect this to be a good approximation if both $\lambda$ and $\lambda\xi$ are small. At this point, it is convenient to introduce the thermodynamic potential density $\Omega$. For a uniform infinite volume system we have the relationship

$$\Omega(T,\xi) = -P(T,\xi) = -\frac{T}{V}\ln Z \tag{7.7}$$

We know from thermodynamical considerations (Landau and Lifshitz [1]; Reif [2]) that in thermal equilibrium $\Omega$ is a minimum with respect to variations in $\xi$, when $\xi$ is treated as a variational parameter. Intuitively, this can be recognized by remembering that in equilibrium the pressure is spatially uniform and that a local fluctuation to a state of lower pressure is obviously unstable. In the mean field approximation,

$$\begin{aligned} \Omega(T,\xi) = \mathrm{U}(\xi) + \int \frac{d^3p}{(2\pi)^3}\Big[\tfrac{1}{2}\omega_1 + \tfrac{1}{2}\omega_2 \\ + T\ln\left(1-\mathrm{e}^{-\beta\omega_1}\right) + T\ln\left(1-\mathrm{e}^{-\beta\omega_2}\right)\Big] \\ \omega_i = \sqrt{\mathbf{p}^2 + \bar{m}_i^2} \end{aligned} \tag{7.8}$$

The vacuum energy density is $\Omega(T=0,\xi)$.

The classical energy density, obtained by neglecting the zero-point energy in the fields, is

$$\Omega_{\mathrm{cl}}(T=0,\ \xi) = \mathrm{U}(\xi) = -c^2\xi^2 + \lambda\xi^4 \tag{7.9}$$

This potential has a minimum at $\xi_0^2 = \xi^2(T=0) = c^2/2\lambda$, as shown in Figure 7.2. The potential energy density has a local maximum at $\xi = 0$. This explains the instability encountered earlier. Instead of expanding about this local maximum, we should expand about the global minimum at $\xi_0$. The mean field masses, that is, the masses of small excitations about

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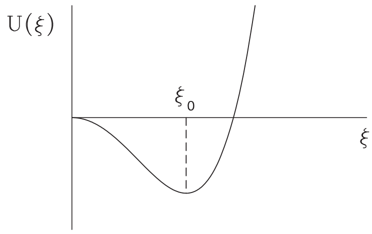


Fig. 7.2

the equilibrium field configuration, are

$$\begin{aligned}\bar{m}_1^2(T=0) &= 2c^2 \\ \bar{m}_2^2(T=0) &= 0\end{aligned} \tag{7.10}$$

These results are rather transparent. If we allow for complex values of $\xi$ then the potential would still have the shape illustrated if we rotated the curve about the vertical axis. So we plot $U$ along the $z$-axis and take the complex $\xi$ plane to define the $x$- and $y$- axes. Since $U$ depends only on $|\xi|$, we obtain the famous "bottom of the wine bottle" shape. Radial excitations of the field have a mass $\sqrt{2}c$, while rotational excitations have zero mass. Since the Lagrangian written in terms of $\Phi$ and $\Phi^*$ has a global U(1) symmetry, it is clear that if we change the phase of the field everywhere in space at the same time there will be no change in the energy of the system. This static, infinite-wavelength, zero-momentum excitation circles around the bottom of the potential in the complex $\xi$ plane. This excitation is called a Goldstone boson. The U(1) symmetry apparent in (7.1) is not so obvious in (7.5). It is said to be spontaneously broken, since the vacuum exhibits a lesser symmetry than the Lagrangian. The real and imaginary components of the field exhibit different masses. The existence of a Goldstone boson in such a case is guaranteed by Goldstone's theorem, which is discussed in more detail in the next section.

There are a number of analogies with more common systems. In a ferromagnetic metal all the spins line up at $T = 0$. Since there is no preferred direction in which they should point, rotational symmetry is spontaneously broken. Spin waves with vanishing momentum carry no energy; their dispersion relation is $\omega = c_s k$. When the ends of a rod are subjected to sufficient force, the lowest-energy state is achieved when the rod is bowed. Since there is no preferred direction in which the rod should bow, rotational symmetry is spontaneously broken. The energy of a rotating bent rod, $\omega = l^2/2I$, vanishes as the angular momentum $l$ goes to zero.

Now we raise the temperature of the system to $T > 0$. When $T^2 \ll \xi_0^2 = c^2/2\lambda$, not much of interest happens. There is an ideal gas of quasiparticles

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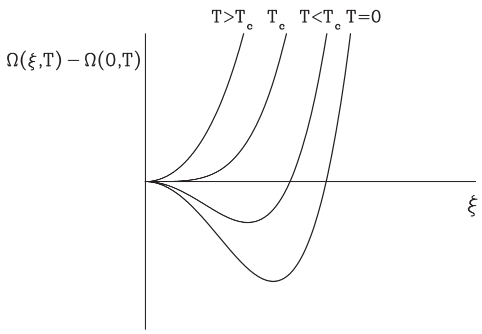


Fig. 7.3

with masses $\bar{m}_1 = \sqrt{2}c$ and $\bar{m}_2 = 0$. The thermal pressure is

$$P_{\text{thermal}} = P_0(T, \bar{m}_1^2) + P_0(T, \bar{m}_2^2) \tag{7.11}$$

where

$$\begin{aligned} P_0(T, \bar{m}^2) &= -T \int \frac{d^3p}{(2\pi)^3} \ln\left(1 - \mathrm{e}^{-\beta\omega}\right) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}^2}{3\omega} \frac{1}{\mathrm{e}^{\beta\omega} - 1} \\ \omega &= \sqrt{\mathbf{p}^2 + \bar{m}^2} \end{aligned} \tag{7.12}$$

When $T$ is not small we must allow for the possibility that thermal fluctuations may change the equilibrium value of the condensate field $\xi$. If the interesting physics occurs when $T^2 \simeq c^2/\lambda \gg c^2$, then we make a high-temperature expansion of $P(T, m^2)$ as (see appendix Section A1.3)

$$\begin{aligned} P_0(T, m^2) = \frac{\pi^2}{90}T^4 - \frac{1}{24}m^2T^2 + \frac{1}{12\pi}m^3T \\ + \frac{m^4}{64\pi^2}\left[\ln\left(\frac{m^2}{16\pi^2T^2}\right) + 2\gamma_{\text{E}} - \frac{3}{2}\right] + \cdots \end{aligned} \tag{7.13}$$

Then, with $P = -\Omega$,

$$\Omega(\xi, T) = \lambda\xi^4 + \left(\frac{1}{3}\lambda T^2 - c^2\right)\xi^2 - \frac{\pi^2}{45}T^4 - \frac{1}{12}c^2T^2 \tag{7.14}$$

Keeping only the first two terms in (7.13) yields (7.14). This is actually a very clever termination of the series, often used in the literature, since (i) it is correct when $T = 0$, (ii) it is a good approximation when $T > c$, and (iii) it is a remarkably transparent function of $\xi$ and $T$. The isotherms of the thermodynamic potential are shown in Figure 7.3. The minimum shifts to smaller values of $\xi$ and becomes less deep, as $T$ increases. At

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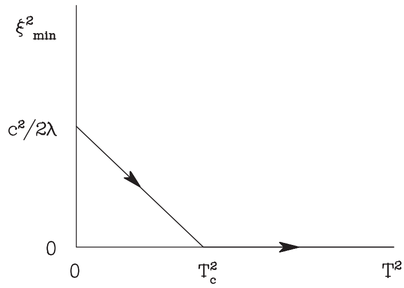


Fig. 7.4

$T_c^2 \equiv 3c^2/\lambda$, the coefficient of $\xi^2$ changes sign and the potential has a minimum at $\xi = 0$. The location of the minimum is

$$\xi^2_{\min}(T) = \begin{cases} c^2/2\lambda - T^2/6 & T \leq T_c \\ 0 & T \geq T_c \end{cases} \tag{7.15}$$

This is shown in Figure 7.4. It can be seen that there is a phase transition at $T_c$. The spontaneously broken U(1) symmetry is restored!

Using (7.15) in (7.14), the pressures in the low- and high-temperature phases are, after normalizing the vacuum pressure and energy density to zero,

$$\begin{aligned} P_<(T) &= \left(\frac{\pi^2}{45} + \frac{\lambda}{36}\right) T^4 - \frac{1}{12} c^2 T^2 \\ P_>(T) &= \frac{\pi^2}{45} T^4 + \frac{1}{12} c^2 T^2 - \frac{c^4}{4\lambda} \end{aligned} \tag{7.16}$$

The pressure and entropy are continuous at $T_c$,

$$\begin{aligned} P_<(T_c) &= P_>(T_c) \\ \frac{dP_<(T_c)}{dT} &= \frac{dP_>(T_c)}{dT} \end{aligned} \tag{7.17}$$

but the heat capacity is discontinuous,

$$\frac{d^2 P_<(T_c)}{dT^2} - \frac{d^2 P_>(T_c)}{dT^2} = \frac{2}{3} c^2 \tag{7.18}$$

Hence this is a second-order phase transition. The physical origin of this symmetry-restoring phase transition is that the ordering inherent in the vacuum, and represented by the accumulation of an infinite number of particles into the zero-momentum state or condensate field $\xi$, is destroyed by thermal fluctuations at high temperatures. The second-order nature of the phase transition is expected from the general Landau theory of phase transitions (Landau and Lifshitz [1]). A first-order transition would arise if a term cubic in $\xi$ were present in $\Omega$, but this is not allowed by the U(1) symmetry.

There are potential flaws in the beautiful scenario just painted. First, the masses in the mean field approximation are

$$\begin{aligned} \bar{m}_1^2 &= 6\lambda\xi_{\text{min}}^2(T) - c^2 = \begin{cases} 2c^2 - \lambda T^2 & T \leq T_\text{c} \\ -c^2 & T \geq T_\text{c} \end{cases} \\ \bar{m}_2^2 &= 2\lambda\xi_{\text{min}}^2(T) - c^2 = \begin{cases} -\frac{1}{3}\lambda T^2 & T \leq T_\text{c} \\ -c^2 & T \geq T_\text{c} \end{cases} \end{aligned} \tag{7.19}$$

We are burdened again by a negative mass-squared at $T > 0$. Also, where is the Goldstone boson when $0 < T \leq T_\text{c}$? Finally, what about the change in the zero-point energy in (7.8) as $\xi$ varies with $T$? We shall return to these questions after a more general discussion of Goldstone's theorem.

## 7.2 Goldstone's theorem

Goldstone's theorem may be stated as follows:

*If a continuous symmetry of the Lagrangian is spontaneously broken, and if there are no long-range forces, then there exists a zero-frequency excitation at zero momentum.*

Here are some examples from nonrelativistic many-body systems [3].

- Ferromagnets. The absence of long-range forces, which may tend to couple spins at large distances, is necessary for the existence of a mode with $\omega \to 0$ as $k \to 0$.
- Superconductors. In the Bardeen–Cooper-Schrieffer (BCS) theory there is a spontaneous breaking of phase invariance associated with the conservation of electron number. However, there is an energy gap (equal to the mass of the Cooper pairs), so there is no Goldstone boson. The reason is that there are long-range electromagnetic forces.
- Superfluids. A low-temperature Bose system is a superfluid. The condensate field, at $T = 0$, is $\langle\Phi\rangle = \xi$, which is related to the particle number density by $n = |\xi|^2$. The phonon spectrum is

$$\omega^2 = \frac{k^2}{2m}\left(\frac{k^2}{2m} + 2nV(\mathbf{k})\right)$$

where $V(\mathbf{k})$ is the Fourier transform of the two-body potential. By definition, a short-range potential has the property that $V(\mathbf{k} = 0)$ is finite and positive. In that case, $\omega \to \sqrt{nV(\mathbf{k} = 0)/m}\,k$ as $k \to 0$. This is not so for a long-range potential. For the Coulomb force, $V(k) = e^2/k^2$ and, as $k \to 0$, $\omega \to e\sqrt{n/m} = \omega_\text{P}$, the plasma frequency.

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We would like a nonperturbative proof of Goldstone's theorem. However, to be concrete, we will construct such a proof in the context of the U(1) scalar field theory discussed in the previous section.

The U(1) symmetry is $\Phi \to \Phi e^{-i\alpha}$, or $\delta\Phi = -i\alpha\Phi$ if $|\alpha| \ll 1$. The conserved current density may be recalled from (2.47). In terms of the shifted field, it is

$$j_\mu = \chi_2 \partial_\mu \chi_1 - \chi_1 \partial_\mu \chi_2 - \sqrt{2}\xi \partial_\mu \chi_2 \tag{7.20}$$

The total charge, $Q = \int d^3x\, j_0(\mathbf{x})$, is conserved; $\dot{Q} = 0$. The change in $\Phi$ due to an infinitesimal change in phase can also be expressed in operator form as

$$\delta\Phi = i\alpha[Q, \Phi] \tag{7.21}$$

That is, the total charge is the generator of the phase transformation. Taking the thermal, or ensemble, average of $\delta\Phi$, we find $\langle \delta\Phi \rangle = -i\alpha \langle \Phi \rangle = -i\alpha\xi$. Taking the thermal average of (7.21), we find an expression for the condensate field,

$$\xi = -\int d^3x\, \langle [j_0(\mathbf{x}, t),\ \Phi(\mathbf{0}, 0)] \rangle \tag{7.22}$$

Now we define the function

$$F^\mu(k_0, \mathbf{k}) = \int d^4x\, e^{ik\cdot x} \langle T\, [j^\mu(x), \Phi(0)] \rangle \tag{7.23}$$

Since $\partial_\mu j^\mu = 0$ and

$$T\, [j^\mu(x)\Phi(0)] = j^\mu(x)\Phi(0)\theta(x^0) + \Phi(0) j^\mu(x)\theta(-x^0) \tag{7.24}$$

it follows that

$$\begin{aligned} k_\mu F^\mu = &-i \int d^4x\, \partial_\mu \Big\{ e^{ik\cdot x} \langle T\, [j^\mu(x), \Phi(0)] \rangle \Big\} \\ &+ i \int d^3x\, e^{-i\mathbf{k}\cdot\mathbf{x}} \langle [j_0(x), \Phi(0)] \rangle \end{aligned} \tag{7.25}$$

If the surface term in (7.25) vanishes then comparison with (7.22) shows that

$$\lim_{\mathbf{k}\to 0} k_\mu F^\mu = -i\xi \tag{7.26}$$

If $\xi \neq 0$, which means that the U(1) symmetry is spontaneously broken, then $F$ has a pole at $k = 0$. This pole corresponds to a zero-frequency excitation at zero momentum.

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It is not difficult to determine $F^\mu$. Substituting (7.20) into (7.23) leads to

$$F^\mu(k) = -\xi k^\mu \int d^4x\, \mathrm{e}^{ik\cdot x} \langle T\,[\chi_2(x)\chi_2(0)]\rangle$$
$$= -i\xi k^\mu D_2(k) \tag{7.27}$$

where $D_2$ is the real time Green's function. Combining (7.26) and (7.27) tells us that the imaginary part of the shifted field has a dispersion relation with the property that $\omega(\mathbf{k}=0)=0$. This is the Goldstone boson.

If the surface term in (7.25) is not zero then no conclusion may be drawn. This is often the case when there are massless spin-1 bosons in the theory. This is a gauge theory. We will discuss what happens in this case later on, focusing especially on the Higgs model and the Glashow–Weinberg–Salam model of the electroweak interaction.

## 7.3 Loop corrections

Now let us turn our attention to $\Omega$ and also to the self-energies of the fields.

In Section 7.1 we neglected the shift in the zero-point energy of the vacuum. Up to an (infinite) additive constant we can write

$$\int \frac{d^3p}{(2\pi)^3}\,\omega = \int \frac{d^4p}{(2\pi)^4}\,\ln(p^2+m^2) \tag{7.28}$$

where $p=(\mathbf{p},p_4)$ is a Euclidean four-vector. Our regularization procedure is simply to place an upper cutoff, $\Lambda_\mathrm{c}$, on the integration over $|p|$. This is what we did in Section 3.4 (see also Chapter 4). Then

$$\int \frac{d^3p}{(2\pi)^3}\,\omega = \frac{1}{64\pi^2}\left[4m^2\Lambda_\mathrm{c}^2 - 2m^4\ln\left(\frac{\Lambda_\mathrm{c}^2}{m^2}\right) - m^4\right] + \text{constant} \tag{7.29}$$

plus terms that vanish as $\Lambda_\mathrm{c}\to\infty$. We may add to the Lagrangian the counterterms

$$\delta c^2\Phi^*\Phi - \delta\lambda(\Phi^*\Phi)^2$$

In general, $\delta c^2$ and $\delta\lambda$ will depend on the other constants in the Lagrangian, and on $c^2$ and $\lambda$ as well as $\Lambda_\mathrm{c}$. The vacuum energy density is

$$\Omega(T=0,\xi) = -(c^2+\delta c^2)\xi^2 + (\lambda+\delta\lambda)\xi^4$$
$$+\frac{1}{64\pi^2}\left[2(\bar m_1^2+\bar m_2^2)\Lambda_\mathrm{c}^2 - \bar m_1^4\ln\left(\frac{\Lambda_\mathrm{c}^2}{\bar m_1^2}\right)\right.$$
$$\left. - \bar m_2^4\ln\left(\frac{\Lambda_\mathrm{c}^2}{\bar m_2^2}\right) - \frac{1}{2}(\bar m_1^4+\bar m_2^4)\right] \tag{7.30}$$

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There is some freedom in choosing $\delta c^2$ and $\delta\lambda$. However, we should insist that $\Omega(T=0,\ \xi)$ be finite (independent of $\Lambda_c$) and that Goldstone's theorem be satisfied ($\bar{m}_2 = 0$). The latter will occur only if $\Omega(T=0,\ \xi)$ has its minimum at $\xi^2 = c^2/2\lambda$. A straightforward calculation yields

$$\begin{aligned} \delta c^2 &= \frac{\lambda\Lambda_c^2}{4\pi^2} + \frac{\lambda c^2}{4\pi^2}\ln\left(\frac{\Lambda_c^2}{2c^2}\right) + c^2\frac{\delta'\lambda}{\lambda} \\ \delta\lambda &= \frac{5\lambda^2}{8\pi^2}\ln\left(\frac{\Lambda_c^2}{2c^2}\right) + \delta'\lambda \end{aligned} \tag{7.31}$$

Here $\delta'\lambda =$ constant $\times\ \lambda^2$ is not determined by the above conditions. The renormalized vacuum energy density is

$$\begin{aligned} \Omega(T=0,\xi) = &-c^2\left(1 - \frac{\lambda}{8\pi^2} + \frac{\delta'\lambda}{\lambda}\right)\xi^2 + \lambda\left(1 - \frac{5\lambda}{16\pi^2} + \frac{\delta'\lambda}{\lambda}\right)\xi^4 \\ &+ \frac{\bar{m}_1^4}{64\pi^2}\ln\left(\frac{\bar{m}_1^2}{2c^2}\right) + \frac{\bar{m}_2^4}{64\pi^2}\ln\left(\frac{\bar{m}_2^2}{2c^2}\right) \end{aligned} \tag{7.32}$$

There are several noteworthy points concerning (7.32). By construction it has its minimum at the same location as the classical energy density. Thus, in the true vacuum $\bar{m}_1^2 = 2c^2$ and $\bar{m}_2^2 = 0$, the same as in the classical approximation. Goldstone's theorem is obeyed. To (7.32) we may add any constant. Thus, not only the location of the minimum but also its depth can be made the same as in the classical approximation. Notice, however, that when $\xi^2 < c^2/2\lambda$ then $\bar{m}_2^2 < 0$ and $\Omega$ has an imaginary part. This is not unreasonable since in that region the system is unstable.

In the high-temperature expansion (7.13) there is also a term of order $m^4 \ln m^2$, with a coefficient of equal magnitude but opposite sign. Thus the order-$m^4 \ln m^2$ terms in the vacuum and high-temperature contributions cancel. Adding together (7.13) and (7.32) gives an improved high-temperature expression for the thermodynamic potential (for now we will neglect the term $-(m_1^3 + m_2^3)T/12$):

$$\begin{aligned} \Omega(T,\xi) = &-\frac{\pi^2}{45}T^4 - \frac{c^2T^2}{12} + \frac{c^4}{32\pi^2}\ln\left(\frac{8\pi^2T^2}{c^2}\ \mathrm{e}^{-2\gamma_\mathrm{E}+3/2}\right) \\ &- c^2\xi^2\left[1 + \frac{\delta'\lambda}{\lambda} + \frac{\lambda}{4\pi^2}\ln\left(\frac{8\pi^2T^2}{c^2}\ \mathrm{e}^{-2\gamma_\mathrm{E}+1}\right) - \frac{\lambda T^2}{3c^2}\right] \\ &+ \lambda\xi^4\left[1 + \frac{\delta'\lambda}{\lambda} + \frac{5\lambda}{8\pi^2}\ln\left(\frac{8\pi^2T^2}{c^2}\ \mathrm{e}^{-2\gamma_\mathrm{E}+1}\right)\right] \end{aligned} \tag{7.33}$$

The appearance of the logarithms is all that really distinguishes this improved potential from its predecessor. (The $\delta'\lambda/\lambda$ terms can be absorbed into the arguments of the logarithms if desired.) Now $\ln(T/c)$

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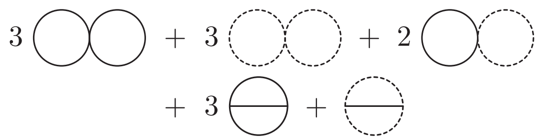
is a slowly varying function compared with $T^2$ or $T^4$. So the shape of the potential is hardly affected. The critical temperature is determined as usual by the vanishing of the coefficient of $\xi^2$. To lowest order, $T_\mathrm{c}^2 = 3c^2/\lambda$, as before. An improved formula is obtained by substituting the lowest-order result in the logarithm:

$$T_\mathrm{c}^2 = \frac{3c^2}{\lambda}\left[1 + \frac{\delta'\lambda}{\lambda} + \frac{\lambda}{4\pi^2}\ln\left(\frac{24\pi^2}{\lambda}\,\mathrm{e}^{-2\gamma_\mathrm{E}+1}\right)\right] \tag{7.34}$$

The correction is of relative order $\lambda \ln \lambda$. For instance, if we take $\delta'\lambda = 0$ and $\lambda = 0.1$ then the correction is only about 2%. It may seem as if the critical temperature depends on the rather arbitrary value of $\delta'\lambda$ but this is not so; the numerical values of $c$ and $\lambda$ depend on the renormalization prescription used to define them, which involves $\delta'\lambda$ through (7.31). In the end, $T_\mathrm{c}$ must be independent of the renormalization prescription.

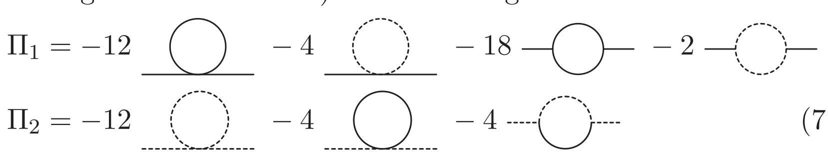
The next problem we face in the mean field approximation is that $\bar{m}_2^2 < 0$ for $T > 0$ and $\bar{m}_1^2 < 0$ for $T > 2T_\mathrm{c}^2/3$. Note that the finite-temperature corrections (7.19) to these masses are negative and proportional to $\lambda T^2$ in the high-temperature limit ($T > c$). The one-loop contributions to the self-energies are of the same order. Therefore, they must be computed.

From the Lagrangian (7.5) we find the two-loop contributions to $\ln Z$ to be




(7.35)

A solid line represents the $\chi_1$ propagator and a broken line represents the $\chi_2$ propagator. There is a factor $-\lambda/4$ at each four-point vertex and a factor $-\sqrt{2}\lambda\xi$ at each three-point vertex. (Note that the 1PR diagrams do not appear on account of the stipulation that $\chi_0(0) = 0$. This can be shown by returning to the diagrammatic rules following from the functional integral in Section 3.2.) The self-energies are




(7.36)

The diagrams involving a three-point vertex, the so-called exchange diagrams, are momentum and frequency dependent. To renormalize, we must add the counterterms $-\delta c^2 + 6\xi^2\delta\lambda$ and $-\delta c^2 + 2\xi^2\delta\lambda$ to $\Pi_1$ and to $\Pi_2$, respectively. In the high-temperature approximation, and at low frequency and momentum, the exchange diagrams may be neglected. This follows simply from power counting. Both types of diagram involve one integration over the loop momentum, but the exchange diagrams involve

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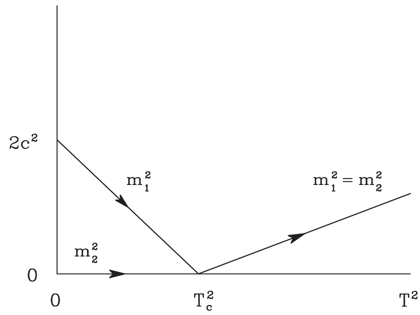


Fig. 7.5

two propagators instead of one. Then

$$\Pi_1^{\text{mat}} = \Pi_2^{\text{mat}} = \frac{1}{3}\lambda T^2 \tag{7.37}$$

Adding these to (7.19), we obtain the masses

$$\begin{aligned} m_1^2 = \bar{m}_1^2 + \Pi_1^{\text{mat}} &= \begin{cases} 2c^2\left(1 - \dfrac{T^2}{T_{\text{c}}^2}\right) & T \le T_{\text{c}} \\ \dfrac{1}{3}\lambda\left(T^2 - T_{\text{c}}^2\right) & T \ge T_{\text{c}} \end{cases} \\ m_2^2 = \bar{m}_2^2 + \Pi_2^{\text{mat}} &= \begin{cases} 0 & T \le T_{\text{c}} \\ \dfrac{1}{3}\lambda\left(T^2 - T_{\text{c}}^2\right) & T \ge T_{\text{c}} \end{cases} \end{aligned} \tag{7.38}$$

The behaviour of the masses as a function of temperature is shown in Figure 7.5. Thus the pathological behavior of the boson propagators has been cured. The vanishing of the masses at the critical point is characteristic of a second-order phase transition. Typically, one finds that the correlation lengths diverge at $T_{\text{c}}$. (The last diagram for $\Pi_1$ in (7.36) actually diverges if we let the external frequency and momentum go to zero and if $T < T_{\text{c}}$, because $m_2 = 0$. That is, the zero-mode contribution is proportional to $\lambda^2\xi^2 T \int dp/p^2$. This is of no physical importance since the mass is defined to be the location of the pole of the real time propagator at zero momentum. The relevant limit in (7.36) is $\Pi_1(\omega = m_1,\ \mathbf{k} = 0)$.)

The lesson learned is that the mean field approximation is not reliable in all respects. It turns out that it correctly predicts a second-order symmetry-restoring phase transition at $T_{\text{c}}^2 = 3c^2/\lambda$. However, it is incorrect in the finer details, such as the finite-temperature behavior of the correlation lengths (boson masses). This is a serious matter, since Goldstone's theorem is violated. At the very least, one should include all loop corrections to the same order in the coupling constants as is retained in the mean field approximation. The reason is that a loop expansion is

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essentially an expansion in powers of the Lagrangian. In order to respect the symmetries of the Lagrangian, one must retain all diagrams through a fixed number of loops.

A better approximation scheme would be to consider the thermodynamic potential $\Omega$ as function of the mean field $\langle\Phi\rangle = \xi$ and as a functional of the boson propagators $\mathcal{D}_1$ and $\mathcal{D}_2$. The mean field would be determined by the minimization condition $\partial\Omega/\partial\xi = 0$, and the propagators would be determined by the Schwinger–Dyson equations. To implement this idea, we would add to the quadratic part of the action $S_0$ the term

$$-\tfrac{1}{2}\beta^2 \sum_n \sum_{\mathbf{p}} \left[\chi_{1;-n}(-\mathbf{p})\Pi_1(\omega_n,\mathbf{p})\chi_{1;n}(\mathbf{p}) + \chi_{2;-n}(-\mathbf{p})\Pi_2(\omega_n,\mathbf{p})\chi_{2;n}(\mathbf{p})\right] \tag{7.39}$$

and subtract the same quantity from $S_{\rm I}$. In the $S_0$ case, (7.39) is to be treated as a counter-term. Recalling (2.36) and the steps leading up to it, we can write the thermodynamic potential as [4]

$$\begin{aligned}\Omega(T,\xi,\mathcal{D}_1,\mathcal{D}_2) &\\ = \mathrm{U}(\xi) - \tfrac{1}{2}T\sum_n \int \frac{d^3p}{(2\pi)^3}&\left[\ln(T^2\mathcal{D}_1) + \ln(T^2\mathcal{D}_2) - \frac{\mathcal{D}_1}{\bar{\mathcal{D}}_1^0} - \frac{\mathcal{D}_2}{\bar{\mathcal{D}}_2^0} + 2\right]\\ &+ \sum_{l=2}^{\infty}\Omega_l(\xi,\mathcal{D}_1,\mathcal{D}_2) + \text{subtractions}\end{aligned} \tag{7.40}$$

Here

$$\begin{aligned}\bar{\mathcal{D}}_1^0 &= \left(\omega_n^2 + \mathbf{p}^2 + \bar{m}_1^2\right)^{-1}\\ \bar{\mathcal{D}}_2^0 &= \left(\omega_n^2 + \mathbf{p}^2 + \bar{m}_2^2\right)^{-1}\end{aligned} \tag{7.41}$$

These are the mean field propagators, and $\Omega_l$ is the sum of all $l$-loop diagrams; in these loop diagrams, the bare propagators are to be replaced with the full propagators. Here, the potential $\Omega$ is an extremum with respect to independent functional variations of $\mathcal{D}_1$ and $\mathcal{D}_2$, on account of the Schwinger–Dyson equations

$$\begin{aligned}\mathcal{D}_1^{-1} - \bar{\mathcal{D}}_1^{0\,-1} &= 2\sum_{l=2}^{\infty}\frac{\delta\Omega_l}{\delta\mathcal{D}_1}\\ \mathcal{D}_2^{-1} - \bar{\mathcal{D}}_2^{0\,-1} &= 2\sum_{l=2}^{\infty}\frac{\delta\Omega_l}{\delta\mathcal{D}_2}\end{aligned} \tag{7.42}$$

These equations determine $\Pi_1$ and $\Pi_2$ self-consistently, just as $\xi$ is determined self-consistently from $\partial\Omega/\partial\xi = 0$.

As a practical matter, the loop sum must be terminated at a finite order. Then the momentum- and frequency-dependent self-energies must

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be determined self-consistently and substituted into (7.40) to compute $\Omega$. The mean field is then determined by minimizing $\Omega$. If only the two-loop diagrams (7.35) are retained and the high-temperature approximation is made, (7.37) and (7.38) follow. Then $\Omega$ may be computed from (7.40) straightforwardly since the propagators are both non-negative for all frequency and momentum. Minimization with respect to $\xi$ will yield $\xi$ as a function of $T$. One finds again that at $T_c^2 = 3c^2/\lambda$ there is a second-order symmetry-restoring phase transition, as predicted by the mean field approximation. This is left as an exercise.

### 7.4 Higgs model

The model discussed so far can be made more interesting by coupling the charged scalar field to the electromagnetic field. The Lagrangian density is

$$\mathcal{L} = (\partial^\mu - ieA^\mu)\Phi^*(\partial_\mu + ieA_\mu)\Phi + c^2\Phi^*\Phi - \lambda(\Phi^*\Phi)^2 - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad (7.43)$$

Anticipating the spontaneous breaking of the U(1) symmetry, which is now a local symmetry, we shift the field by setting

$$\Phi = \xi + \chi \quad (7.44)$$

and stipulate that $\langle\chi\rangle = 0$. Apart from terms linear in $\chi$, we obtain

$$\mathcal{L} = -\mathrm{U}(\xi) + \mathcal{L}_0 + \mathcal{L}_\mathrm{I} \quad (7.45)$$

where

$$\begin{aligned}\mathcal{L}_0 = {}& \tfrac{1}{2}(\partial_\mu\chi_1)(\partial^\mu\chi_1) - \tfrac{1}{2}\bar{m}_1^2\chi_1^2 + \tfrac{1}{2}(\partial_\mu\chi_2)(\partial^\mu\chi_2) - \tfrac{1}{2}\bar{m}_2^2\chi_1^2 \\ & - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + e^2\xi^2A^\mu A_\mu - \sqrt{2}e\xi\chi_2\partial_\mu A^\mu \\ \mathcal{L}_\mathrm{I} = {}& -\sqrt{2}\lambda\xi\left(\chi_1^2 + \chi_2^2\right)\chi_1 - \tfrac{1}{4}\lambda\left(\chi_1^2 + \chi_2^2\right)^2 \\ & + eA^\mu(\chi_1\partial_\mu\chi_2 - \chi_2\partial_\mu\chi_1) \\ & + e^2A^\mu A_\mu\left[\sqrt{2}\xi\chi_1 + \tfrac{1}{2}\left(\chi_1^2 + \chi_2^2\right)\right]\end{aligned}$$

Here $\bar{m}_1^2$, $\bar{m}_2^2$, and U($\xi$) are as defined in Section 7.1. It would appear from $\mathcal{L}_0$ that the electromagnetic field has developed a mass $\sqrt{2}e\xi$. However, this must be carefully considered because of the mixing between $\chi_2$ and $A_\mu$.

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To find the spectrum of excitations at $T = 0$ it is useful to make the change of variables

$$\begin{aligned} \Phi &= \left(\xi + 2^{-1/2}\phi\right) \exp\left(\frac{i\eta}{\sqrt{2}\xi}\right) \\ A'_\mu &= A_\mu + \frac{\partial_\mu \eta}{\sqrt{2}e\xi} \end{aligned} \tag{7.46}$$

where $\phi$ and $\eta$ are two independent real fields. Substitution into (7.43) yields

$$\mathcal{L} = -\mathrm{U}(\xi) + \mathcal{L}'_0 + \mathcal{L}'_\mathrm{I} \tag{7.47}$$

where

$$\mathcal{L}'_0 = \tfrac{1}{2}\partial_\mu\phi\, \partial^\mu\phi - \tfrac{1}{2}\bar{m}_1^2\phi^2 - \tfrac{1}{4}F'^{\mu\nu}F'_{\mu\nu} + e^2\xi^2 A'^\mu A'_\mu$$

$$\mathcal{L}'_\mathrm{I} = -\sqrt{2}\lambda\xi\phi^3 - \tfrac{1}{4}\lambda\phi^4 + e^2\left(\sqrt{2}\xi + \tfrac{1}{2}\phi\right)\phi A'^\mu A'_\mu$$

Notice that all reference to the field $\eta$ has gone! Minimizing the classical energy density $\mathrm{U}(\xi)$ gives an equilibrium condensate $\xi^2 = c^2/2\lambda$, the same as before. Thus, at $T = 0$, we have a real scalar field with mass $\sqrt{2}c$ and a vector field with mass $ec/\sqrt{\lambda}$. Counting the number of degrees of freedom, we have one for the former and three for the latter. This is the same as without spontaneous symmetry breaking, namely two for the $\Phi$ field and two for the massless $A_\mu$ field. There is no Goldstone boson; the Goldstone theorem does not apply, because $A_\mu$ is a vector field. The generation of mass for the vector field via spontaneous symmetry breaking is known as the Higgs mechanism. It is a central concept in modern gauge theories.

The choice of variables in (7.46) is not very appropriate for a mean field approximation at high temperature, because we expect $\xi$ to decrease with increasing $T$ and eventually to vanish above a critical temperature. Therefore we return to (7.45) to study the thermodynamics.

At $T = 0$ it can be shown that the $\chi_2$ field in (7.45) does not represent an observable particle in scattering experiments [5]. In more picturesque language, it is said that the vector field increases its number of polarization degrees of freedom from two to three and becomes massive by eating the would-be Goldstone boson.

The partition function is

$$Z = \int [dA_\mu]\,[d\Phi]\,[d\Phi^*]\,\delta(F)\,\det\left(\frac{\partial F}{\partial\alpha}\right)\exp\left(\int_0^\beta d\tau \int d^3x\,\mathcal{L}\right) \tag{7.48}$$

One convenient choice of gauge is the so-called $R_\rho$-gauge,

$$F = \partial^\mu A_\mu - \sqrt{2}e\xi\rho\chi_2 - f(\mathbf{x}, \tau) \tag{7.49}$$

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in the limit $\rho \to 0$. Under an infinitesimal gauge transformation

$$\begin{aligned} \Phi &\to \Phi e^{ie\alpha} \approx \left(\xi + \frac{\chi_1 + i\chi_2}{\sqrt{2}}\right)(1 + ie\alpha) \\ A^\mu &\to A^\mu - \partial^\mu \alpha \end{aligned} \tag{7.50}$$

we have

$$\frac{\partial F}{\partial \alpha} = -\partial^2 - e^2 \xi \left(2\xi + \sqrt{2}\chi_1\right)\rho \to -\partial^2 \tag{7.51}$$

Furthermore, multiplying the right-hand side of (7.48) by

$$\exp\left(-\frac{1}{2\rho}\int_0^\beta d\tau \int d^3x\, f^2\right)$$

and functionally integrating over $f$ gives a $\beta$-independent correction. Hence

$$Z = \lim_{\rho\to 0} \det(-\partial^2) \int [dA_\mu]\,[d\Phi]\,[d\Phi^*] \exp\left(\int_0^\beta d\tau \int d^3x\, \mathcal{L}_{\text{eff}}\right) \tag{7.52}$$

where

$$\mathcal{L}_{\text{eff}} = -\mathrm{U}(\xi) + \mathcal{L}_0 + \mathcal{L}_{\mathrm{I}} - \frac{1}{2\rho}\left(\partial^\mu A_\mu - \sqrt{2} e \xi \rho \chi_2\right)^2$$

Close scrutiny of (7.52) brings out the following points. The factor $\det\left(-\partial^2\right)$ cancels two specious degrees of freedom. The gauge-fixing term has a part that is independent of $\rho$ and that, in fact, cancels the mixing term between $\chi_2$ and $A_\mu$ in (7.45). The limit $\rho \to 0$ ensures that only those gauge-field configurations with $\partial^\mu A_\mu = 0$ contribute to the partition function.

A high-temperature mean field approximation similar to (7.13) and (7.14) can be carried out, with the result

$$\Omega(\xi, T) = \lambda \xi^4 + \left[\left(\frac{\lambda}{3} + \frac{e^2}{4}\right) T^2 - c^2\right] \xi^2 - \frac{2\pi^2}{45} T^4 - \frac{1}{12} c^2 T^2 \tag{7.53}$$

This predicts a second-order symmetry-restoring phase transition at $T_c^2 = 12c^2/(4\lambda + 3e^2)$. Of course, the particle masses exhibit the pathological behavior typical of the mean field approximation and it is necessary to calculate the one-loop self-energies to obtain a more respectable behavior.

Since the Higgs model contains two independent dimensionless coupling constants, new phenomena may occur. If $\lambda \gtrsim e^4$ then the qualitative behavior of the phase transition sketched above is not altered by higher-order loop corrections. If $\lambda \lesssim e^4$ then the mass of the vector meson is

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comparable with or greater than $T_c$, and the second-order phase transition may even become a first-order one. In fact, when $\lambda \to 3e^4/32\pi^2$, quantum corrections cause $T_c$ to decrease to zero, and for $\lambda < 3e^4/32\pi^2$ there is no spontaneous symmetry breaking even at $T = 0$. The interested reader is referred to the review of Lindé [6].

If $c = 0$ then we are dealing with massless scalar electrodynamics, not the Higgs model. Surprisingly, spontaneous symmetry breaking occurs here also. It is driven by the one-loop quantum correction to the vacuum energy density, the shift in the zero-point energy of the vacuum. This phenomenon was discovered by Coleman and Weinberg [7]. The finite-temperature behavior of the Coleman–Weinberg model is left as an exercise.

## 7.5 Exercises

7.1 Choose $\delta'\lambda$ in (7.32) so that the depth of the minimum is the same as in the classical theory. Then plot the classical and one-loop quantum vacuum energy densities versus $\xi$ for $\lambda = 0.1, 0.01, 0.001$.

7.2 Retaining the two-loop diagrams (7.35) and using the high-temperature approximation, as discussed at the end of Section 7.3, calculate $T_c$.

7.3 An alternative to the mean field expansion is an ordinary perturbative expansion based on the $T = 0$ value of the condensate field $\xi_0$. This scheme has the disadvantage that it is not self-consistent, but the advantage that one need not do an expansion in terms of full propagators since no tachyons appear in the perturbative expansion. In this case $\langle\chi\rangle$ will not vanish at $T > 0$. Using only the one-loop diagrams, show that $\langle\Phi\rangle = \xi_0 + \langle\chi\rangle$ vanishes at $T_c^2 = 3c^2/\lambda$.

7.4 Read the paper Coleman and Weinberg [7]. Verify their result that there is spontaneous symmetry breaking at $T = 0$ in massless scalar electrodynamics. Show that the symmetry is restored at high temperature, and calculate $T_c$.

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