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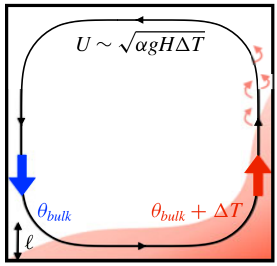
Thermal forcing and 'classical' and 'ultimate' regimes of Rayleigh–Bénard convection



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The fundamental challenge to characterize and quantify thermal transport in the strongly nonlinear regime of Rayleigh–Bénard convection – the buoyancy-driven flow of a horizontal layer of fluid heated from below – has perplexed the fluid dynamics community for decades. Rayleigh proposed controlling the temperature of thermally conducting boundaries in order to study the onset of convection, in which case vertical heat transport gauges the system response. Conflicting experimental results for ostensibly similar set-ups have confounded efforts to discriminate between two competing theories for how boundary layers and interior flows interact to determine transport through the convecting layer asymptotically far beyond onset. In a conceptually new approach, Bouillaut, Lepot, Aumaître and Gallet (*J. Fluid Mech.*, vol. 861, 2019, R5) devised a procedure to radiatively heat a portion of the fluid domain bypassing rigid conductive boundaries and allowing for dissociation of thermal and viscous boundary layers. Their experiments reveal a new level of complexity in the problem suggesting that heat transport scaling predictions of both theories may be realized depending on details of the thermal forcing.

Key words: turbulent convection

1. Introduction

Rayleigh (1916) proposed a model for Bénard's turn of the 20th century experiments consisting of the Boussinesq approximation to the Navier–Stokes equations of motion, wherein the density ρ is fixed in all but the temperature-dependent buoyancy force term and the fluid's material properties (viscosity ν, specific heat c and thermal diffusion and expansion coefficients κ and α) are assumed constant, applied to a layer of fluid confined between parallel horizontal iso-thermal no-penetration plates. Even though the actual thermo-capillary mechanism underlying Bénard's observations – the Bénard–Marangoni instability – was only properly formulated as a hydrodynamic problem decades later by Pearson (1958), Rayleigh's model for buoyancy-driven thermal convection is still called the Rayleigh–Bénard problem.

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A key feature of interest is the relation between the temperature drop ΔT across a fluid layer of thickness h and the resulting heat flux Q. In Rayleigh's model the temperature drop is controlled at the boundaries, and the first result of his analysis was to identify the non-dimensional group determining instability of the quiescent conduction state: $g\alpha\Delta Th^3/\nu\kappa$ where g is the acceleration of gravity. Today we call this the Rayleigh number Ra, leaving the fluid's Prandtl number $Pr = \nu/\kappa$ as the other non-dimensional parameter characterizing the particular system at hand. Heat transport is traditionally measured in units of the conductive heat flux via the Nusselt number $Nu = Qh/\rho c\kappa\Delta T$, and the challenge is, for a given domain, to determine Nu as a function of Ra and Pr thereby quantifying the effective thermal conductivity of the convecting layer. The strongly nonlinear regime of paramount importance for geophysical and astrophysical applications corresponds to large Rayleigh numbers and two distinct theories, conventionally referred to as the 'classical' and 'ultimate' theories, remain in contention for the asymptotic behaviour of Nu as $Ra \to \infty$. Mathematical analysis has failed to prove, and experimental studies have failed to definitively rule out, either of these asymptotic theories for $O(1)$ values of Pr.

The work of Bouillaut *et al.* (2019), based on the novel experimental approach of Lepot, Aumaître & Gallet (2018) in which the fluid is radiatively heated rather than relying on conduction through the rigid boundaries, sheds new light on the conundrum. Boundaries and boundary layers play key roles in the alternative 'classical' and 'ultimate' theories for the extreme limit of Rayleigh–Bénard convection and direct internal heating allows for the disentanglement of velocity and temperature boundary layers. The experimental data suggest that, depending on details of the structure and strength of the thermal forcing, the predictions of either competing theory may be observed.

2. Overview

The 'classical' theory asserts that $Nu \sim Ra^{1/3}$. It was simultaneously proposed by Priestly (1954), who argued that Q should be independent of h for turbulent convection in large aspect ratio domains, and Malkus (1954), whose complicated maximal dissipation theory also predicted that the scaling is uniform in Pr. A decade later Howard (1964) reinterpreted the uniform-in-Pr prediction as a marginally stable thermal boundary layer argument, explicitly neglecting the potential effect of a velocity boundary layer.

The 'ultimate' theory asserts $Nu \sim Pr^{1/2}Ra^{1/2}$ for $Pr \leqslant O(1)$. It was originally proposed by Spiegel (1963) based on the idea that buoyant fluid elements transport heat at the free-fall velocity $U \sim (g\alpha\Delta Th)^{1/2}$ so that Q becomes independent of the material transport coefficients ν and κ. (Spiegel's theory actually predates this reference as evidenced in the testimony of Batchelor 1961.) Spiegel referred to this approach as a 'mixing length' theory without reference to the nature of boundaries or boundary layers. But soon thereafter his postdoc mentor Kraichnan (1962) considered ultra-high Ra situations when velocity boundary layers at the rigid plates presumably transition to shear turbulence, postulating logarithmic corrections to the asymptotic $Ra^{1/2}$ scaling. The moniker 'ultimate' was first used by Chavanne *et al.* (1997) referring to Kraichnan's modification of Spiegel's theory and has since been adopted by the community for an asymptotic state with predominant $1/2$ scaling. Moreover, $Nu \sim Ra^{1/2}$ scaling, albeit uniform in Pr, is mathematically 'ultimate' in the sense that it is a rigorous upper bound for heat transport in Rayleigh's model (Howard 1963; Doering & Constantin 1996).

Rather than fixing the temperature and inserting and removing heat at the top and bottom boundaries, Bouillaut *et al.* (2019) heat the bottom of the layer by illuminating the fluid, which is treated with optically absorbing dye, from below. The optical absorption length ℓ can be varied by adjusting the dye concentration to control the spatial profile of heat injection into the container. The container is thermally insulated on all boundaries so the bulk temperature increases linearly with time as the experiment proceeds. As long as the Boussinesq approximation remains valid, however, the difference between the local temperature and the linearly increasing-in-time bulk temperature behaves as if the system is heated near the bottom and cooled in the bulk above. In this set-up, distinct from the set-up of Rayleigh's model, the heat flux Q is controlled and temperature drop ΔT across the layer must be measured. That is, both the Nusselt and Rayleigh numbers are emergent quantities.

Controlling the heat flux through, rather than the temperatures at, the rigid boundaries has been considered before. While this variation of Rayleigh's model profoundly changes the nature of the bifurcation at onset (Hurle, Jakeman & Pike 1967) it does not apparently affect the *Nu*–*Ra* relation for high *Ra* turbulent convection (Johnston & Doering 2009). Internal heating or cooling – regulating heat flux into or out of the system in the bulk – while correspondingly extracting or injecting heat via conduction at at least one rigid boundary has also been considered (Goluskin 2016). But in all these cases the interplay of thermal and velocity boundary layers cannot be separated somewhere in the system.

Bouillaut *et al.* (2019) control the thermal injection length scale directly via the optical absorption length, introducing the fundamentally new dimensionless parameter ℓ/h into the problem. The limit of small ℓ/h corresponds to fixing the heat flux at the rigid bottom boundary while finite ℓ/h maintains heat input in a portion of the domain well outside any potentially shrinking velocity boundary layer. Their experimental data for various values of *Nu*, *Ra* and ℓ/h collapse in the form $(\ell/h)^2 Nu = f[(\ell/h)^6 Ra]$ with scaling function f satisfying $f[x] \sim x^{1/3}$ for $x \ll 1$ and $f[x] \sim x^{1/2}$ for $x \gg 1$. (Impressively, the data collapse appears over nearly twenty decades of $(\ell/h)^6 Ra$.) This implies 'classical' $Nu \sim Ra^{1/3}$ scaling when the absorption length $\ell \ll \sqrt{h\delta}$, where $\delta \equiv h/2Nu$ is the apparent thermal boundary layer thickness, and 'ultimate' $Nu \sim Ra^{1/2}$ scaling for fixed non-zero ℓ/h as $Ra \to \infty$.

3. Future

The work of Bouillaut *et al.* (2019) opens new directions for Rayleigh–Bénard research. It should stimulate new experimental investigations aimed both at independent confirmation and at understanding Prandtl number influence on the empirical scaling function. The corresponding class of mathematical models for Rayleigh–Bénard convection, with internal heat sources and sinks rather than conduction boundaries, presents new challenges for computation, theory and analysis.

The most obvious theoretical approach to bypass complicating boundary layer effects is to consider the Boussinesq equations with an imposed thermal gradient in a fully periodic domain (Borue & Orszag 1997), but that turns out to be an unphysical idealization: there is no limit to the resulting flows' energy due to unbounded runaway solutions that inevitably pollute simulations and obviate analysis (Calzavarini *et al.* 2006). The generalized models proposed by Lepot *et al.* (2018) and Bouillaut *et al.* (2019), however, are physically well defined in the sense that energy in all solutions remains uniformly bounded for all times (Fantuzzi & Doering 2018; Lepot 2018), even in idealized fully periodic domains (Muite *et al.* 2017). Perhaps surprisingly these studies also show that, in terms of sensible definitions of Nu and Ra in this setting, asymptotic heat transport scaling as high as $Nu \sim Ra$ [sic] may be realized. That is, the 'mixing-length' theory $Nu \sim Ra^{1/2}$ scaling is no longer 'ultimate' in the sense of 'maximal' for these internally thermally driven systems.

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