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WEIGHTED COMPOSITION OPERATORS ON THE BLOCH SPACE

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We characterise bounded and compact weighted composition operators on the Bloch space and the little Bloch space. The results generalise the known corresponding results on composition operators and pointwise multipliers on the Bloch space and the little Bloch space.

1. INTRODUCTION

Let D be the open unit disk in the complex plane. Let u be a fixed analytic function on D and φ an analytic self-map of D. We can define a linear operator uC_φ on the space of analytic functions on D, called a *weighted composition operator*, by

$$uC_\varphi f = uf \circ \varphi.$$

It is easy to see that an operator defined in this manner is linear. We can regard this operator as a generalisation of a multiplication operator and a composition operator. In this note we study the boundedness and the compactness of weighted composition operators on the Bloch space and the little Bloch space. Recall that a function f analytic in D is said to belong to the Bloch space $\mathcal{B}$ if

$$\sup_{z\in D}(1-|z|^2)|f'(z)| < \infty$$

and to the little Bloch space $\mathcal{B}_0$ if

$$\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0.$$

Under the norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in D}(1-|z|^2)|f'(z)|,$$

$\mathcal{B}$ becomes a Banach space and $\mathcal{B}_0$ is a closed subspace of $\mathcal{B}$. See [**7**, Chapter 5] for more about the Bloch space.

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Recall that, in [**1**], Arazy (and Brown and Shields [**2**] also) characterised the boundedness of multiplication operators on $\mathcal{B}$. Madigan and Matheson [**4**] characterised the boundedness and the compactness of composition operators on $\mathcal{B}$ and $\mathcal{B}_0$. Our results can be viewed as generalisations of their results. For more information on composition operators, see [**3**] and [**5**].

We collect some basic properties of functions in the Bloch space and the little Bloch space here. Recall that
$$\beta(z,w) = \frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)}$$
is the Bergman metric on D, where $\rho(z,w) = |z-w|/|1-\bar{z}w|$ and $z, w \in D$. It is well-known that the following hold ([**6**] and [**7**]):

$$\text{(1)} \qquad \left|f(z) - f(w)\right| \leqslant \|f\|_{\mathcal{B}}\beta(z,w) \quad \text{for} \quad f \in \mathcal{B}$$

and

$$\text{(2)} \qquad \lim_{|z|\to 1}\frac{f(z)}{\beta(0,z)} = 0 \quad \text{for} \quad f \in \mathcal{B}_0.$$

From (1), we can show that for $f \in \mathcal{B}$

$$\text{(3)} \qquad \left|f(z)\right| \leqslant \frac{1}{\log 2}\|f\|_{\mathcal{B}}\log\frac{2}{1-|z|^2}.$$

We shall use these estimates in the proofs of theorems below.

2. The case of the Bloch space

In this section we characterise bounded and compact weighted composition operators on the Bloch space.

Theorem 1. *Let u be an analytic function on the unit disk D and φ an analytic self-map of D. Then uC_φ is bounded on the Bloch space $\mathcal{B}$ if and only if the following are satisfied:*

(i) $\sup_{z\in D}(1-|z|^2)|u'(z)|\log\left(2\big/\left(1-|\varphi(z)|^2\right)\right) < \infty$;

(ii) $\sup_{z\in D}\left((1-|z|^2)\big/\left(1-|\varphi(z)|^2\right)\right)|u(z)\varphi'(z)| < \infty$.

Proof: Suppose that uC_φ is bounded on the Bloch space. Then we can easily obtain the following by taking the constant function and $f(z) = z$ in $\mathcal{B}$ respectively:

$$\text{(4)} \qquad u \in \mathcal{B};$$

$$\text{(5)} \qquad \sup_{z\in D}(1-|z|^2)\left|u(z)\varphi'(z)\right| < \infty.$$

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For $\lambda \in D$, let $f(z) = \log 2/\left(1 - \overline{\varphi(\lambda)}z\right)$. Then $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \leqslant 3$. So

$$\begin{aligned} 3\|uC_\varphi\| &\geqslant \|uC_\varphi f\|_{\mathcal{B}} \\ &\geqslant \left(1-|\lambda|^2\right)|u'(\lambda)| \log \frac{2}{1-|\varphi(\lambda)|^2} - \frac{1-|\lambda|^2}{1-|\varphi(\lambda)|^2}|u(\lambda)\varphi'(\lambda)|. \end{aligned}$$

That is,

$$3\|uC_\varphi\| + \frac{1-|\lambda|^2}{1-|\varphi(\lambda)|^2}|u(\lambda)\varphi'(\lambda)| \geqslant \left(1-|\lambda|^2\right)|u'(\lambda)| \log \frac{2}{1-|\varphi(\lambda)|^2}.$$

Thus it is sufficient to prove that the estimate (ii) is true. Here we take the function $f(z) = 1/\left(1 - \overline{\varphi(\lambda)}z\right)$ for $\lambda \in D$. Then $f \in \mathcal{B}$ and

$$\|f\|_{\mathcal{B}} \leqslant 1 + \frac{2}{1-|\varphi(\lambda)|}.$$

It is clear that

$$\|uC_\varphi f\|_{\mathcal{B}} \geqslant \left(1-|\lambda|^2\right)\left|\frac{|u'(\lambda)|}{1-|\varphi(\lambda)|^2} - |u(\lambda)|\frac{|\overline{\varphi(\lambda)}\varphi'(\lambda)|}{\left(1-|\varphi(\lambda)|^2\right)^2}\right|.$$

So

$$\|uC_\varphi\|\left(1+\frac{2}{1-|\varphi(\lambda)|}\right) \geqslant \frac{1-|\lambda|^2}{\left(1-|\varphi(\lambda)|^2\right)^2}|u(\lambda)\overline{\varphi(\lambda)}\varphi'(\lambda)| - \frac{1-|\lambda|^2}{1-|\varphi(\lambda)|^2}|u'(\lambda)|.$$

Therefore,

$$5\|uC_\varphi\| + \left(1-|\lambda|^2\right)|u'(\lambda)| \geqslant \frac{1-|\lambda|^2}{1-|\varphi(\lambda)|^2}|u(\lambda)\varphi(\lambda)\varphi'(\lambda)|.$$

Thus, for a fixed δ, $0 < \delta < 1$, by (4),

$$\sup\left\{\frac{1-|\lambda|^2}{1-|\varphi(\lambda)|^2}\,|u(\lambda)\varphi'(\lambda)| : \lambda \in D,\ |\varphi(\lambda)| > \delta\right\} < \infty. \tag{6}$$

For $\lambda \in D$ such that $|\varphi(\lambda)| \leqslant \delta$, we have

$$\frac{1-|\lambda|^2}{1-|\varphi(\lambda)|^2}|u(\lambda)\varphi'(\lambda)| \leqslant \frac{1}{1-\delta^2}\left(1-|\lambda|^2\right)|u(\lambda)\varphi'(\lambda)|$$

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and so, by (5),

$$\sup\left\{\frac{1-|\lambda|^2}{1-|\varphi(\lambda)|^2}|u(\lambda)\varphi'(\lambda)| : \lambda \in D,\ |\varphi(\lambda)| \leqslant \delta\right\} < \infty. \tag{7}$$

Consequently, by (6) and (7),

$$\sup_{\lambda\in D}\frac{1-|\lambda|^2}{1-|\varphi(\lambda)|^2}|u(\lambda)\varphi'(\lambda)| < \infty$$

and so,

$$\sup_{\lambda\in D}(1-|\lambda|^2)|u'(\lambda)|\log\frac{2}{1-|\varphi(\lambda)|^2} < \infty.$$

Conversely, suppose that the conditions (i) and (ii) hold. For a function $f \in \mathcal{B}$, we have the following inequality:

$$\begin{aligned}
&(1-|z|^2)|(uC_\varphi f)'(z)| \\
&\quad= (1-|z|^2)\left|u'(z)f(\varphi(z)) + u(z)f'(\varphi(z))\varphi'(z)\right| \\
&\quad\leqslant (1-|z|^2)\left|u'(z)f(\varphi(z))\right| + \frac{1-|z|^2}{1-|\varphi(z)|^2}|u(z)\varphi'(z)|\left(1-|\varphi(z)|^2\right)\left|f'(\varphi(z))\right| \\
&\quad\leqslant (1-|z|^2)|u'(z)|\frac{1}{\log 2}\|f\|_{\mathcal{B}}\log\frac{2}{1-|\varphi(z)|^2} + \frac{1-|z|^2}{1-|\varphi(z)|^2}|u(z)\varphi'(z)|\|f\|_{\mathcal{B}},
\end{aligned}$$

where we use (3) in the last inequality.

So the conditions imply that the righthand side is bounded by some constant times $\|f\|_{\mathcal{B}}$. Consequently uC_φ is bounded on $\mathcal{B}$. □

Remark. Condition (i) in Theorem 1 implies

$$u \in \mathcal{B} \tag{8}$$

and

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)|u'(z)| = 0. \tag{9}$$

Next we shall consider compactness.

Theorem 2. *Let u be an analytic function on the unit disk D and φ an analytic self-map of D. Suppose that uC_φ is bounded on $\mathcal{B}$. Then uC_φ is compact on $\mathcal{B}$ if and only if the following are satisfied:*

(i) $\displaystyle\lim_{|\varphi(z)|\to 1}(1-|z|^2)|u'(z)|\log\left(2/\left(1-|\varphi(z)|^2\right)\right) = 0$;

(ii) $\displaystyle\lim_{|\varphi(z)|\to 1}\left((1-|z|^2)/\left(1-|\varphi(z)|^2\right)\right)|u(z)\varphi'(z)| = 0$.

PROOF: Suppose uC_φ is compact on $\mathcal{B}$. Let $\{z_n\}$ be a sequence in D such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Let $f_n(z) = \left(1 - |\varphi(z_n)|^2\right)/\left(1 - \overline{\varphi(z_n)}z\right)$. Then $f_n \in \mathcal{B}$, $\|f_n\|_{\mathcal{B}} \leqslant 5$ and f_n converges to 0 uniformly on compact subsets of D.

Since uC_φ is compact on $\mathcal{B}$, we have

$$\|uC_\varphi f_n\|_{\mathcal{B}} \to 0 \quad \text{as} \quad n \to \infty.$$

Thus

$$\begin{aligned}\|uC_\varphi f_n\|_{\mathcal{B}} &= \sup_{z\in D}\left(1 - |z|^2\right)\left|(uC_\varphi f_n)'(z)\right| \\ &\geqslant \left(1 - |z_n|^2\right)\left|u'(z_n) + u(z_n)\frac{1 - |\varphi(z_n)|^2}{\left(1 - |\varphi(z_n)|^2\right)^2}\overline{\varphi(z_n)}\varphi'(z_n)\right| \\ &\geqslant \left|\left(1 - |z_n|^2\right)|u'(z_n)| - \frac{1 - |z_n|^2}{1 - |\varphi(z_n)|^2}\left|u(z_n)\overline{\varphi(z_n)}\varphi'(z_n)\right|\right|.\end{aligned}$$

From (9) we get

$$\lim_{|\varphi(z_n)|\to 1} \frac{1 - |z_n|^2}{1 - |\varphi(z_n)|^2}\left|u(z_n)\varphi'(z_n)\right| = 0. \tag{10}$$

Next let

$$f_n(z) = \left(\log\frac{2}{1 - |\varphi(z_n)|^2}\right)^{-1}\left(\log\frac{2}{1 - \overline{\varphi(z_n)}z}\right)^2$$

for a sequence $\{z_n\}$ in D such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Then $\{f_n\}$ is a bounded sequence in the Bloch space and $f_n(z) \to 0$ uniformly on every compact subset of D. By a method similar to that above,

$$\begin{aligned}0 \leftarrow &\|uC_\varphi f_n\|_{\mathcal{B}} \\ &\geqslant \left|\left(1 - |z_n|^2\right)|u'(z_n)|\log\frac{2}{1 - |\varphi(z_n)|^2} - 2\left(1 - |z_n|^2\right)|u(z_n)|\frac{|\varphi(z_n)\varphi'(z_n)|}{1 - |\varphi(z_n)|^2}\right|.\end{aligned}$$

So, by (10), we get

$$\lim_{|\varphi(z_n)|\to 1}\left(1 - |z_n|^2\right)|u'(z_n)|\log\frac{2}{1 - |\varphi(z_n)|^2} = 0. \tag{11}$$

The converse can be proved by a method similar to that in the proof of [**4**, Theorem 2]. We omit the details here. ∎

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3. The case of the little Bloch space

In this section we consider the case of $\mathcal{B}_0$ and give some examples of functions u and φ satisfying the hypotheses of our theorems.

Theorem 3. *Let u be an analytic function on the unit disk D and φ an analytic self-map of D. Then uC_φ is bounded on the little Bloch space $\mathcal{B}_0$ if and only if the following are all satisfied:*

(i) $\sup_{z\in D}|u'(z)|(1-|z|^2)\log\Big(2/\big(1-|\varphi(z)|^2\big)\Big) < \infty$;

(ii) $\sup_{z\in D}|u(z)\varphi'(z)|(1-|z|^2)/\big(1-|\varphi(z)|^2\big) < \infty$;

(iii) $u \in \mathcal{B}_0$;

(iv) $\lim_{|z|\to 1}|u(z)\varphi'(z)|(1-|z|^2) = 0$.

Proof: First, suppose that uC_φ is bounded on $\mathcal{B}_0$. Then (i) and (ii) can be proved exactly in the same way as in the proof of Theorem 1 since the test functions which we used there are also in the little Bloch space $\mathcal{B}_0$.

By taking $f(z) = c$ (constant), we get $u \in \mathcal{B}_0$, which is (iii).

For proving (iv), let $f(z) = z$. We get from $uC_\varphi \in \mathcal{B}_0$ that

$$\lim_{|z|\to 1}|u(z)\varphi'(z) + u'(z)\varphi(z)|(1-|z|^2) = 0. \tag{12}$$

Since $u \in B_0$ and $|\varphi(z)| < 1$, we get that

$$\lim_{|z|\to 1}|u'(z)\varphi(z)|(1-|z|^2) = 0.$$

From this fact and (12) we get (iv).

Next we suppose (i)–(iv) are satisfied. Take any $\varepsilon > 0$. Let $f \in \mathcal{B}_0$. Then, by (2), there is $\delta_1 \in (0,1)$ such that for any $z \in D$, $|z| > \delta_1$, we have $|f(z)| < \varepsilon \log 2/(1-|z|^2)$. Thus for $|\varphi(z)| > \delta_1$, by (i) we can find a constant $C_1 > 0$ such that

$$\Big|u'(z)f\big(\varphi(z)\big)\Big|(1-|z|^2) < \varepsilon|u'(z)|(1-|z|^2)\log\frac{2}{1-|\varphi(z)|^2} \leqslant C_1\varepsilon. \tag{13}$$

On the other hand, since, by (iii), $u \in \mathcal{B}_0$, we know that for the above ε, there is $\delta_2 \in (0,1)$ such that $|z| > \delta_2$ implies $|u'(z)|(1-|z|^2) < \varepsilon$. Thus for $|\varphi(z)| \leqslant \delta_1$, if $|z| > \delta_2$, we have a constant $C_2 > 0$ such that

$$\Big|u'(z)f\big(\varphi(z)\big)\Big|(1-|z|^2) \leqslant \frac{1}{\log 2}\|f\|_{\mathcal{B}}|u'(z)|(1-|z|^2)\log\frac{2}{1-\delta_1^2} \leqslant C_2\varepsilon. \tag{14}$$

By combining (13) and (14), we see that whenever $|z| > \delta_2$ we have

$$\Big|u'(z)f\big(\varphi(z)\big)\Big|(1-|z|^2) \leqslant \max(C_1, C_2)\varepsilon. \tag{15}$$

Since $f \in \mathcal{B}_0$, there is $\delta_3 \in (0,1)$ such that $|z| > \delta_3$ implies that $|f'(z)| < \varepsilon/(1-|z|^2)$. Thus, for $|\varphi(z)| > \delta_3$, by (ii), we know that there is a constant $C_3 > 0$ such that

$$\Big|u(z)f'\big(\varphi(z)\big)\varphi'(z)\Big|(1-|z|^2) < \varepsilon|u(z)\varphi'(z)|\frac{1-|z|^2}{1-|\varphi(z)|^2} \leqslant C_3\varepsilon. \tag{16}$$

On the other hand, by (iv), there is $\delta_4 \in (0,1)$ such that $|z| > \delta_4$ implies

$$|u(z)\varphi'(z)|(1-|z|^2) < \varepsilon.$$

Thus, for $|\varphi(z)| \leqslant \delta_3$, and $|z| > \delta_4$, we have a constant $C_4 > 0$ such that

$$\Big|u(z)f'\big(\varphi(z)\big)\varphi'(z)\Big|(1-|z|^2) \leqslant \|f\|_{\mathcal{B}}|u(z)\varphi'(z)|\frac{1-|z|^2}{1-\delta_3^2} \leqslant C_4\varepsilon. \tag{17}$$

By combining (15) and (17), we see that whenever $|z| > \delta_4$,

$$\Big|u(z)f'\big(\varphi(z)\big)\varphi'(z)\Big|(1-|z|^2) < \max(C_3, C_4)\varepsilon. \tag{18}$$

By combining (16) and (18), we see that for $\delta = \max(\delta_2, \delta_4)$, if $|z| > \delta$ then there is a constant $C > 0$ such that

$$\Big|u'(z)f\big(\varphi(z)\big) + u(z)f'\big(\varphi(z)\big)\varphi'(z)\Big|(1-|z|^2) < C\varepsilon,$$

which means

$$\lim_{|z|\to 1}|(uC_\varphi f)'(z)|(1-|z|^2) = 0.$$

Thus $uC_\varphi f \in \mathcal{B}_0$. The proof is complete. ∎

We use the following lemma to give a criterion for compactness.

Lemma. [4] *A closed set K in $\mathcal{B}_0$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}(1-|z|^2)|f'(z)| = 0.$$

Theorem 4. *Let u be an analytic function on the unit disk D and φ an analytic self-map of D. Then uC_φ is compact on $\mathcal{B}_0$ if and only if the following are satisfied:*

(i) $\displaystyle\lim_{|z|\to 1}|u'(z)|(1-|z|^2)\log\Big(2/\big(1-|\varphi(z)|^2\big)\Big) = 0;$

(ii) $\displaystyle\lim_{|z|\to 1}|u(z)\varphi'(z)|(1-|z|^2)/\big(1-|\varphi(z)|^2\big) = 0.$

PROOF: By the lemma, uC_φ is compact on $\mathcal{B}_0$ if and only if

$$\lim_{|z|\to 1} \sup_{\|f\|_{\mathcal{B}}\leqslant 1} (1-|z|^2)\left|(uC_\varphi f)'(z)\right| = 0. \tag{19}$$

Suppose that (i) and (ii) are satisfied. By (i), $u \in \mathcal{B}_0$. For a function $f \in \mathcal{B}_0$, we have the following inequality:

$$\begin{aligned}(1-|z|^2)\left|(uC_\varphi f)'(z)\right| &= (1-|z|^2)\Big|u'(z)f\big(\varphi(z)\big) + u(z)f'\big(\varphi(z)\big)\varphi'(z)\Big| \\ &\leqslant (1-|z|^2)\left|u'(z)\right|\frac{1}{\log 2}\log\frac{2}{1-\left|\varphi(z)\right|^2}\|f\|_{\mathcal{B}} + \frac{1-|z|^2}{1-\left|\varphi(z)\right|^2}\left|u(z)\varphi'(z)\right|\|f\|_{\mathcal{B}}.\end{aligned}$$

By (i) and (ii), the above inequality implies (19). Thus uC_φ is compact on $\mathcal{B}_0$.

Conversely, suppose that uC_φ is compact on $\mathcal{B}_0$. Using the same test functions as in the proof of Theorem 2, we see

$$\lim_{|\varphi(z)|\to 1}\left|u'(z)\right|(1-|z|^2)\log\frac{2}{1-\left|\varphi(z)\right|^2} = 0 \tag{20}$$

and

$$\lim_{|\varphi(z)|\to 1}\left|u(z)\varphi'(z)\right|\frac{1-|z|^2}{1-\left|\varphi(z)\right|^2} = 0. \tag{21}$$

Since uC_φ is bounded on $\mathcal{B}_0$, Theorem 3 implies that $u \in \mathcal{B}_0$ and

$$\lim_{|z|\to 1}\left|u(z)\varphi'(z)\right|(1-|z|^2) = 0. \tag{22}$$

It is easy to show that ($u \in \mathcal{B}_0$ and (20)) is equivalent to (i), and ((21) and (22)) is equivalent to (ii). ∎

REMARK. The conditions in Theorem 4 include the conditions of the boundedness of uC_φ.

Finally we give some examples. In the trivial cases that $u(z)$ is constant and $\varphi(z) = z$, our theorems give necessary and sufficient conditions for multiplication operators and composition operators to be bounded and compact on $\mathcal{B}$ and $\mathcal{B}_0$ respectively (see [**1**, **2**, **4**]). It seems that the result for compact multiplication operators on the Bloch space and little Bloch space has not appeared in the literature. Therefore, we single it out as a corollary. For an analytic function u on D, we denote by M_u the multiplication operator, that is, $M_u f(z) = u(z)f(z)$.

Corollary 1. *Let u be an analytic function on the unit disk D. Then the following statements are equivalent:*

(i) M_u *is a compact operator on* $\mathcal{B}$;
(ii) M_u *is a compact operator on* $\mathcal{B}_0$;
(iii) $u = 0$.

Here are some other examples.

Example 1. Let $u(z) = \log 2/(1-z)$ and $\varphi(z) = (1-z)/2$. Then u does not induce a bounded multiplication operator on $\mathcal{B}$, but uC_φ is bounded on $\mathcal{B}$.

Example 2. Let $u(z) = 1 - z$ and $\varphi(z) = (1+z)/2$. Then neither u induces a compact multiplication operator nor φ induces a compact composition operator on $\mathcal{B}$ and $\mathcal{B}_0$. But uC_φ is compact on $\mathcal{B}$ and $\mathcal{B}_0$.

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