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Characterizations of Continuous and Discrete q-Ultraspherical Polynomials

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Abstract. We characterize the continuous q-ultraspherical polynomials in terms of the special form of the coefficients in the expansion $\mathcal{D}_q P_n(x)$ in the basis $\{P_n(x)\}$, $\mathcal{D}_q$ being the Askey–Wilson divided difference operator. The polynomials are assumed to be symmetric, and the connection coefficients are multiples of the reciprocal of the square of the L^2 norm of the polynomials. A similar characterization is given for the discrete q-ultraspherical polynomials. A new proof of the evaluation of the connection coefficients for big q-Jacobi polynomials is given.

1 Introduction

The problem of characterizing classes of orthogonal polynomials is an old one and has extensive literature. Al-Salam wrote an important survey article [1] covering the literature on characterization theorems up to 1990, the date of its publication. Many of these characterization theorems deal with the classical orthogonal polynomials of Hermite, Laguerre, and Jacobi, and later the wider class of orthogonal polynomials considered by Hahn; see [1]. The Hahn class contains the big q-Jacobi polynomials that generalize Jacobi polynomials. The big q-Jacobi polynomials contain a symmetric orthogonal polynomial sequence: the discrete q-ultraspherical polynomials. The ultraspherical (Gegenbauer) polynomials are the spherical harmonics on Euclidean spaces, [17].

Two noteworthy contributions from the 1970's are the works [2, 3]. In the first, Al-Salam and Chihara characterized orthogonal polynomials having a certain convolution property and discovered what has become known as the Al-Salam–Chihara polynomials [13, 14]. The weight function for these polynomials was found later in [9]. Another proof is in [10]. The second paper by Al-Salam and Chihara [3] gives an interesting characterization of the q-Pollaczek polynomials, which was conjectured by Andrews and Askey.

Another important orthogonal polynomial sequence is the sequence of continuous q-ultraspherical polynomials. It is a one parameter generalization of the spherical harmonics. They first appeared in the work of L. J. Rogers in the 1890's. He used them to prove the Rogers–Ramanujan identities. They appeared in 1941 in

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the works of Feldheim [11] and Lanzewizky [15] who independently showed that the only orthogonal polynomials in the Fejer class ([18]) are the ultraspherical and q-ultraspherical polynomials. There weight function was found in [8]. They are also special Askey–Wilson polynomials, so another proof follows from [10].

Recently Lasser and Obermaier observed that the ultraspherical polynomials, normalized to be 1 at $x = 1$, have the property

$$\frac{d}{dx}P_n(x) = \sigma_n \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} h_{n-2k-1}P_{n-2k-1}(x), \tag{1.1}$$

where σ_n is a constant depending on n and the h_k's come from the orthogonality relation

$$\int_{\mathbb{R}} P_m(x)P_n(x)d\mu(x) = \delta_{m,n}/h_n,$$

with orthogonality measure μ; see [16]. They showed that this property characterizes the ultraspherical polynomials among symmetric orthogonal polynomials, which satisfy a three term recurrence relation

$$xP_n(x) = a_nP_{n+1} + c_nP_{n-1}, \quad n \geq 1,$$

with $a_n + c_n = 1$. Since

$$\sigma_n = \int_{\mathbb{R}} \frac{d}{dx}P_n(x)P_{n-2k-1}(x)d\mu(x), \quad k = 0, 1, \ldots, \lfloor (n-1)/2 \rfloor,$$

this characterization is due to the constancy of the Fourier coefficients of $\frac{d}{dx}P_n(x)$. A second characterization of the ultraspherical polynomials given in [16] is based on the ratio of coefficients of the three term recurrence relation, namely on the property

$$\frac{c_n}{c_{n-1}} = \frac{n}{2c_{n-1} + n - 1} = \frac{s_n}{(s_{n+1} - s_{n-1})c_{n-1} + s_{n-1}}, \quad n \geq 2, \tag{1.2}$$

where

$$\frac{d}{dx}x^n = s_nx^{n-1}.$$

Properties (1.1) and (1.2) are very curious, and we discovered that they are shared by many other systems of orthogonal polynomials. In the present work we give two q-analogues of the Lasser-Obermaier result. Section 2 contains all the notations and preliminary results needed in our analysis.

Askey and Wilson solved the connection coefficient problem for the Askey–Wilson polynomials in [10]. The big q-Jacobi polynomials are limiting cases of the Askey-Wilson polynomials, [14, §3.5], so their connection coefficient can be found from the Askey–Wilson result. In Section 3, however, we include a direct and independent evaluation of the connection coefficients for big q-Jacobi polynomials, and then identify the connection coefficients in the special case of the discrete q-ultraspherical polynomials. The Askey–Wilson proof uses technical special functions, but our proof is much simpler and is more elementary than the Askey–Wilson proof.

2 Preliminaries

We shall follow the notation and terminology of q-series as in [7,12,13]. In particular the q-shifted factorials are

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a_1,a_2,\dots,a_m;q)_n = \prod_{j=1}^{m}(a_j;q)_n.$$

Here $n = 1,2,\dots$ or ∞, when $|q| < 1$, which we shall always assume. Moreover, $(a;q)_0 := 1$. A basic hypergeometric function is

$${}_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\dots,a_{r+1} \\ b_1,b_2,\dots,b_r \end{matrix}\,\middle|\, q,z\right) = \sum_{n=0}^{\infty} \frac{(a_1,\dots,a_{r+1};q)_n}{(q,b_1,\dots,b_r;q)_n}\, z^n.$$

We shall make use of two special sums, the q-analogue of the Chu–Vandermonde theorem, [12, (II.6)],

$$ {}_2\phi_1(q^{-n},a;c;q) = \frac{(c/a;q)_n}{(c;q)_n}\, a^n, \tag{2.1}$$

and the Andrews terminating analogue of Watson's theorem [4], [12, (II.17)],

$$ {}_4\phi_3\left(\begin{matrix} q^{-n}, b^2q^{n+1}, c, -c \\ qb, -qb, c^2 \end{matrix}\,\middle|\, q,q\right) = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ \frac{c^n(q,b^2q^2/c^2;q^2)_{n/2}}{(q^2b^2,qc^2;q^2)_{n/2}}, & \text{if } n \text{ is even}. \end{cases} \tag{2.2}$$

The q-difference operator is

$$(D_q f)(x) = \frac{f(x)-f(qx)}{x-qx}, x \neq 0.$$

The product rule for D_q is

$$ D_q(fg)(x) = f(x)D_q g(x) + g(qx)D_q f(x). \tag{2.3}$$

The q-integral is defined as an infinite Riemann sum, via

$$\int_0^a f(x)d_qx := (1-q)a\sum_{n=0}^{\infty} q^n f(aq^n), \quad \int_a^b f(x)d_qx := \int_0^b f(x)d_qx - \int_0^a f(x)d_qx.$$

When $g(a/q) = g(b/q) = 0$, the integration by parts for the q-integral is

$$q\int_a^b D_q f(x)g(x)d_qx = -\int_a^b f(x)D_{q^{-1}}g(x)d_qx,$$

[13, (11.4.6)].

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One special q-integral we shall use is ([12, (2.10.20)])

$$\int_a^b \frac{(qt/a, qt/b; q)_\infty}{(ct/a, dt/b; q)_\infty} d_q t = \frac{b(1-q)(q, a/b, qb/a, cd; q)_\infty}{(c, d, bc/a, ad/b; q)_\infty}. \tag{2.4}$$

Let $x = (z + 1/z)/2$ and set $f(x) = \breve{f}(z)$. The Askey–Wilson divided difference operator $\mathcal{D}_q$ is defined by

$$(\mathcal{D}_q f)(x) = \frac{\breve{f}(q^{1/2}z) - \breve{f}(q^{-1/2}z)}{\breve{e}(q^{1/2}z) - \breve{e}(q^{-1/2}z)}, \quad e(x) := x.$$

The Askey–Wilson operator is a degree reducing operator as can be seen from its action on the Chebyshev polynomials of the first kind, namely

$$\mathcal{D}_q T_n(x) = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} U_{n-1}(x). \tag{2.5}$$

Indeed (2.5) shows that, at least on polynomials, $\mathcal{D}_q \to \frac{d}{dx}$ as $q \to 1$. Also note that $\mathcal{D}_q$ is invariant under $q \to 1/q$.

The product rule for the Askey–Wilson operator is

$$\mathcal{D}_q(fg) = \mathcal{A}_q f \mathcal{D}_q g + \mathcal{A}_q g \mathcal{D}_q f, \tag{2.6}$$

where the averaging operator $\mathcal{A}_q$ is defined by

$$(\mathcal{A}_q f)(x) = \frac{1}{2}\left(\breve{f}(q^{1/2}z) + \breve{f}(q^{-1/2}z)\right).$$

The operator $\mathcal{A}_q$ is a degree preserving operator as can be seen from

$$\mathcal{A}_q T_n(x) = \frac{q^{n/2} + q^{-n/2}}{2} T_n(x). \tag{2.7}$$

The big q-Jacobi polynomials are defined by

$$P_n(x; \mathbf{a}) = P_n(x; a_1, a_2; a_3) = {}_3\phi_2\left(\left.\begin{matrix} q^{-n}, a_1 a_2 q^{n+1}, x \\ q a_1, q a_3 \end{matrix}\right| q, q\right),$$

where $\mathbf{a}$ stands for the vector (a_1, a_2, a_3). They satisfy the orthogonality relation [6, 12–14],

$$\int_{qa_3}^{qa_1} P_m(x; \mathbf{a}) P_n(x; \mathbf{a}) \frac{(x/a_1, x/a_3; q)_\infty}{(x, a_2 x/a_3; q)_\infty} d_q x = \frac{\delta_{m,n}}{h_n(\mathbf{a})},$$

$$\begin{aligned} 1/h_n(\mathbf{a}) := a_1 q(1-q) &\frac{(q, a_1 a_2 q^2, a_3/a_1, q a_1/a_3; q)_\infty}{(q a_1, q a_2, q a_3, q a_1 a_2/a_3; q)_\infty} \\ &\times \frac{(1 - q a_1 a_2)}{(1 - a_1 a_2 q^{2n+1})} \frac{(q, q a_2, q a_1 a_2/a_3; q)_n}{(q a_1, q a_1 a_2, q a_3; q)_n} (-a_1 a_3)^n q^{n(n+3)/2}, \end{aligned}$$

for $qa_1, qa_2 \in (0,1), a_3 < 0$.

The q-Chu–Vandermonde sum (2.1) shows that the values of a big q-Jacobi polynomial at the end points qa_1, qa_3 are

$$P_n(qa_1;\mathbf{a}) = (-a_3)^n q^{\binom{n+1}{2}} \frac{(qa_1a_2/a_3;q)_n}{(qa_3;q)_n}, \tag{2.8}$$
$$P_n(qa_3;\mathbf{a}) = (-a_1)^n q^{\binom{n+1}{2}} \frac{(qa_2;q)_n}{(qa_1;q)_n}.$$

Moreover, it is clear that $P_n(1;\mathbf{a}) = 1$. The weight function is

$$w(x;\mathbf{a}) = \frac{(x/a_1, x/a_3;q)_\infty}{(x, xa_2/a_3;q)_\infty}. \tag{2.9}$$

The Rodrigues type formula is ([6, 13, 14])

$$w(x;\mathbf{a})P_n(x;\mathbf{a}) = \frac{(a_1a_3)^n q^{n(n+1)}(1-q)^n}{(qa_1, qa_3;q)_n} (D_q)^n w(x;q^n\mathbf{a}).$$

Moreover,

$$D_{q^{-1}} P_n(x;\mathbf{a}) = \frac{q^{1-n}(1-q^n)(1-a_1a_2q^{n+1})}{(1-q)(1-qa_1)(1-qa_3)} P_{n-1}(x;q\mathbf{a}). \tag{2.10}$$

The continuous q-ultraspherical polynomials are ([8], [13, §13.2–13.3]),

$$C_n(\cos\theta;\beta|q) = \sum_{k=0}^{n} \frac{(\beta;q)_k(\beta;q)_{n-k}}{(q;q)_k(q;q)_{n-k}} e^{i\theta(n-2k)}. \tag{2.11}$$

The representation (2.11) is equivalent to the ${}_2\phi_1$ representation

$$C_n(\cos\theta;\beta\,|\,q) = \frac{(\beta;q)_n\, e^{in\theta}}{(q;q)_n}\, {}_2\phi_1\left(\left. \begin{matrix} q^{-n}, \beta \\ q^{1-n}/\beta \end{matrix} \right| q, qe^{-2i\theta}/\beta \right). \tag{2.12}$$

The orthogonality relation of the continuous q-ultraspherical polynomials is [13, §13.2],

$$\int_{-1}^{1} C_m(x;\beta\,|\,q)C_n(x;\beta\,|\,q)w(x\,|\,\beta)\,dx = \frac{2\pi(\beta, q\beta;q)_\infty}{(q,\beta^2;q)_\infty} \frac{(1-\beta)(\beta^2;q)_n}{(1-\beta q^n)(q;q)_n} \delta_{m,n},$$

which holds for $|\beta| < 1$, with

$$w(\cos\theta\,|\,\beta) = \frac{(e^{2i\theta}, e^{-2i\theta};q)_\infty}{(\beta e^{2i\theta}, \beta e^{-2i\theta};q)_\infty} (\sin\theta)^{-1}. \tag{2.13}$$

When $\beta > 1$, point masses appear at $x = \pm(\beta^{1/2} + \beta^{-1/2})/2$. We cannot find the value of $C_n(\pm 1;\beta \mid q)$ for any β in closed form, but it is clear from (2.11) that

$|C_n(x;\beta\,|\,q)| \le C_n(1;\beta\,|\,q)$ for $x \in [-1,1]$. The only points, other than $x = 0$, at which the polynomials can be evaluated are $x = \pm(\beta^{1/2} + \beta^{-1/2})/2$, which seem to be the natural end points of the interval of orthogonality. At these points $e^{-i\theta} = \sqrt{\beta}$, the ${}_2\phi_1$ representation (2.12), and the q-Chu–Vandermonde sum (2.1) give

$$C_n(\pm(\beta^{1/2} + \beta^{-1/2})/2;\beta|q) = (\pm 1)^n \beta^{-n/2}\frac{(\beta^2;q)_n}{(q;q)_n}. \tag{2.14}$$

The action of the Askey–Wilson operator on $C_n(x;\beta|q)$ is given by

$$\mathcal{D}_q C_n(x;\beta\mid q) = 2q^{-(n-1)/2}\frac{1-\beta}{1-q}C_{n-1}(x;q\beta\mid q). \tag{2.15}$$

The following connection coefficient problem was solved by L. J. Rogers in 1894:

$$C_n(x;\gamma\,|\,q) = \sum_{k=0}^{\lfloor n/2\rfloor}\frac{\beta^k(\gamma/\beta;q)_k(\gamma;q)_{n-k}}{(q;q)_k(q\beta;q)_{n-k}}\,\frac{(1-\beta q^{n-2k})}{(1-\beta)}\,C_{n-2k}(x;\beta\,|\,q). \tag{2.16}$$

See [13] and [10] for references and proofs.

If $\{P_n(x)\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials with respect to a positive Borel measure μ, then it will satisfy a three term recurrence relation of the form

$$xP_n(x) = a_nP_{n+1}(x) + b_nP_n(x) + c_nP_{n-1}(x), \quad n \ge 0,$$

with $b_n \in \mathbb{R}$ and $a_{n-1}c_n > 0, n > 0$. The orthogonality implies

$$a_nh_n = c_{n+1}h_{n+1}, \tag{2.17}$$

where h_n is defined through

$$1/h_n = \int_{\mathbb{R}}(P_n(x))^2\,d\mu(x). \tag{2.18}$$

The relationship (2.17) will be used repeatedly in the sequel.

3 Connection Relation for Big q-Jacobi Polynomials

In this section we give an explicit representation for the connection coefficients of the big q-Jacobi polynomials.

Let

$$P_n(x;\mathbf{b}) = \sum_{k=0}^{n} a_{n,k}(\mathbf{a},\mathbf{b})P_k(x;\mathbf{a}). \tag{3.1}$$

Then

$$\begin{aligned}
a_{n,k}(\mathbf{a},\mathbf{b})/h_k(\mathbf{a}) &= \int_{qa_3}^{qa_1} P_n(x;\mathbf{b})P_k(x;\mathbf{a})w(x;\mathbf{a})\,d_qx \\
&= \frac{(a_1a_3)^k q^{k(k+1)}(1-q)^k}{(qa_1,qa_3;q)_k}\int_{qa_3}^{qa_1} P_n(x;\mathbf{b})(D_q)^k w(x;q^k\mathbf{a})\,d_qx \\
&= \frac{(-a_1a_3)^k q^{k^2}(1-q)^k}{(qa_1,qa_3;q)_k}\int_{qa_3}^{qa_1} (D_{q^{-1}})^k P_n(x;\mathbf{b})w(x;q^k\mathbf{a})\,d_qx \\
&= \frac{(-a_1a_3)^k q^{\binom{k+1}{2}+k(k-n)}(q;q)_n(b_1b_2q^{n+1};q)_k}{(qa_1,qa_3;q)_k(q;q)_{n-k}(qb_1,qb_3;q)_k} \\
&\quad\times \int_{qa_3}^{qa_1} P_{n-k}(x;q^k\mathbf{b})w(x;q^k\mathbf{a})\,d_qx.
\end{aligned}$$

We now evaluate the last integral. Clearly

$$\int_{qa_3}^{qa_1} P_{n-k}(x;q^k\mathbf{b})w(x;q^k\mathbf{a})d_qx = \\ \sum_{s=0}^{n-k}\frac{(q^{k-n},b_1b_2q^{n+k+1};q)_s}{(q,q^{k+1}b_1,q^{k+1}b_3;q)_s}q^s\int_{qa_3}^{qa_1}\frac{(q^{-k}x/a_1,q^{-k}x/a_3;q)_\infty}{(xq^s,xa_2/a_3;q)_\infty}\,d_qx.$$

Since the integrand vanishes at $x = a_1q^j$ and $x = a_3q^j$, $0 \le j \le k$, we conclude that

$$\begin{aligned}
&\int_{qa_3}^{qa_1}\frac{(q^{-k}x/a_1,q^{-k}x/a_3;q)_\infty}{(xq^s,xa_2/a_3;q)_\infty}\,d_qx \\
&= \int_{q^{k+1}a_3}^{q^{k+1}a_1}\frac{(q^{-k}x/a_1,q^{-k}x/a_3;q)_\infty}{(xq^s,xa_2/a_3;q)_\infty}\,d_qx \\
&= \frac{a_1q^{k+1}(1-q)(q,a_3/a_1,qa_1/a_3,a_1a_2q^{2k+2+s};q)_\infty}{(a_3q^{k+s+1},q^{k+1}a_1a_2/a_3,a_1q^{k+s+1},a_2q^{k+1};q)_\infty},
\end{aligned}$$

where we applied (2.4). Therefore

$$\begin{aligned}
a_{n,k}(\mathbf{a},\mathbf{b})/h_k(\mathbf{a}) &= \frac{(q,a_3/a_1,qa_1/a_3,a_1a_2q^{2k+2};q)_\infty(b_1b_2q^{n+1};q)_k}{(a_3q^{k+1},q^{k+1}a_1a_2/a_3,a_1q^{k+1},a_2q^{k+1};q)_\infty} \\
&\quad\times a_1q^{k+1}(1-q)\frac{(-a_1a_3)^k q^{\binom{k+1}{2}+k(k-n)}(q;q)_n}{(q;q)_{n-k}(aq_1,qa_3,qb_1,qb_3;q)_k} \\
&\quad\times {}_4\phi_3\left(\left.\begin{matrix} q^{k-n},b_1b_2q^{n+k+1},a_1q^{k+1},a_3q^{k+1} \\ q^{k+1}b_1,q^{k+1}b_3,a_1a_2q^{2k+2}\end{matrix}\right|q,q\right).
\end{aligned}$$

This establishes the following theorem.

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Theorem 3.1 *The connection coefficients for big q-Jacobi polynomials in* (3.1) *are given by*

$$a_{n,k}(\mathbf{a},\mathbf{b}) = q^{k(k-n)} \frac{(q;q)_n\,(qa_1,qa_3,b_1b_2q^{n+1};q)_k}{(q;q)_{n-k}(q,qb_1,qb_3,q^{k+1}a_1a_2;q)_k} \times {}_4\phi_3\left(\begin{matrix} q^{k-n}, b_1b_2q^{n+k+1}, a_1q^{k+1}, a_3q^{k+1} \\ q^{k+1}b_1, q^{k+1}b_3, a_1a_2q^{2k+2} \end{matrix} \,\middle|\, q,q \right). \tag{3.2}$$

Corollary 3.2 (Andrews and Askey [5]) *The little q-Jacobi polynomials*

$$p_n(x;a,b) = \frac{(qa;q)_n}{(q;q)_n} {}_2\phi_1(q^{-n}, abq^{n+1}; qa; q, qz)$$

have the connection relation

$$p_n(x;b_1,b_2) = \sum_{k=0}^{n} c_{n,k}(a_1,a_2,b_1,b_2)p_k(x;a_1,a_2),$$

where

$$c_{n,k}(a_1,a_2,b_1,b_2) = \frac{q^{k(k-n)}(b_1b_2q^{n+1};q)_k(qb_1;q)_n}{(q;q)_{n-k}(qb_1,q^{k+1}a_1a_2;q)_k} \times {}_3\phi_2\left(\begin{matrix} q^{k-n}, b_1b_2q^{n+k+1}, a_1q^{k+1} \\ q^{k+1}b_1, a_1a_2q^{2k+2} \end{matrix} \,\middle|\, q,q \right).$$

Proof It is clear that

$$p_n(x;a_1,a_2) = \frac{(qa_1;q)_n}{(q;q)_n} \lim_{a_3\to\infty} P_n(qa_3x;a_1,a_2,a_3).$$

In (3.1) we take $b_3 = a_3$, replace x by qa_3x and let $a_3 \to \infty$. The result follows from (3.2). ■

The case of discrete q-ultraspherical polynomials is

$$C_n(x;\alpha:q) = P_n(x;\alpha q^{-1/2}, \alpha q^{-1/2}, -\alpha q^{-1/2}).$$

The three term recurrence relation for the big q-Jacobi polynomials is

$$(x-1)P_n(x;\mathbf{a}) = A_nP_{n+1}(x;\mathbf{a}) - (A_n+B_n)P_n(x;\mathbf{a}) + B_nP_{n-1}(x;\mathbf{a}),$$

where

$$A_n = \frac{(1-a_1q^{n+1})(1-a_3q^{n+1})(1-a_1a_2q^{n+1})}{(1-a_1a_2q^{2n+1})(1-a_1a_2q^{2n+2})},$$

$$B_n = -a_1a_3q^{n+1}\frac{(1-q^n)(1-a_2q^n)(1-(a_1a_2/a_3)q^n)}{(1-a_1a_2q^{2n})(1-a_1a_2q^{2n+1})}.$$

In the case of the discrete q-ultraspherical polynomials, the recurrence relation becomes

$$x(1-\alpha^2q^{2n})C_n(x;\alpha:q) = (1-\alpha^2q^n)C_{n+1}(x;\alpha:q)+\alpha^2q^n(1-q^n)C_{n-1}(x;\alpha:q).$$

Moreover, $C_n(1;\alpha:q)=1$ and the end point evaluations are

$$C_n(\alpha\sqrt{q};\alpha:q)=\alpha^nq^{n^2/2},\quad C_n(-\alpha\sqrt{q};\alpha:q)=(-1)^n\alpha^nq^{n^2/2}.$$

In this case the orthogonality relation becomes

$$\int_{-\alpha\sqrt{q}}^{\alpha\sqrt{q}} C_n(x;\alpha:q)C_n(x;\alpha:q)\frac{(qx/\alpha^2;q^2)_\infty}{(x^2;q^2)_\infty}\,d_qx=\frac{\delta_{m,n}}{h_n(\alpha)},$$
$$\frac{1}{h_n(\alpha)}=\alpha^{2n}q^{\binom{n+1}{2}}\,\frac{(1-\alpha^2)(q;q)_n}{(1-\alpha^2q^{2n})(\alpha^2;q)_n}\times 2\alpha\sqrt{q}(1-q)\frac{(q^2\alpha^2,q^2;q^2)_\infty}{(q\alpha^2;q^2)_\infty}.$$

Since the weight function is even, the connection coefficients $a_{n,k}$ are zero when $n-k$ is odd. Thus

$$C_n(x;\beta:q)=\sum_{k=0}^{\lfloor n/2\rfloor} b_{n,k}(\alpha,\beta)C_{n-2k}(x;\alpha:q),$$

and the ${}_4\phi_3$ is summed by (2.2). The result is

$$b_{n,k}(\alpha,\beta)=q^{2k(2k-n)}\frac{(q;q)_n\,(q\alpha^2;q^2)_{n-2k}(\beta^2q^n;q)_{n-2k}}{(q;q)_{2k}(q,\alpha^2q^{n-2k};q)_{n-2k}(q\beta^2;q^2)_{n-2k}} \times\frac{(\alpha q^{n-2k+1/2})^{2k}(q,\beta^2/\alpha^2;q^2)_k}{(\beta^2q^{2n-4k+1},\alpha^2q^{2n-4k+2};q^2)_k}.$$

The above expression simplifies to

$$b_{n,k}(\alpha,\beta)=q^k\alpha^{2k}\frac{(q;q)_n(\beta^2;q^2)_{n-k}(\beta^2/\alpha^2;q^2)_k}{(q^2;q^2)_k(q;q)_{n-2k}(q^2\alpha^2;q^2)_{n-k}}\,\frac{(1-\alpha^2q^{2n-4k})(\alpha^2;q)_{n-2k}}{(1-\alpha^2)(\beta^2;q)_n}.$$

4 A Characterization of Discrete q-Ultraspherical Polynomials

To state our characterization of the discrete q-ultraspherical polynomials we need to renormalize the polynomials to be equal to 1 at the right end point of the interval of orthogonality and renormalize the weight function to have total mass 1. In view of (2.8) and (2.9), we set

$$P_n(x;\alpha:q)=\alpha^{-n}q^{-n^2/2}C_n(x;\alpha:q),$$
$$\tilde{w}(x;\alpha)=\frac{1}{2\alpha\sqrt{q}(1-q)}\,\frac{(q\alpha^2;q^2)_\infty}{(q^2\alpha^2,q^2;q^2)_\infty}\,\frac{(qx/\alpha^2;q^2)_\infty}{(x^2;q^2)_\infty},$$

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so that

$$xP_n(x;\alpha:q) = a_nP_{n+1}(x;\alpha:q) + c_nP_{n-1}(x;\alpha:q),$$
$$a_n = \alpha q^{n+1/2}\frac{(1-\alpha^2q^n)}{1-\alpha^2q^{2n}}, \quad c_n = \alpha q^{1/2}\frac{(1-q^n)}{(1-\alpha^2q^{2n})}.$$

Note that $a_n + c_n = \alpha q^{1/2}$. Now the orthogonality relation becomes

$$\int_{-\alpha q^{1/2}}^{\alpha q^{1/2}} P_m(x;\alpha:q)P_n(x;\alpha:q)\tilde{w}(x;\alpha)d_qx = \frac{\delta_{m,n}}{h_n(\alpha)},$$
$$h_n(\alpha) = q^{\binom{n}{2}}\frac{(1-\alpha^2q^{2n})(\alpha^2;q)_n}{(1-\alpha^2)(q;q)_n}.$$

Applying (2.10) we see that the connection relation becomes

$$D_{q^{-1}}P_n(x;\alpha:q) = q^{-(n^2+n-1)/2}\frac{(1-\alpha^2q^n)(q;q)_n}{\alpha(1-q)(q\alpha^2;q)_n} \times \sum_{k=0}^{\lfloor (n-1)/2\rfloor} h_{n-2k-1}(\alpha)\,P_{n-2k-1}(x;\alpha:q). \tag{4.1}$$

We next state a characterization theorem, which is the main result of this section. Note that the h_k's are defined by (2.18).

Theorem 4.1 *Let $\{P_n(x)\}_{n=0}^{\infty}$ be a polynomial sequence orthogonal with respect to a positive Borel measure μ. Assume that $\{P_n(x)\}_{n=0}^{\infty}$ is recursively defined by $P_0(x) = 1$ and*

$$xP_n(x) = a_nP_{n+1}(x) + c_nP_{n-1}(x), \quad n \geq 0, \tag{4.2}$$

with

$$0 < a_n + c_n = A = \alpha q^{1/2} < 1, \quad c_0 = 0, \quad c_1 = \alpha q^{1/2}\frac{1-q}{1-\alpha^2q^2},$$

where $0 < q < 1$. Moreover, set $D_{q^{-1}}(x^n) = s_nx^{n-1}$, that is

$$s_n = q^{1-n}\frac{1-q^n}{1-q}.$$

Then the following three statements are equivalent.

(i) *For all $n \in \mathbb{N}$ we have*

$$D_{q^{-1}}P_n(x) = \sigma_n \sum_{k=0}^{\lfloor (n-1)/2\rfloor} h_{n-2k-1}P_{n-2k-1}(x),$$

where the constant σ_n depends only on n.

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(ii) *For all $n \geq 2$ it holds that*

$$\frac{c_n}{c_{n-1}} = \frac{q^{-1}As_n}{(s_{n+1} - s_{n-1})c_{n-1} + As_{n-1}}.$$

(iii) *The polynomials $\{P_n(x)\}_{n=0}^{\infty}$ are the discrete q-ultraspherical polynomials $\{P_n(x;\alpha : q)\}_{n=0}^{\infty}$.*

In this case the constants in (i) *are given by $\sigma_n = \frac{s_n}{c_n h_n}$.*

Proof First we assume (iii). Then for $n \geq 2$ we get

$$\begin{aligned}\frac{c_n}{c_{n-1}} &= \frac{\alpha q^{1/2}(1-q^n)}{c_{n-1}(1-\alpha^2 q^{2n})} = \frac{\alpha q^{1/2}(1-q^n)}{c_{n-1}(1-q^2+q^2-\alpha^2 q^{2n})} \\ &= \frac{q^{-1}As_n}{(s_{n+1}-s_{n-1})c_{n-1}+As_{n-1}}.\end{aligned}$$

This shows that (iii) implies (ii). Conversely, this also shows that (ii) implies (iii). We derive (i) from (iii) by applying (4.1). Hence it remains to prove that (i) implies (ii). We set

$$D_{q^{-1}}P_n(x) = \sum_{k=0}^{n-1} \omega_n(k)P_k(x)h_k.$$

Comparing the coefficients of x^{n-1} gives $s_n = \omega_n(n-1)h_{n-1}a_{n-1}$. In other words

$$\sigma_n = \omega_n(n-1) = \frac{s_n}{a_{n-1}h_{n-1}} = \frac{s_n}{c_n h_n}.$$

Moreover, set

$$P_n(x) = \sum_{k=0}^{n} \epsilon_n(k)P_k(x)h_k.$$

Applying the q-product rule (2.3) we get

$$D_{q^{-1}}(xP_n(x)) = q^{-1}xD_{q^{-1}}P_n(x) + P_n(x).$$

Acting with $D_{q^{-1}}$ on (4.2) results in

$$D_{q^{-1}}(xP_n(x)) = a_n D_{q^{-1}}P_{n+1}(x) + c_n D_{q^{-1}}P_{n-1}(x).$$

Thus

$$D_{q^{-1}}P_{n+1}(x) = \tfrac{1}{a_n}(P_n(x) + q^{-1}xD_{q^{-1}}P_n(x) - c_n D_{q^{-1}}P_{n-1}(x)).$$

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Again by comparing coefficients of $P_k(x)$ one gets

$$\omega_{n+1}(k) = \frac{1}{a_n}\Big(\epsilon_n(k) + q^{-1}\big(a_k\omega_n(k+1) + c_k\omega_n(k-1)\big) - c_n\omega_{n-1}(k)\Big).$$

If we assume (i), then $\sigma_{n+1} = \omega_{n+1}(n) = \omega_{n+1}(n-2)$ for $n \geq 2$. This yields

$$\begin{aligned}\frac{s_{n+1}}{a_n h_n} &= \frac{\epsilon_n(n-2) + q^{-1}(a_{n-2}\omega_n(n-1) + c_{n-2}\omega_n(n-3)) - c_n\omega_{n-1}(n-2)}{a_n} \\ &= \frac{1}{a_n}\left(\frac{A}{q}\frac{s_n}{c_n h_n} - c_n\frac{s_{n-1}}{c_{n-1}h_{n-1}}\right).\end{aligned}$$

Therefore

$$s_{n+1} = \frac{As_n}{qc_n} - \frac{a_{n-1}s_{n-1}}{c_{n-1}},$$

which gives

$$\frac{c_{n-1}s_{n+1} + (A - c_{n-1})s_{n-1}}{c_{n-1}} = \frac{As_n}{qc_n},$$

and finally yields

$$\frac{c_n}{c_{n-1}} = \frac{q^{-1}As_n}{(s_{n+1} - s_{n-1})c_{n-1} + As_{n-1}}.$$

This proves that (i) implies (ii), and the proof is complete. ■

Let $\{P_n(x)\}_{n=0}^{\infty}$ be a sequence of symmetric orthogonal polynomials with respect to a positive Borel measure μ. Moreover, assume that $\{P_n(x)\}_{n=0}^{\infty}$ is generated by $P_0(x) = 1$ and (4.2), with $0 < a_n + c_n = A < 1$, $c_0 = 0$ and $c_1 < A$. Then the assumptions of Theorem 4.1 are fulfilled with

$$q = \frac{A - c_1}{A - A^2 c_1} \quad \text{and} \quad \alpha = Aq^{-1/2}.$$

5 A Characterization of Continuous q-Ultraspherical Polynomials

To state our characterization of the continuous q-ultraspherical polynomials we need to renormalize the polynomials to be equal to 1 at the point $(\beta^{1/2} + \beta^{-1/2})/2$ and renormalize the weight function to have total mass 1. In view of (2.14) and (2.13) we set

$$P_n(x;\beta \mid q) = \frac{(q;q)_n}{(\beta^2;q)_n}\beta^{n/2}C_n(x;\beta \mid q),$$

$$\tilde{w}(x;\beta) = \frac{(e^{2i\theta}, e^{-2i\theta};q)_\infty}{(\beta e^{2i\theta}, \beta e^{-2i\theta};q)_\infty}\frac{(\beta^2, q;q)_\infty}{(\beta, \beta q;q)_\infty}\frac{1}{2\pi\sqrt{1-x^2}}, \quad x = \cos\theta,$$

so that

$$xP_n(x;\beta \mid q) = a_n P_{n+1}(x;\beta \mid q) + c_n P_{n-1}(x;\beta \mid q),$$

$$a_n = \frac{1-\beta^2 q^n}{2\beta^{1/2}(1-\beta q^n)}, \quad c_n = \frac{\beta^{1/2}(1-q^n)}{2(1-\beta q^n)}.$$

Note that $a_n + c_n = \frac{\beta^{1/2}+\beta^{-1/2}}{2}$. Now the orthogonality relation becomes

$$\int_{-1}^{1} P_m(x;\beta \,|\, q) P_n(x;\beta \,|\, q) \tilde{w}(x;\beta) dx = \frac{\delta_{m,n}}{h_n(\beta)},$$

$$h_n(\beta) = \frac{(\beta^2;q)_n (1-\beta q^n)}{(q;q)_n (1-\beta)} \beta^{-n}.$$

Applying (2.15) and (2.16) we see that the connection relation becomes

$$\mathcal{D}_q P_n(x;\beta \,|\, q) = 2\beta^{(2n-1)/2} q^{(1-n)/2} \frac{1-\beta}{1-q} \frac{(q;q)_n}{(\beta^2;q)_n} \times \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} h_{n-2k-1}(\beta)\, P_{n-2k-1}(x;\beta \,|\, q). \tag{5.1}$$

The proof of our characterization of the continuous q-ultraspherical polynomials relies on the following lemma.

Lemma 5.1 *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an orthogonal polynomial sequence initially defined by $P_0(x) = 1$ and generated by a three term recurrence relation of the form*

$$xP_n(x) = a_n P_{n+1}(x) + c_n P_{n-1}(x), \quad n \geq 0, \tag{5.2}$$

where $c_0 = 0$. Moreover, let

$$\mathcal{A}_q P_n(x) = \sum_{k=0}^{n} \alpha_n(k) P_k(x) h_k. \tag{5.3}$$

Then we have

$$\alpha_n(n) = \frac{q^{n/2} + q^{-n/2}}{2h_n}, \quad \text{and}$$

$$\alpha_n(n-2) = \frac{(1-q)(q^{(n-2)/2} - q^{-n/2})}{2h_{n-2}} \; \frac{2^{-2}n - \sum_{k=1}^{n-1} a_{k-1}c_k}{a_{n-2}a_{n-1}}. \tag{5.4}$$

Proof Recall that $\mathcal{A}_q$ acts as a multiplier operator on the Chebyshev polynomials of the first kind $T_n(x)$; see (2.7). It holds $T_0(x) = 1$, $T_1(x) = x$, and it is easily seen by induction that

$$T_n(x) = 2^{n-1}x^n - 2^{n-3}nx^{n-2} + \cdots, \quad n \geq 2.$$

We next expand P_n in the Chebyshev polynomials $\{T_n(x)\}$ then act with $\mathcal{A}_q$ on the result. Set

$$P_n(x) = \frac{x^n}{a_0 a_1 \cdots a_{n-1}} + d_{n,2} x^{n-2} + \cdots .$$

Substituting $P_n(x)$ in (5.2) and comparing coefficients of x^{n-1} results in

$$a_0 a_1 \cdots a_{n-1} d_{n,2} = a_0 a_1 \cdots a_n d_{n+1,2} + a_{n-1} c_n,$$

which yields by induction

$$d_{n,2} = \frac{-\sum_{k=1}^{n-1} a_{k-1} c_k}{a_0 a_1 \cdots a_{n-1}}.$$

Now set

$$P_n(x) = \frac{2^{1-n}}{a_0 a_1 \cdots a_{n-1}} T_n(x) + e_{n,2} T_{n-2}(x) + \cdots .$$

Comparing coefficients of x^{n-2} yields

$$d_{n,2} = 2^{n-3} e_{n,2} - \frac{2^{-2} n}{a_0 a_1 \cdots a_{n-1}}, \quad \text{or} \quad e_{n,2} = 2^{3-n} \frac{2^{-2} n - \sum_{k=1}^{n-1} a_{k-1} c_k}{a_0 a_1 \cdots a_{n-1}}.$$

Finally set

$$\mathcal{A}_q P_n(x) = \frac{q^{n/2} + q^{-n/2}}{2} P_n(x) + f_{n,2} P_{n-2}(x) + \cdots .$$

Since

$$\mathcal{A}_q P_n(x) = \frac{q^{n/2} + q^{-n/2}}{2} \frac{2^{1-n}}{a_0 a_1 \cdots a_{n-1}} T_n(x) + \frac{q^{(n-2)/2} + q^{(2-n)/2}}{2} e_{n,2} T_{n-2}(x) + \cdots$$

and

$$T_n(x) = 2^{n-1} a_0 a_1 \cdots a_{n-1} P_n(x) - 2^{n-1} a_0 a_1 \cdots a_{n-1} e_{n,2} T_{n-2}(x) + \cdots ,$$

comparing coefficients of $P_{n-2}(x)$ results in

$$\begin{aligned} f_{n,2} &= \Big(\frac{q^{(n-2)/2} + q^{(2-n)/2}}{2} - \frac{q^{n/2} + q^{-n/2}}{2} \Big) e_{n,2} 2^{n-3} a_0 a_1 \cdots a_{n-3} \\ &= \frac{(1-q)(q^{(n-2)/2} - q^{-n/2})}{2} \, \frac{2^{-2} n - \sum_{k=1}^{n-1} a_{k-1} c_k}{a_{n-2} a_{n-1}}, \end{aligned}$$

and the proof is complete. ∎

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The following characterization theorem is the main result of this section. Note that the h_k's are defined by (2.18).

Theorem 5.2 *Let $\{P_n(x)\}_{n=0}^{\infty}$ be an orthogonal polynomial sequence with respect to a positive Borel measure μ. Assume that $\{P_n(x)\}_{n=0}^{\infty}$ is recursively defined by $P_0(x) = 1$ and*

$$xP_n(x) = a_n P_{n+1}(x) + c_n P_{n-1}(x), \quad n \geq 0, \tag{5.5}$$

with

$$a_n + c_n = B = \frac{\beta^{1/2} + \beta^{-1/2}}{2} > 1, \tag{5.6}$$

$$c_0 = 0, \quad c_1 = \frac{\beta^{1/2}(1-q)}{2(1-\beta q)} < B, \tag{5.7}$$

where $0 < q < 1$ and $0 < \beta$. Moreover, set $\mathcal{D}_q T_n(x) = s_n U_{n-1}(x)$, that is

$$s_n = q^{-(n-1)/2} \frac{1-q^n}{1-q}.$$

Then the following three statements are equivalent.

(i) *For all $n \in \mathbb{N}$ we have*

$$\mathcal{D}_q P_n(x) = \sigma_n \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} h_{n-2k-1} P_{n-2k-1}(x),$$

where the constant σ_n depends only on n.

(ii) *For all $n \geq 2$ it holds that*

$$\frac{c_n}{c_{n-1}} = \frac{\frac{\beta^{1/2} q^{-1/2} + \beta^{-1/2} q^{1/2}}{2} s_n}{(s_{n+1} - s_{n-1}) c_{n-1} + B s_{n-1}}.$$

(iii) *The polynomials $\{P_n(x)\}_{n=0}^{\infty}$ are the continuous q-ultraspherical polynomials $\{P_n(x; \beta \mid q)\}_{n=0}^{\infty}$.*

In this case the constants in (i) *are given by $\sigma_n = \frac{s_n}{c_n h_n}$.*

Proof Since $\{P_n(x)\}_{n=0}^{\infty}$ is an orthogonal polynomial sequence with respect to a positive Borel measure μ, it holds that $a_0 c_1 > 0$. Thus $a_0 = B > 1$ implies $c_1 > 0$. Therefore (5.7) implies $\beta < q^{-1/2}$. Furthermore, (5.6) yields $\beta \neq 1$. Thus our setting implies $0 < \beta < q^{-1/2}$ and $\beta \neq 1$.

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First we assume (iii). Then for $n \geq 2$ we get

$$\begin{aligned}\frac{c_n}{c_{n-1}} &= \frac{\beta^{1/2}(1-q^n)}{2(1-\beta q^n)c_{n-1}} = \frac{\frac{\beta^{1/2}q^{-1/2}}{2}s_n}{\frac{q^{-n/2}}{1-q}(1-\beta q^n)c_{n-1}} \\ &= \frac{\frac{\beta^{1/2}q^{-1/2}+\beta^{-1/2}q^{1/2}}{2}s_n}{\frac{q^{-n/2}}{1-q}(1-\beta q^n)c_{n-1}\frac{\beta+q}{\beta}} = \frac{\frac{\beta^{1/2}q^{-1/2}+\beta^{-1/2}q^{1/2}}{2}s_n}{s_{n-1}\frac{1-\beta q^n}{q-\beta q^n}\frac{\beta+q}{2\beta^{1/2}}} \\ &= \frac{\frac{\beta^{1/2}q^{-1/2}+\beta^{-1/2}q^{1/2}}{2}s_n}{s_{n-1}(1+\frac{1-q}{q-\beta q^n})(B+\frac{q-1}{2\beta^{1/2}})} = \frac{\frac{\beta^{1/2}q^{-1/2}+\beta^{-1/2}q^{1/2}}{2}s_n}{(s_{n+1}-s_{n-1})c_{n-1}+Bs_{n-1}}.\end{aligned}$$

This shows that (iii) implies (ii). This also shows that (ii) implies (iii). We derive (i) from (iii) by applying (5.1). Hence it remains to prove that (i) implies (ii).

We set

$$\mathcal{D}_q P_n(x) = \sum_{k=0}^{n-1} \omega_n(k) P_k(x) h_k.$$

Representing both sides in terms of Chebyshev polynomials of the second kind and comparing coefficients of $U_{n-1}(x)$ gives $s_n = \omega_n(n-1)h_{n-1}a_{n-1}$. In other words

$$\sigma_n = \omega_n(n-1) = \frac{s_n}{a_{n-1}h_{n-1}} = \frac{s_n}{c_n h_n}.$$

Applying the product rule (2.6) for the Askey–Wilson operator we get

$$\mathcal{D}_q(xP_n(x)) = \frac{q^{1/2}+q^{-1/2}}{2} x\mathcal{D}_q P_n(x) + \mathcal{A}_q P_n(x).$$

Acting by $\mathcal{D}_q$ on (5.5) results in

$$\mathcal{D}_q(xP_n(x)) = a_n \mathcal{D}_q P_{n+1}(x) + c_n \mathcal{D}_q P_{n-1}(x).$$

Thus

$$\mathcal{D}_q P_{n+1}(x) = \frac{1}{a_n}\Big(\mathcal{A}_q P_n(x) + \frac{q^{1/2}+q^{-1/2}}{2} x\mathcal{D}_q P_n(x) - c_n \mathcal{D}_q P_{n-1}(x)\Big).$$

Comparing coefficients of $P_k(x)$ one gets

$$\begin{aligned}&\omega_{n+1}(k) = \\ &\qquad \frac{1}{a_n}\Big(\alpha_n(k) + \frac{q^{1/2}+q^{-1/2}}{2}\big(a_k\omega_n(k+1) + c_k\omega_n(k-1)\big) - c_n\omega_{n-1}(k)\Big),\end{aligned}$$

where $\alpha_n(k)$ is defined by (5.3). If we assume (i), then $\sigma_{n+1} = \omega_{n+1}(n) = \omega_{n+1}(n-2)$ for $n \geq 2$. This yields

$$\frac{s_{n+1}}{a_n h_n} = \frac{1}{a_n}\left(\alpha_n(n-2) + \frac{q^{1/2}+q^{-1/2}}{2} B \frac{s_n}{c_n h_n} - c_n \frac{s_{n-1}}{c_{n-1} h_{n-1}}\right).$$

Then

$$\frac{s_{n+1}c_{n-1} + s_{n-1}a_{n-1}}{c_{n-1}} = h_n \alpha_n(n-2) + \frac{q^{1/2}+q^{-1/2}}{2} B \frac{s_n}{c_n}.$$

Applying (5.4) and $h_n/h_{n-2} = (a_{n-2}a_{n-1})/(c_{n-1}c_n)$ we get

$$\begin{aligned}\frac{c_n}{c_{n-1}} &= \frac{h_n c_n \alpha_n(n-2) + \frac{q^{1/2}+q^{-1/2}}{2} B s_n}{(s_{n+1}-s_{n-1})c_{n-1} + B s_{n-1}} \\ &= \frac{\frac{1-q}{2c_{n-1}}(q^{(n-2)/2} - q^{-n/2})(2^{-2}n - \sum_{k=1}^{n-1} a_{k-1}c_k) + \frac{q^{1/2}+q^{-1/2}}{2} B s_n}{(s_{n+1}-s_{n-1})c_{n-1} + B s_{n-1}}.\end{aligned}$$

Due to the first equality of the proof it remains to show that $c_k = \frac{\beta^{1/2}(1-q^k)}{2(1-\beta q^k)}$, $k = 1, \dots, n-1$, implies

$$\frac{1-q}{2c_{n-1}}(q^{(n-2)/2} - q^{-n/2})(2^{-2}n - \sum_{k=1}^{n-1} a_{k-1}c_k) + \frac{q^{1/2}+q^{-1/2}}{2} B s_n = \frac{\beta^{1/2}q^{-1/2} + \beta^{-1/2}q^{1/2}}{2} s_n.$$

A simple induction results in

$$\sum_{k=1}^{n-1} a_{k-1}c_k = \frac{(n-1) - nq + q^n + \beta(1 - nq^{n-1} + (n-1)q^n)}{4(1-\beta q^{n-1})(1-q)}.$$

Therefore

$$\begin{aligned}&\frac{1-q}{2}(q^{(n-2)/2} - q^{-n/2}) \frac{2^{-2}n - \sum_{k=0}^{n-1} a_{k-1}c_k}{c_{n-1}} = \\ &\qquad (q^{(n-2)/2} - q^{-n/2}) \frac{(1-q)(1-\beta)(1-q^n)}{4\beta^{1/2}(1-q)(1-q^{n-1})} = \frac{(q-1)(1-\beta)}{4\beta^{1/2}q^{1/2}} s_n,\end{aligned}$$

which finally yields the desired relationship. This proves that (i) implies (ii), and the proof is complete. ■

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Let $\{P_n(x)\}_{n=0}^{\infty}$ be a symmetric sequence of polynomials orthogonal with respect to a positive Borel measure μ. Moreover assume that $\{P_n(x)\}_{n=0}^{\infty}$ is generated by $P_0(x) = 1$ and (5.5), with $a_n + c_n = B > 1$, $c_0 = 0$ and $c_1 < B$. Then there exist two solutions of (5.6) with respect to β, namely

$$\beta_1 = 2B(B - \sqrt{B^2 - 1}) - 1 < 1, \quad \text{and} \quad \beta_2 = 2B(B + \sqrt{B^2 - 1}) - 1 > 1.$$

Furthermore, there exists one solution of (5.7) with respect to q, that is

$$q = \frac{\sqrt{\beta} - 2c_1}{\sqrt{\beta} - 2\beta c_1}.$$

Then the assumptions of Theorem 5.2 are fulfilled if

$$0 < c_1 < \frac{\sqrt{\beta_1}}{2}, \quad \beta = \beta_1, \quad q = \frac{\sqrt{\beta_1} - 2c_1}{\sqrt{\beta_1} - 2\beta_1 c_1}$$

or

$$\frac{\sqrt{\beta_2}}{2} < c_1 < B, \quad \beta = \beta_2, \quad q = \frac{\sqrt{\beta_2} - 2c_1}{\sqrt{\beta_2} - 2\beta_2 c_1}.$$

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