Can. J. Math., Vol. XXIII, No. 4, 1971, pp. 704–711

GOING DOWN IN POLYNOMIAL RINGS

STEPHEN McADAM

Introduction. In this paper, $R \subset T$ will be commutative domains having a common identity.

Definition. Suppose that R is a subdomain of T.

(i) If P is a prime ideal of R and Q is a prime ideal of T, we say that Q lies over P if $Q \cap R = P$.

(ii) If every prime of R has a prime of T lying over it, we say that $R \subset T$ has lying over.

(iii) If there is a unique prime of T lying over P in R, we say that P is unibranched in T.

(iv) If every prime of R is unibranched in T we say that $R \subset T$ is unibranched.

(v) We say that $R \subset T$ has going down if whenever $P \subset P'$ are primes of R and Q' is prime in T with $Q' \cap R = P'$, there is a prime Q of T with $Q \subset Q'$ and $Q \cap R = P$.

Our goal is to prove the following three theorems. Here x and x_i are indeterminates.

THEOREM A. *Let R be a domain and T be a domain between R and its quotient field. Suppose that $R[x] \subset T[x]$ has going down. Then for any prime P of R, either P is unibranched in T or $PT = T$. If $R \subset T$ also has lying over, it is unibranched.*

In Theorems B and C we take $R[x_1, \ldots, x_n] = R$ *if* $n = 0$.

THEOREM B. *Let $R \subset T$ be domains. The following are equivalent.*

(i) *$R[x_1] \subset T[x_1]$ is unibranched.*

(ii) *For some $n \geqq 1$, $R[x_1, \ldots, x_n] \subset T[x_1, \ldots, x_n]$ is unibranched.*

(iii) *For all $n \geqq 0$, $R[x_1, \ldots, x_n] \subset T[x_1, \ldots, x_n]$ is unibranched.*

THEOREM C. *Let R be a domain with T a domain between R and its integral closure. The following are equivalent.*

(i) *$R[x_1] \subset T[x_1]$ is unibranched.*

(ii) *For some $n \geqq 1$, $R[x_1, \ldots, x_n] \subset T[x_1, \ldots, x_n]$ is unibranched.*

(iii) *For all $n \geqq 0$, $R[x_1, \ldots, x_n] \subset T[x_1, \ldots, x_n]$ is unibranched.*

(iv) *$R[x_1, x_2] \subset T[x_1, x_2]$ has going down.*

(v) *For some $n \geqq 2$, $R[x_1, \ldots, x_n] \subset T[x_1, \ldots, x_n]$ has going down.*

(vi) *For all $n \geqq 0$, $R[x_1, \ldots, x_n] \subset T[x_1, \ldots, x_n]$ has going down.*

Received December 16, 1970. This work, in conjunction with [**2**], forms the author's doctoral dissertation, written at the University of Chicago under the guidance of Professor I. Kaplansky.

https://doi.org/10.4153/CJM-1971-079-5 Published online by Cambridge University Press

Remarks. Theorem A was first proved for an integral extension by I. Kaplansky. The proof to be given here is a first cousin to his method. In view of Theorem A, it is natural to ask whether going down in $R[x] \subset T[x]$ implies unibranchedness in $R[x] \subset T[x]$. It happens that the case of one indeterminate will elude us. However, in an integral extension if we have going down upon adjoining two or more indeterminates, the situation becomes highly tractable. That situation is the topic of Theorem C. Finally, the results of this paper are primarily of interest in the non-Noetherian case since [**2**] proves the following: let R be a Noetherian domain and let T be a domain between R and its integral closure. Suppose that $R \subset T$ has going down. Then any prime of R of rank greater than one is unibranched in T. This result supersedes Theorem A and parts of Theorem C for the case of R Noetherian, $R \subset T$ integral.

Theorem A.

LEMMA 1. *Let $R \subset T$ have going down and let P be a prime of R. Then either $PT = T$ or there is a prime of T which lies over P.*

Proof. If $PT \neq T$, there is a maximal ideal N of T with $PT \subset N$. Let $M = N \cap R$. Then $P \subset M = N \cap R$. The conclusion follows from going down.

LEMMA 2. *Suppose that $R[x] \subset T[x]$ has going down. Then so does $R \subset T$.*

Proof. Let $P \subset P'$ be primes of R, and let Q' be a prime of T with $Q' \cap R = P'$. Then $PR[x] \subset P'R[x] = Q'T[x] \cap R[x]$. The hypothesis implies that there is a prime $N \subset T[x]$ such that $N \subset Q'T[x]$ and $N \cap R[x] = PR[x]$. Let $Q = N \cap T$. Then $Q \cap R = P$.

THEOREM A. *Let R be a domain, and T be a domain between R and its quotient field. Suppose that $R[x] \subset T[x]$ has going down. Then for any prime P of R, either P is unibranched in T or $PT = T$. If $R \subset T$ also has lying over, it is unibranched.*

Proof. By Lemma 2, $R \subset T$ has going down. By Lemma 1 it is enough to show that there can not be two distinct primes of T both lying over P. Suppose that this were not true. Say Q_1 and Q_2 are prime in T and $Q_1 \cap R = P = Q_2 \cap R$. Also assume that $Q_1 \not\subset Q_2$. Let $u \in Q_1 - Q_2$. Observe that $(Q_i, x)T[x] \cap R[x] = (P, x)R[x]$ for $i = 1, 2$. Also $(x - u)T[x] \subset (Q_1, x)T[x]$ since $u \in Q_1$. Thus, $(x - u)T[x] \cap R[x] \subset (Q_1, x)T[x] \cap R[x] = (P, x)R[x]$. By going down, since $(Q_2, x)T[x]$ lies over $(P, x)R[x]$, there is a prime of $T[x]$ which is contained in $(Q_2, x)T[x]$ and which lies over $(x - u)T[x] \cap R[x]$. Let N be such a prime. We have $N \subset (Q_2, x)T[x]$ and $N \cap R[x] = (x - u)T[x] \cap R[x]$. Suppose that $u = a/b$ with a and b in R. Then $bx - a$ is in $(x - u)T[x] \cap R[x]$ so that $b(x - u) = bx - a \in N$. Note that $b \notin N$, since if $b \in N$ then $b \in N \cap R[x] \subset (x - u)T[x]$. This is not so. Thus $x - u \in N \subset (Q_2, x)T[x]$ which implies that $u \in Q_2$. This is a contradiction.

The last statement is immediate since lying over excludes the possibility that $PT = T$.

This completes the proof of Theorem A.

Theorem C. Before going into the proof of Theorem B, which will be the most involved part of this paper, we show how A and B together imply C.

THEOREM C. *Let T be a domain between R and the integral closure of R. The following are equivalent.*

(i) $R[x_1] \subset T[x_1]$ *is unibranched.*

(ii) *For some* $n \geqq 1$, $R[x_1, \ldots, x_n] \subset T[x_1, \ldots, x_n]$ *is unibranched.*

(iii) *For all* $n \geqq 0$, $R[x_1, \ldots, x_n] \subset T[x_1, \ldots, x_n]$ *is unibranched.*

(iv) $R[x_1, x_2] \subset T[x_1, x_2]$ *has going down.*

(v) *For some* $n \geqq 2$, $R[x_1, \ldots, x_n] \subset T[x_1, \ldots, x_n]$ *has going down.*

(vi) *For all* $n \geqq 0$, $R[x_1, \ldots, x_n] \subset T[x_1, \ldots, x_n]$ *has going down.*

Here $R[x_1, \ldots, x_n] = R$ *if* $n = 0$.

Proof. The equivalence of (i), (ii) and (iii) is given by Theorem B. (vi) implies (v) trivially, and (v) implies (iv) by Lemma 2 and induction. Since $R \subset T$ is an integral extension, so is $R[x_1] \subset T[x_1]$, which therefore has lying over. Theorem A shows (iv) implies (i). Finally, since

$$R[x_1, \ldots, x_n] \subset T[x_1, \ldots, x_n]$$

is integral, it has going up. In the presence of going up, unibranchedness is easily seen to imply going down. Thus (iii) implies (vi). This completes the proof of Theorem C, modulo Theorem B.

Remark. In the proof of Theorem C, the only use made of the fact that $R \subset T$ was an integral extension, was in getting $R[x_1, \ldots, x_n] \subset T[x_1, \ldots, x_n]$ to have lying over and going up. Since going up easily implies lying over (see [**1**, Theorem 42]) it would suffice for Theorem C to assume that $R \subset T$ is such that $R[x_1, \ldots, x_n] \subset T[x_1, \ldots, x_n]$ has going up for all n. Interestingly, it is not difficult to show if $R \subset T$ has lying over then so does $R[x] \subset T[x]$. We will do this. The author does not know if the same is true of going up.

Prime behavior in $R[x] \subset T[x]$**.** The proof of Theorem B depends heavily on a knowledge of how primes in $T[x]$ contract to $R[x]$. In this section we develop the needed information.

Notation. Let P be prime in R. We will use P^* to denote $PR[x]$. Suppose also that F_P is the quotient field of R/P, and let $\alpha(x)$ be a monic irreducible element of $F_P[x]$. We will write $\langle P, \alpha(x) \rangle$ to denote $\{g(x) \in R[x] | \alpha(x)$ divides

$\bar{g}(x)\}$, where $\bar{g}(x)$ is the result of reducing $g(x)$ modulo P. Whenever $\langle P, \alpha(x)\rangle$ occurs, it will be understood that $\alpha(x)$ is monic irreducible in $F_P[x]$.

THEOREM 1. *Let P be prime in R and let M be a prime of $R[x]$ with $M \cap R = P$. Then either $M = P^*$ or M is of the form $\langle P, \alpha(x)\rangle$. $\langle P, \alpha(x)\rangle \subset \langle P, \alpha'(x)\rangle$ implies that $\langle P, \alpha(x)\rangle = \langle P, \alpha'(x)\rangle$ which in turn implies that $\alpha(x) = \alpha'(x)$. Finally, P^* is properly contained in each $\langle P, \alpha(x)\rangle$, each of which is prime.*

Proof. The proof is essentially contained in [**1**, 1–5].

Notation. A prime of the form $\langle P, \alpha(x)\rangle$ in $R[x]$ will be referred to as an *upper* to P. That is, an upper to P is any prime of $R[x]$ meeting R at P, except the prime $P^* = PR[x]$. Now assume that Q is prime in T and that $Q \cap R = P$. We will adopt analogous notation for primes of $T[x]$ which contract to Q in T, except that here we will use K_Q to denote the quotient field of T/Q. Thus the primes of $T[x]$ contracting to Q are $Q^* = QT[x]$, and $\langle Q, \beta(x)\rangle$ where $\beta(x)$ is a monic irreducible polynomial in $K_Q[x]$. Furthermore, when $Q \cap R = P$, we will consider R/P a subdomain of T/Q, and F_P a subfield of K_Q. The reduction modulo P of elements in $R[x]$ will also be considered a restriction of the reduction of elements of $T[x]$ modulo Q.

THEOREM 2. *Suppose that $R \subset T$ and that P is a prime of R. Let $\langle P, \alpha(x)\rangle$ be an upper to P in $R[x]$.*

(i) *The primes of $T[x]$ which lie over $\langle P, \alpha(x)\rangle$ are just $\{\langle Q, \beta(x)\rangle | Q$ is a prime in T lying over P and $\beta(x)$ divides $\alpha(x)$ in $K_Q[x]\}$.*

(ii) *The primes of $T[x]$ lying over P^* are just $\{Q^* | Q$ is a prime of T lying over $P\} \cup \{\langle Q, \beta(x)\rangle | Q$ is a prime of T lying over P and $\beta(x)$ does not divide any polynomial of $F_P[x]\}$.*

Proof. (i) Suppose that Q is prime in T and that $\langle Q, \beta(x)\rangle$ is an upper to Q such that $\beta(x)|\alpha(x)$ in $K_Q[x]$. We will first show that $\langle Q, \beta(x)\rangle \cap R[x] = \langle P, \alpha(x)\rangle$. Note that $\langle Q, \beta(x)\rangle \cap T = Q$ and $Q \cap R = P$ imply that $(\langle Q, \beta(x)\rangle \cap R[x]) \cap R = P$. Thus, $\langle Q, \beta(x)\rangle \cap R[x]$ is either P^* or an upper to P. If we show that $\langle P, \alpha(x)\rangle \subset \langle Q, \beta(x)\rangle \cap R[x]$ then, since $\langle P, \alpha(x)\rangle$ is an upper to P, Theorem 1 will show that $\langle P, \alpha(x)\rangle = \langle Q, \beta(x)\rangle \cap R[x]$. Suppose that $g(x) \in \langle P, \alpha(x)\rangle$. Then by definition, $g(x) \in R[x]$ and $g(x)$ modulo P is divisible by $\alpha(x)$ in $F_P[x]$. However, $F_P[x] \subset K_Q[x]$ and $\beta(x)|\alpha(x)$ in $K_Q[x]$, so $g(x)$ modulo Q is divisible by $\beta(x)$ in $K_Q[x]$. Hence $g(x) \in \langle Q, \beta(x)\rangle$, and we are finished. Conversely, suppose that N is a prime of $T[x]$ lying over $\langle P, \alpha(x)\rangle$. Let $N \cap T = Q$. Since $\langle P, \alpha(x)\rangle \cap R = P$, $Q \cap R = P$. Obviously, $Q^* \cap R[x] = P^* \neq \langle P, \alpha(x)\rangle$, so N must be an upper to Q. Suppose that $N = \langle Q, \beta(x)\rangle$. We will show that $\beta(x)|\alpha(x)$ in $K_Q[x]$. Since $\alpha(x) \in F_P[x]$ and F_P is the quotient field of R/P, we can write

$$\alpha(x) = x^n + \frac{\bar{r}_{n-1}}{\bar{s}_{n-1}} x^{n-1} + \ldots + \frac{\bar{r}_0}{\bar{s}_0},$$

where r_i and s_i are in R, the bar means modulo P, and $s_i \notin P$. Let $s = s_{n-1}s_{n-2} \ldots s_0$ and

$$f(x) = sx^n + \frac{sr_{n-1}}{s_{n-1}}x^{n-1} + \ldots + \frac{sr_0}{s_0}.$$

Then $f(x) \in R[x]$ and $f(x)$ modulo P is $\bar{s}\alpha(x)$. Hence $f(x) \in \langle P, \alpha(x)\rangle$. Therefore $f(x) \in \langle Q, \beta(x)\rangle$ so $\beta(x)$ divides $f(x)$ modulo Q. But since $s \notin P = Q \cap R$, $\bar{s} \neq 0$ and $\beta(x)|\bar{s}\alpha(x)$ implies that $\beta(x)|\alpha(x)$. This completes the proof of (i).

(ii) Suppose that Q is prime in T and that $Q \cap R = P$. Then clearly $Q^* \cap R[x] = P^*$. Suppose now that $\langle Q, \beta(x)\rangle$ is an upper to Q and that $\beta(x)$ does not divide any polynomial in $F_P[x]$. Since $\langle Q, \beta(x)\rangle \cap T = Q$ and $Q \cap R = P$, we have that $(\langle Q, \beta(x)\rangle \cap R[x]) \cap R = P$, so $\langle Q, \beta(x)\rangle \cap R[x]$ is either P^* or an upper to P. However by (i), it can not be an upper to P for if it were $\langle P, \alpha(x)\rangle$, we would have $\beta(x)|\alpha(x)$. Thus $\langle Q, \beta(x)\rangle \cap R[x] = P^*$. Finally, suppose that N is a prime in $T[x]$ and that $N \cap R[x] = P^*$. If $N \cap T = Q$ then we must have $Q \cap R = P$. If $N = Q^*$, we are finished. Otherwise, N is of the form $\langle Q, \beta(x)\rangle$. We will show that $\beta(x)$ does not divide any polynomial of $F_P[x]$. If this were false, we could assume that $\beta(x)$ divided the monic irreducible polynomial $\alpha(x) \in F_P[x]$. However, by (i) we would then have that $N \cap R[x] = \langle Q, \beta(x)\rangle \cap R[x] = \langle P, \alpha(x)\rangle$, contradicting $N \cap R[x] = P^*$. This completes the proof.

PROPOSITION 1. *Let R be a subdomain of T. Then $R \subset T$ has lying over if and only if $R[x] \subset T[x]$ does.*

Proof. Suppose that $R \subset T$ has lying over. Let P be a prime of R and let $\langle P, \alpha(x)\rangle$ be an upper to P in $R[x]$. Also, let Q be a prime in T lying over P. Then Q^* in $T[x]$ lies over P^* in $R[x]$. Furthermore, $F_P[x] \subset K_Q[x]$ so we may let $\beta(x)$ be a monic irreducible factor of $\alpha(x)$ in $K_Q[x]$. According to Theorem 2, $\langle Q, \beta(x)\rangle \cap R[x] = \langle P, \alpha(x)\rangle$. Thus, lying over holds with respect to P^* and $\langle P, \alpha(x)\rangle$. Since any prime of $R[x]$ is of one of these two types, $R[x] \subset T[x]$ has lying over. Conversely, if $R[x] \subset T[x]$ has lying over and P is prime in R then let N be a prime of $T[x]$ such that $N \cap R[x] = P^*$. Clearly $N \cap T$ is a prime of T lying over P.

Theorem B.

Definition. Call the domain extension $R \subset T$ a U-extension if: (i) $R \subset T$ is unibranched, and (ii) for every prime Q of T with $Q \cap R = P$, K_Q is an algebraic purely inseparable extension of F_P.

THEOREM 3. *Let $R \subset T$ be domains. $R[x] \subset T[x]$ is unibranched if and only if $R \subset T$ is a U-extension.*

Proof. Suppose that $R[x] \subset T[x]$ is unibranched, and that P is prime in R. If Q and Q' are distinct primes of T both lying over P, then Q^* and Q'^* are distinct in $T[x]$ and both lie over P^*, which is a contradiction. Thus there is at most one prime in T lying over P. Also $R[x] \subset T[x]$, being unibranched, implies that it has lying over. By Proposition 1, $R \subset T$ has lying over. Thus $R \subset T$ is unibranched. Now let Q be prime in T with $Q \cap R = P$. Suppose that $F_P \subset K_Q$ is not algebraic. Then there is an irreducible $\beta(x) \in K_Q[x]$ with the property that $\beta(x)$ does not divide any polynomial of $F_P[x]$. By Theorem 2, $\langle Q, \beta(x) \rangle$ lies over P^*. However Q^* also lies over P^*, contradicting unibranchedness. Thus $F_P \subset K_Q$ is an algebraic extension. Suppose now that $F_P \subset K_Q$ is not purely inseparable. Then there is a monic irreducible $\alpha(x) \in F_P[x]$ which has distinct monic irreducible factors $\beta_1(x)$ and $\beta_2(x)$ in $K_Q[x]$. By Theorem 2, both $\langle Q, \beta_1(x) \rangle$ and $\langle Q, \beta_2(x) \rangle$ lie over $\langle P, \alpha(x) \rangle$, again contradicting unibranchedness. Thus $R[x] \subset T[x]$ unibranched implies that $R \subset T$ is a U-extension.

Conversely, let $R \subset T$ be a U-extension. If P is a prime of R and Q is the unique prime of T lying over P, then Q^* lies over P^*. Theorem 2 says that any other prime of $T[x]$ which lies over P^* must be of the form $\langle Q, \beta(x) \rangle$, where $\beta(x)$ does not divide any polynomial of $F_P[x]$. However, K_Q being an algebraic extension of F_P means that there is no such $\beta(x)$ in $K_Q[x]$. Therefore, there is a unique prime of $T[x]$ lying over P^*. Finally, let $\langle P, \alpha(x) \rangle$ be an upper to P. Since $F_P \subset K_Q$ is purely inseparable, $\alpha(x)$ has a unique monic irreducible factor $\beta(x)$ in $K_Q[x]$. Theorem 2 says that $\langle Q, \beta(x) \rangle$ is the unique prime of $T[x]$ lying over $\langle P, \alpha(x) \rangle$.

The proof of Theorem B will be accomplished by showing that being a U-extension is stable under adjoining indeterminates. First, however, we show that unibranchedness in $R \subset T$ does not imply unibranchedness in $R[x] \subset T[x]$. Thus, in Theorem B we must assume that $R[x_1] \subset T[x_1]$ is unibranched to be able to conclude that unibranchedness remains upon adjoining more indeterminates.

Example. Let T be the domain of all power series in indeterminate t with complex coefficients. Let R be the subdomain of those power series having real constant term. Both R and T are local, one dimensional, so that $R \subset T$ is trivially unibranched. In both R and T, the unique maximal ideal is generated by t. However $R/tR \approx \mathbf{R}$ (the reals), while $T/tT \approx \mathbf{C}$ (the complex numbers). Thus T/tT is not a purely inseparable extension of R/tR, and Theorem 3 tells us that $R[x] \subset T[x]$ is not unibranched.

We now turn our attention towards showing that $R \subset T$ is a U-extension if and only if $R[x] \subset T[x]$ is a U-extension.

LEMMA 3. *Let $F \subset K$ be fields. If K is an algebraic purely inseparable extension of F, then $K(x)$ is an algebraic purely inseparable extension of $F(x)$.*

Proof. If the characteristic of F is 0, then $F = K$ and $F(x) = K(x)$. If the characteristic of F is $p > 0$, then for any $k \in K$ there is a positive integer n

such that $k^{p^n} \in F$. Since raising to p^nth powers is an endomorphism, we see that for any $g(x) \in K[x]$ we have $g(x)^{p^n} \in F[x]$ for some n. This extends to $K(x)$ and $F(x)$, yielding our result.

LEMMA 4. *Let $R \subset T$, and let Q be a prime of T with $Q \cap R = P$. Suppose that N is an upper to Q in $T[x]$ and that $N \cap R[x] = M$. Further, suppose that K_Q is an algebraic purely inseparable extension of F_p. Then the quotient field of $T[x]/N$ is an algebraic purely inseparable extension of the quotient field of $R[x]/M$.*

Proof. By an appropriate localization and reduction modulo P, P^*, Q, and Q^*, the problem reduces to the following: let $F \subset K$ be fields with K an algebraic purely inseparable extension of F. Suppose that N is a non-zero prime in $K[x]$ and that $N \cap F[x] = M$. Then $K[x]/N$ is an algebraic purely inseparable extension of $F[x]/M$. If the characteristic of F is 0, then $F = K$ and this fact is trivial. If the characteristic of F is $p > 0$, then for any $k \in K$ there is an n such that $k^{p^n} \in F$. Thus, for $g(x) \in K[x]$ there is an n such that $g(x)^{p^n} \in F[x]$ and $(g(x) + N)^{p^n} \in F[x]/M$.

THEOREM 4. *Let $R \subset T$ be domains. $R \subset T$ is a U-extension if and only if $R[x] \subset T[x]$ is a U-extension.*

Proof. Suppose that $R[x] \subset T[x]$ is a U-extension. Then by definition, $R[x] \subset T[x]$ is unibranched and by Theorem 3, $R \subset T$ is a U-extension. Conversely, suppose that $R \subset T$ is a U-extension. By Theorem 3, $R[x] \subset T[x]$ is unibranched. Therefore to show that $R[x] \subset T[x]$ is a U-extension, we need only pick a prime N in $T[x]$ with $N \cap R[x] = M$ and show that the quotient field of $T[x]/N$ is an algebraic purely inseparable extension of the quotient field of $R[x]/M$. Let $N \cap T = Q$ and $M \cap R = P$. If $N = Q^*$, then $M = P^*$ and $R[x]/M = R[x]/P^* \approx (R/P)[x] \subset (T/Q)[x] \approx T[x]/Q^* = T[x]/N$. Since $R \subset T$ is a U-extension, K_Q is an algebraic purely inseparable extension of F_P. By Lemma 3, $K_Q(x)$ is an algebraic purely inseparable extension of $F_P(x)$. Now $K_Q(x)$ is the quotient field of $(T/Q)[x] \approx T[x]/N$ and $F_P(x)$ is the quotient field of $(R/P)[x] \approx R[x]/M$. Thus in case $N = Q^*$, we are finished. If $N \neq Q^*$ then N is an upper to Q in $T[x]$. Lemma 4 says that the quotient field of $T[x]/N$ is an algebraic purely inseparable extension of $R[x]/M$. This completes the proof.

THEOREM B. *Let $R \subset T$ be domains. The following are equivalent.*
(i) *$R[x_1] \subset T[x_1]$ is unibranched.*
(ii) *For some $n \geqq 1$, $R[x_1, \ldots, x_n] \subset T[x_1, \ldots, x_n]$ is unibranched.*
(iii) *For all $n \geqq 0$, $R[x_1, \ldots, x_n] \subset T[x_1, \ldots, x_n]$ is unibranched.*
Here $R[x_1, \ldots, x_n] = R$ if $n = 0$.

Proof. (iii) $\Rightarrow$ (ii) is immediate. Suppose that (ii) is true. If

$$R[x_1, \ldots, x_n] \subset T[x_1, \ldots, x_n]$$

is unibranched then by Theorem 3, $R[x_1, \ldots, x_{n-1}] \subset T[x_1, \ldots, x_{n-1}]$ is a U-extension and hence unibranched. In this way induction yields that $R[x_1] \subset T[x_1]$ if unibranched. Thus (ii) $\Rightarrow$ (i). Finally, if $R[x_1] \subset T[x_1]$ is unibranched, then $R \subset T$ is a U-extension and is unibranched. Also, Theorem 4 and induction shows that $R[x_1, \ldots, x_{n-1}] \subset T[x_1, \ldots, x_{n-1}]$ is a U-extension for all n, which according to Theorem 3 gives that

$$R[x_1, \ldots, x_n] \subset T[x_1, \ldots, x_n]$$

is unibranched. This shows that (i) $\Rightarrow$ (iii).

Example. Let T be the domain of power series in indeterminate t with complex coefficients. Let R be the subdomain of those power series whose coefficient of t is 0. Both R and T are local, one dimensional. $R \subset T$ is trivially unibranched. The non-zero prime of T is tT while that of R is $(t^2, t^3)R$. Both T/tT and $R/(t^2, t^3)R$ are isomorphic to the complex numbers. That is, $R/(t^2, t^3)R = T/tT$. Thus $R \subset T$ is a U-extension. By Theorems 3 and B, $R[x_1, \ldots, x_n] \subset T[x_1, \ldots, x_n]$ is unibranched for all $n \geqq 0$.

Bibliography

1. I. Kaplansky, *Commutative rings* (Allyn and Bacon, Boston, 1970).

2. S. McAdam, *Going down* (to appear).

The University of Texas,
Austin, Texas

https://doi.org/10.4153/CJM-1971-079-5 Published online by Cambridge University Press