# ON CENTRAL TRACES AND GROUPS OF SYMMETRIES
# OF ORDER UNIT BANACH SPACES

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## 1. Introduction

A central trace on an order-unit Banach space $A(K)$ is a centre-valued module homomorphism invariant under the group of symmetries of $A(K)$.

The concept of central traces has been crucial in the theory of types for convex sets established in (4), (5). In von Neumann algebras, they are precisely the canonical centre-valued traces and their existence hinges on a fundamental theorem (Dixmier's approximation process) in von Neumann algebras. On the other hand, the existence of central traces in finite dimensional spaces is an easy consequence of Ryll-Nard-zewski's fixed point theorem (5).

The main objective of this note is to demonstrate, by means of the Bochner integral, that if a group of symmetries is uniformly almost periodic, then there is a positive operator on $A(K)$ which resembles a central trace. In particular, if the group of all symmetries is uniformly almost periodic as well as transitive and also, if $K$ satisfies Størmer's axiom, then there is a central trace on $A(K)$. The results proved here for compact convex sets, extending some of those in (8) on topological dynamics and $C^*$-algebras, appear to be more unified and transparent.

## 2. Central traces and groups of symmetries

Let $K$ be a compact convex subset of a locally convex space and let $A(K)$ be the Banach space of continuous affine functions on $K$ with the constant 1-function $e$ as order-unit. We shall refer to (1) for undefined terminology.

$K$ is said to satisfy *Størmer's axiom* if the closed convex hull $\overline{\mathrm{co}} \cup_a F_a$ of a family $\{F_a\}$ of closed split faces of $K$ is also a split face. The set of extreme points of $K$ will be denoted by $\partial K$. For each extreme point $k$, let $F_k$ be the smallest closed split face of $K$ containing $k$. As usual, $K$ will be identified, whenever it is convenient, with the vaguely compact convex state space $\{\varphi \in A(K)^*: \varphi(e) = \|\varphi\| = 1\}$ of $A(K)$.

Let $O(A)$ denote the set of all order-bounded linear operators on $A(K)$, that is all linear operators $T: A(K) \to A(K)$ such that $-\|T\|I \leqslant T \leqslant \|T\|I$ where $I$ is the identity operator on $A(K)$. The set $Z = \{Te: T \in O(A)\}$ is the *centre* of $A(K)$. For each $a$ in $A(K)$ and $T$ in $O(A)$, we have

$$(Ta)(k) = (Te)(k) \cdot a(k) \tag{2.1}$$

where $k$ is any extreme point of $K$.

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A *symmetry* of $A(K)$ is an isometric order-isomorphism $\sigma: A(K) \to A(K)$ such that $\sigma(e) = e$ and $\sigma T = T\sigma$ for each $T$ in $O(A)$. A *central trace* on $A(K)$ is a positive linear operator $\Gamma: A(K) \to Z$ such that $\Gamma(e) = e$, $T\Gamma = \Gamma T$ for each $T$ in $O(A)$ and, $\Gamma\sigma = \Gamma$ for each symmetry $\sigma$ of $A(K)$ (See (4) and (5)).

The set $S$ of all symmetries of $A(K)$ form a group with respect to function composition. The identity of $S$ is the identity operator $I$ on $A(K)$. For each $\sigma$ in $S$, we let $\sigma^*: A(K)^* \to A(K)^*$ be the dual of $\sigma$; then $\sigma^*$ restricts to an affine homeomorphism from $K$ onto itself. We shall also denote by $\sigma^*$ this restriction to $K$ if no confusion will arise.

Let $G$ be a group of symmetries of $A(K)$ and let $A^A$ denote the product space $A(K)^{A(K)}$ with the product topology so that a net $\{\sigma_\alpha\}$ in $G \subseteq A^A$ converges to an element $\tau$ in $A^A$ if and only if $\|\sigma_\alpha(a) - \tau(a)\| \to 0$ for each $a$ in $A(K)$. The uniform boundedness principle shows that $\tau: A(K) \to A(K)$ is a positive bounded operator with $\|\tau\| \leqslant 1$ and also $\tau^* K \subseteq K$. Moreover, $G$ is a topological group in the product topology. In fact, it is easily seen that the group multiplication is continuous. Further, if $\sigma_\alpha \to \sigma$ in $G$, then by isometry, for each $a$ in $A(K)$, $\|\sigma_\alpha^{-1} a - \sigma^{-1} a\| = \|\sigma_\alpha^{-1}(a - \sigma_\alpha \sigma^{-1} a)\| = \|a - \sigma_\alpha \sigma^{-1} a\| = \|\sigma(\sigma^{-1} a) - \sigma_\alpha(\sigma^{-1} a)\| \to 0$. So the inverse operation is also continuous on $G$.

Next the action $K \times G \to K: (k, \sigma) \to \sigma^*(k)$ of $G$ on $K$ makes $(K, G)$ into a transformation group. We say that $G$ *acts transitively on* $K$ if for each extreme point $k$, $\partial F_k$ is contained in the closure of $\{\sigma^*(k): \sigma \in G\}$, or equivalently, if $F_k = \overline{\text{co}}\{\sigma^*(k): \sigma \in G\}$.

Plainly, if every extreme point of $K$ is a split face, then any group $G$ of symmetries is transitive since for each $k$ in $\partial K$, $F_k = \{k\}$ and for each $\sigma$ in $G$, $k = \sigma^* x$ for some $x$ in $\partial K$.

**Example 1.**  Let $K$ be the state space of a $C^*$-algebra $A$ with identity. Then $A(K)$ can be identified with the self-adjoint part of $A$. It is evident that the inner automorphisms of $A$ are symmetries. Indeed using a theorem of Kadison (9, Theorem 10), one can show that the symmetries of $A(K)$ are just (the real parts of) the automorphisms of $A$ leaving the centre of $A$ fixed (see (5)). Moreover, it also follows from Kadison's transitivity theorem (10) that $S$ acts transitively on $K$.

Let $G$ be a group of symmetries of $A(K)$. An element $a$ in $A(K)$ is called *G-invariant* if $\sigma(a) = a$ for all $\sigma$ in $G$. We shall denote by $A_G$ the set of $G$-invariant elements in $A(K)$. A subset $F$ of $K$ is called *G-invariant* if $\sigma^* F \subseteq F$ for all $\sigma$ in $G$. The set of all $G$-invariant elements in $K$ will be denoted by $K_G$ which is clearly a compact convex set.

Let $G_0 = \{\sigma \in S: \sigma^{**}$ is a symmetry of $A(K)^{**}\}$. The following proposition generalises a well-known result in $C^*$-algebras concerning the invariant and split faces.

**Proposition 1.**  *Let $G$ be a subgroup of $G_0$. Let $F$ be a split face of $K$. Then $F$ is G-invariant. Conversely, if $G$ is transitive and $K$ satisfies Størmer's axiom, then every closed G-invariant face is a split face.*

**Proof.**  Let $F$ be a split face of $K$. Then there is a central projection $E: A(K)^* \to$

$A(K)^*$ such that $F = EA(K)^* \cap K$. Let $\sigma \in G$. Then $\sigma^{**} E^* = E^* \sigma^{**}$ since $\sigma^{**}$ is a symmetry of $A(K)^{**}$. It follows that $\sigma^* E = E \sigma^*$ and hence $\sigma^* F \subseteq F$. So $F$ is $G$-invariant.

Conversely, suppose $F$ is a closed $G$-invariant face. Let $k \in \partial F$. Then, by transitivity of $G$, $F_k = \overline{\text{co}}\{\sigma^*(k) : \sigma \in G\}$ which is contained in $F$ as $F$ is $G$-invariant. Therefore $F = \overline{\text{co}} \cup_{k \in \partial F} F_k$ is a split face by Størmer's axiom.

**Example 2.** Evidently, $G_0 = S$ if $K$ is finite dimensional. Let us take $K$ to be a square or a disc in the plane. Then it is not difficult to see that the centre of $K$ is the only element of $K_S$.

**Example 3.** Let $K$ be the state space of a function algebra. Since each extreme point of $K$ is a split face, it follows from the above proposition and the Krein-Milman Theorem that $K = K_{G_0}$. Thus $K_{G_0}$ need not be a Choquet simplex in general.

**Example 4.** Let $K$ be the state space of a $C^*$-algebra with identity and let $U$ be the group of all inner automorphisms of $A$. Then $(A, U)$ form a $U$-abelian system (see (14, 3.1.11) for definition) and as $U \subseteq G_0$, $(A, G_0)$ form a $G_0$-abelian system as well. It follows that $K_{G_0}$ is a Choquet simplex (see (11)). Note that if $A$ is a type $I$ von Neumann algebra, then $U = S$ (see (14; 2.9.32)).

It would be interesting to characterise those compact convex sets $K$ for which $K_{G_0}$ are Choquet simplexes.

Let us observe that the centre of a $C^*$-algebra is the set of elements which are invariant under the inner automorphisms. This is a particular instance of the following more general result in convexity.

**Proposition 2.** *Let $G$ be a transitive group of symmetries of $A(K)$ and let $Z$ be the centre of $A(K)$. Suppose $K$ satisfies Størmer's axiom. Then $Z = A_G$.*

**Proof.** Since, for each $z$ in $Z$, there is an order-bounded operator $T$ in $O(A)$ such that $z = Te$, we have $\sigma z = \sigma Te = T\sigma e = Te = z$ for all $\sigma$ in $G$. So $Z$ is contained in $A_G$.

Let us recall that $Z$ is identifiable with the space of facially continuous functions on $\partial K$ (see (1)). Now suppose $a \in A(K)$ is $G$-invariant. We show that $a$ is facially continuous on $\partial K$. Let $r$, $t$ be any real numbers and let $W = \{w \in \partial K : r \leqslant a(w) \leqslant t\}$. We prove that $W$ is facially closed in $\partial K$. Let $F = \overline{\text{co}} \cup_{w \in W} F_w$. Then $F$ is a split face by Størmer's axiom. We assert that $W = \partial F$. Let us first prove that $a$ is constant on $\partial F_w$ for each $w$ in $W$. In fact, if $k$ is an element in $\partial F_w$, then the transitivity of $G$ gives $k \in \{\sigma^* w : \sigma \in G\}^-$. But $a$ is $G$-invariant, so $a\sigma^*(w) = \sigma a(w) = a(w)$ for all $\sigma$ in $G$. Therefore $a(k) = a(w)$ by continuity. Hence $a$ is a constant on $\partial F_w$ and it follows that $\partial F_w \subseteq W$. Now the affineness and continuity of $a$ give $\partial F \subseteq W$. Thus $W = \partial F$ as asserted. This proves that $a$ is facially continuous on $\partial K$, that is, $a \in Z$. As $a \in A_G$ was arbitrary, we conclude that $Z = A_G$.

## 3. Uniform almost periodicity

In this section, we shall apply some ideas from (8) to study the groups of

symmetries of $A(K)$ and to prove the existence of an analogue of central traces by using the Bochner integral.

Let $G$ be a group of symmetries of $A(K)$. The transformation group $(K, G)$ is said to be *uniformly almost periodic* if for each $f$ in $C(K)$, $f$ is almost periodic (with respect to $G$), that is $\{f\sigma^*: \sigma \in G\}$ has compact closure in $C(K)$ (see (7)).

We first extend a result of Green in (8).

**Proposition 3.** *The transformation group $(K, G)$ is uniformly almost periodic if and only if $G$ has compact closure in $A(K)^{A(K)}$ in the product topology. Under this condition, the closure $G^-$ is a topological group of symmetries of $A(K)$.*

**Proof.** First suppose $(K, G)$ is uniformly almost periodic. For each $a$ in $A(K)$, let $[a] = \{\sigma a: \sigma \in G\}$ be the orbit of $a$. Then, by uniform almost periodicity, $[a]^- = \{a\sigma^*: \sigma \in G\}^-$ is compact in $C(K)$ and hence compact in $A(K)$. So $\Pi_{a \in A(K)} [a]^-$ is compact in $A^A$ in the product topology. Let $\tau \in G^- \subseteq A^A$. Then there is a net $\{\sigma_\alpha\}$ in $G$ such that $\|\sigma_\alpha a - \tau a\| \to 0$ for each $a$ in $A(K)$. Therefore $\tau a \in [a]^-$ for each $a$ in $A(K)$. It follows that $G^- \subseteq \Pi_{a \in A(K)} [a]^-$ is compact.

Conversely, let $G^-$ be compact in $A^A$. Let us first prove that $\{\sigma^*: \sigma \in G^-\}$ is compact in $K^K$ in the uniform topology, that is, for any net $\{\sigma_\alpha^*\}$, there is a subnet $\{\sigma_\beta^*\}$ converging to $\tau^*$, where $\tau \in G^-$, in that

$$\sup_{k \in K} |\sigma_\beta^*(k)(a) - \tau^*(k)(a)| \to 0$$

for each $a$ in $A(K)$. As $G^-$ is compact in $A^A$, there is a subnet $\{\sigma_\beta\}$ of $\{\sigma_\alpha\}$ converging to an element $\tau \in G^-$ in the product topology. It follows that $\sup |\sigma_\beta^*(k)(a) - \tau^*(k)(a)| = \sup |\sigma_\beta(a)(k) - \tau(a)(k)| \leqslant \|\sigma_\beta(a) - \tau(a)\| \to 0$. This proves that $\{\sigma^*: \sigma \in G^-\}$ is compact in $K^K$ in the uniform topology and so it is equicontinuous (2; Corollary 3 p. 292). Hence $\{\sigma^*: \sigma \in G\}$ is also equicontinuous in $K^K$. It follows that $\{f\sigma^*: \sigma \in G\}$ has compact closure in $C(K)$ for each $f$ in $C(K)$ (7; 4.4, 4.5). Therefore $(K, G)$ is uniformly almost periodic.

Finally we show that $G^-$ is contained in the group $S$ of all symmetries of $A(K)$ under this condition. Let $\tau \in G^-$ and let $\{\sigma_\alpha\}$ be a net in $G$ converging to $\tau$. We have noted before that $\tau$ is a positive bounded operator on $A(K)$. Let us show that $\tau$ has an inverse. Indeed, by compactness of $G^-$, there is a subnet $\{\sigma_\beta^{-1}\}$ of the net $\{\sigma_\alpha^{-1}\}$ converging to some $\sigma$ in $G^-$. Therefore $\tau\sigma = \tau \lim_\beta \sigma_\beta^{-1} = \lim_\beta \sigma_\beta \sigma_\beta^{-1} = I = \sigma\tau$. So $\tau$ has an inverse in $G$. Now it is routine to verify that $\tau$ is a symmetry of $A(K)$. The proof is complete.

**Corollary 4.** *Let $(K, G)$ be uniformly almost periodic. Then $K$ contains a $G$-invariant element. In particular, if $(K, S)$ is uniformly almost periodic and $A(K)$ has trivial centre, then there is a central trace on $A(K)$.*

**Proof.** Since $\{\sigma^*: \sigma \in G\} \subseteq K^K$ is an equicontinuous group of affine mappings, Kakutani's fixed point theorem asserts that there is an element $k$ in $K$ such that $\sigma^*(k) = k$ for all $\sigma$ in $G$ (see (6, p. 457)). Further, if the centre $Z$ of $A(K)$ is trivial, the operator $\Gamma: A(K) \to Z$ defined by $\Gamma(a) = a(k)e$ is obviously a central trace.

We now construct a Bochner integral on a uniformly almost periodic group of symmetries.

**Theorem 5.** *Let $G$ be a group of symmetries of $A(K)$ and let $(K, G)$ be uniformly almost periodic. Then there is a positive linear operator $\Gamma: A(K) \to A_G$ satisfying the following conditions:*

(i) $\Gamma(e) = e$;

(ii) $\Gamma T = T\Gamma$ *for each $T$ in $O(A)$;*

(iii) $\Gamma\sigma = \Gamma$ *for each $\sigma$ in $G$.*

**Proof.** By Proposition 3, $G^-$ is a compact topological group. Let $\mu$ be the normalised Haar measure on $G^-$. For each $a$ in $A(K)$, the continuous function $\sigma \mapsto \sigma(a): G^- \to A(K)$ is separably-valued since $G^-$ is compact. Also, it is clearly weakly measurable. Furthermore, the real function $\sigma \mapsto \|\sigma(a)\|$ is bounded and hence $\mu$-integrable on $G^-$. It follows that $\sigma \mapsto \sigma(a)$ is Bochner $\mu$-integrable (**16**; Theorem 1, p. 133). Let

$$\Gamma(a) = \int_{G^-} \sigma(a)d\mu(\sigma) \ (a \in A(K))$$

be the Bochner integral. Then, by (**16**; Corollary 2, p. 134), for each $\tau$ in $G$, we have $\tau(\Gamma a) = \tau \int_{G^-} \sigma(a)d\mu(\sigma) = \int_{G^-} \tau\sigma(a)d\mu(\sigma) = \int_{G^-} \sigma(a)d\mu(\sigma) = \Gamma(a)$ by the invariance of $\mu$. This shows that $\Gamma(a)$ is $G$-invariant, that is, $\Gamma(a) \in A_G$.

It is evident that $\Gamma: A(K) \to A_G$ is positive and linear. To complete the proof, it remains to verify conditions (i), (ii) and (iii). First, condition (i) follows easily from the fact that $\mu$ is normalised. Next, let $T \in O(A)$, then again by (**16**; Corollary 2, p. 134), we have $T\Gamma(a) = T \int_{G^-} \sigma(a)d\mu(\sigma) = \int_{G^-} T\sigma(a)d\mu(\sigma) = \int_{G^-} \sigma(Ta)d\mu(\sigma) = \Gamma(Ta)$. So $T\Gamma = \Gamma T$. Finally, as before, the invariance of $\mu$ implies that $\Gamma\sigma = \Gamma$ for each $\sigma$ in $G$. This concludes the proof.

**Corollary 6.** *Let $S$ be the group of all symmetries of $A(K)$. Suppose that $(K, S)$ is transitive, uniformly almost periodic and also, $K$ satisfies Stormer's axiom. Then there is a central trace on $A(K)$.*

**Proof.** By Proposition 2, $A_S$ is the centre of $A(K)$ and so the operator $\Gamma$ in Theorem 5 is a central trace on $A(K)$.

We shall give some examples at the end of the paper.

## 4. Tensor products

We consider in this section the tensor products of uniformly almost periodic groups. We shall use (**13**) and (**15**) as our references to tensor products of compact convex sets. Let $K_1$ and $K_2$ be two compact convex sets and let $K_1 \otimes K_2$ be their injective tensor product. Let us recall that each extreme point of $K_1 \otimes K_2$ is the product $k_1 \otimes k_2$ of two uniquely determined extreme points $k_1$ of $K_1$ and $k_2$ of $K_2$ (cf. (**13**)). The injective tensor product of $A(K_1)$ and $A(K_2)$ will be denoted by $A(K_1) \otimes A(K_2)$ which is identified as the space $A(K_1 \otimes K_2)$ of continuous affine

functions on $K_1 \otimes K_2$ with $e_1 \otimes e_2$ as the constant 1-function where $e_1$ is the order-unit in $A(K_1)$ and $e_2$ the order-unit in $A(K_2)$.

Next if $\sigma_1$; $A(K_1) \rightarrow A(K_1)$ and $\sigma_2: A(K_2) \rightarrow A(K_2)$ are bounded operators, we define $\sigma_1 \otimes \sigma_2: A(K_1) \otimes A(K_2) \rightarrow A(K_1) \otimes A(K_2)$ to be the unique bounded operator on $A(K_1) \otimes A(K_2)$ such that $(\sigma_1 \otimes \sigma_2)(a_1 \otimes a_2) = \sigma_1(a_1) \otimes \sigma_2(a_2)$ for $a_1 \in A(K_1)$ and $a_2 \in A(K_2)$ (cf. 15).

**Proposition 7.** *Let* $\sigma_1: A(K_1) \rightarrow A(K_1)$ *and* $\sigma_2: A(K_2) \rightarrow A(K_2)$ *be symmetries. Then* $\sigma_1 \otimes \sigma_2: A(K_1) \otimes A(K_2) \rightarrow A(K_1) \otimes A(K_2)$ *is a symmetry.*

**Proof.** By the functorial property of the injective tensor products (15; p. 74), $\sigma_1 \otimes \sigma_2$ is an order-isometric isomorphism and also, $(\sigma_1 \otimes \sigma_2)(e_1 \otimes e_2) = \sigma_1 e_1 \otimes \sigma_2 e_2 = e_1 \otimes e_2$. It remains to prove that $\sigma_1 \otimes \sigma_2$ commutes with the order-bounded operators. Let $T: A(K_1) \otimes A(K_2) \rightarrow A(K_1) \otimes A(K_2)$ be an order-bounded operator. Then by (3, Proposition 9), we may assume that there exist order-bounded operators $P_n: A(K_1) \rightarrow A(K_1)$ and $Q_n: A(K_2) \rightarrow A(K_2)$ such that

$$\|P_n e_1 \otimes Q_n e_2 - T(e_1 \otimes e_2)\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Let $f$ be an element of $A(K_1) \otimes A(K_2)$. Then we have

$$\|(P_n \otimes Q_n)(f) - Tf\| = \sup_{k_1 \otimes k_2 \in \partial(K_1 \otimes K_2)} |(P_n \otimes Q_n)(f)(k_1 \otimes k_2) - (Tf)(k_1 \otimes k_2)|$$

$$= \sup |(P_n \otimes Q_n)(e_1 \otimes e_2)(k_1 \otimes k_2)f(k_1 \otimes k_2) - T(e_1 \otimes e_2)(k_1 \otimes k_2)f(k_1 \otimes k_2)|$$

by (2.1)

$$= \sup |(P_n e_1 \otimes Q_n e_2)(k_1 \otimes k_2) - T(e_1 \otimes e_2)(k_1 \otimes k_2)\|f(k_1 \otimes k_2)|$$

$$\leqslant \|f\| \|P_n e_1 \otimes Q_n e_2 - T(e_1 \otimes e_2)\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

It follows that

$$(\sigma_1 \otimes \sigma_2)(Tf) = \lim_n (\sigma_1 \otimes \sigma_2)(P_n \otimes Q_n)(f)$$

$$= \lim_n (\sigma_1 P_n \otimes \sigma_2 Q_n)(f)$$

$$= \lim_n (P_n \sigma_1 \otimes Q_n \sigma_2)(f)$$

$$= \lim_n (P_n \otimes Q_n)(\sigma_1 \otimes \sigma_2)(f)$$

$$= T(\sigma_1 \otimes \sigma_2)(f).$$

This proves that $(\sigma_1 \otimes \sigma_2)T = T(\sigma_1 \otimes \sigma_2)$. Therefore $\sigma_1 \otimes \sigma_2$ is a symmetry.

Let $(K_1, G_1)$ and $(K_2, G_2)$ be uniformly almost periodic groups. Then $G_1^-$ and $G_2^-$ are compact topological groups by Proposition 3. Let

$$G = G_1 \otimes G_2 = \{\sigma_1 \otimes \sigma_2: \sigma_1 \in G_1, \sigma_2 \in G_2\}.$$

Then it follows from Proposition 7 that $G$ is a group of symmetries of $A(K_1) \otimes A(K_2)$. Let $A = A(K_1) \otimes A(K_2)$. Define a map $G_1^- \times G_2^- \rightarrow A^A$ by $(\sigma_1, \sigma_2) \mapsto \sigma_1 \otimes \sigma_2$. Then a

simple application of the uniform boundedness principle (**6**, p. 55) shows that the map is continuous and its range is just $(G_1 \otimes G_2)^-$ by compactness of $G_1^-$ and $G_2^-$. It follows that $(G_1 \otimes G_2)^-$ is compact in the product topology. Thus we have proved the following result.

**Theorem 8.** *Let $(K_1, G_1)$ and $(K_2, G_2)$ be uniformly almost periodic groups. Then $(K_1 \otimes K_2, G_1 \otimes G_2)$ is a uniformly almost periodic group.*

We conclude the paper with a few illustrative examples.

**Example 5.** Let $K_1$ be a Bauer simplex. Then the group $S_1$ of symmetries of $A(K_1)$ consists of only one element, namely, the identity operator on $A(K_1)$. So $(K_1, S_1)$ is uniformly almost periodic.

**Example 6.** Let $K_2$ be a square in the plane. Then, since there is no non-trivial order-bounded operators on $A(K_2)$, the symmetries $S_2$ of $A(K_2)$ are precisely the order-isometric automorphisms. Therefore $\{\sigma^*: \sigma \in S_2\}$ consists of all the affine autohomeomorphisms of $K_2$ and is just the dihedral group of order 8 (see (**12**, p. 55)). It follows that $(K_2, S_2)$ is uniformly almost periodic.

**Example 7.** Let $A$ be a UHF algebra and $K_3$ its state space. Let $\{A_n\}$ be its defining sequence with $A_n = M_1 \otimes \cdots \otimes M_n$ where each $M_i$ is a finite dimensional factor. Let $U$ be the group of all unitaries in $\cup_{n=1}^{\infty} A_n$ which have the form $u_1 \otimes \cdots \otimes u_k$ where $u_i$ is a unitary in $M_i$. Let $S_3$ be the group of (the real parts of) all inner automorphisms of $A$ induced by the elements of $U$. Then it was shown in (**8**) that $(K_3, S_3)$ is uniformly almost periodic.

Now we can use Theorem 8 to construct a few more uniformly almost periodic groups.

**Example 8.** In the above examples, the groups $(K_i \otimes K_j, S_i \otimes S_j)$ $i, j = 1, 2, 3$, are uniformly almost periodic.

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