



RESEARCH ARTICLE

Inductive local-global conditions and generalised Harish-Chandra theory

Damiano Rossi 

Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA; E-mail: damiano.rossi@rutgers.edu.

Received: 30 September 2024; **Revised:** 15 August 2025; **Accepted:** 15 August 2025

2020 Mathematics Subject Classification: *Primary* – 20C15, 20C20, 20C33

Abstract

We study new properties of generalised Harish-Chandra theory aiming at explaining the inductive local-global conditions for finite groups of Lie type in nondefining characteristic. In particular, we consider a parametrisation of generalised Harish-Chandra series that is compatible with Clifford theory and with the action of automorphisms on irreducible characters and we reduce it to the verification of certain requirements on stabilisers and extendibility of characters. This parametrisation is used by the author in a separate paper to obtain new conjectures for finite reductive groups that can be seen as geometric realisations of the local-global counting conjectures and their inductive conditions. As a by-product, we extend the parametrisation of generalised Harish-Chandra series given by Broué–Malle–Michel to the nonunipotent case by assuming maximal extendibility.

Contents

1	Introduction	2
2	Preliminaries	5
2.1	Regular embeddings	5
2.2	Automorphisms	6
2.3	Restrictions on primes	7
2.4	<i>e</i> -Harish-Chandra theory	8
2.5	A nonblockwise version of Parametrisation C	9
3	Parametrisation of nonunipotent <i>e</i>-Harish-Chandra series	9
3.1	<i>e</i> -Harish-Chandra theory for groups with connected centre	9
3.2	Consequences of equivariant maximal extendibility	14
3.3	<i>e</i> -Harish-Chandra series and regular embeddings	17
4	The criteria	20
4.1	The criterion for Parametrisation 2.7	21
4.2	The criterion for Parametrisation C	25
5	Stabilisers, extendibility and consequences	31
5.1	Proof of Theorem D	31
5.2	Results for groups of type A and C	33
	References	35

1. Introduction

Generalised Harish-Chandra theory is a powerful tool that lies at the heart of many results in modular representation theory of finite reductive groups in nondefining characteristic. First introduced by Fong–Srinivasan [FS86] for classical groups and then fully developed by Broué–Malle–Michel [BMM93] in the unipotent case, this theory extends the classical Harish-Chandra theory formulated by Harish-Chandra [HC70] and further studied by Howlett–Lehrer [HL80] by replacing Harish-Chandra induction with Deligne–Lusztig induction.

The aim of this paper is twofold. First, we extend generalised Harish-Chandra theory, and in particular the parametrisation of generalised Harish-Chandra series given in [BMM93, Theorem 3.2], to the nonunipotent case by assuming certain requirements on character extendibility. Secondly, we introduce a new compatibility of this parametrisation with Clifford theory, which we realise by requiring certain conditions on isomorphisms of character triples, and with the action of automorphisms on irreducible characters. The motivation for our second aim comes from two natural questions. On the one hand, our work suggests a more conceptual explanation for the validity of the so-called inductive conditions for the long-standing local-global conjectures in representation theory of finite groups. Namely, for finite reductive groups in nondefining characteristic these conditions are consequences of Parametrisation C below. On the other hand, Parametrisation C together with [Ros24a, Theorem A] provide a uniform description of the characters of finite reductive groups in terms of generalised Harish-Chandra theory. This description is used in [Ros24a, Theorem F] to prove two new conjectures for finite reductive groups that yield a geometric analogue of Dade’s conjecture by replacing ℓ -local structures in the finite reductive group with certain e -local structures inherent to the underlying algebraic group (see [Ros24a, Conjecture C and Conjecture D] as well as [Ros24c, Theorem A and Theorem B]). An explanation for the connection between ℓ -local and e -local structures has been obtained in [Ros23] and [Ros24b] using methods from algebraic topology. The Clifford theoretic properties introduced in Parametrisation C play a fundamental role in the proof of [Ros24a, Theorem F]. Here, we also recall that the interest towards a proof of the above-mentioned geometric version of Dade’s Conjecture for finite reductive groups is driven by a reduction theorem for Dade’s Conjecture to finite quasi-simple groups (see [Spä17] and [Ros]).

More precisely, let $\mathbf{G}$ be a connected reductive group defined over an algebraic closure $\mathbb{F}$ of a field of characteristic p and $F : \mathbf{G} \rightarrow \mathbf{G}$ a Frobenius endomorphism endowing $\mathbf{G}$ with an $\mathbb{F}_q$ -structure for some power q of p . Fix a prime ℓ different from p and denote by e the multiplicative order of q modulo ℓ (modulo 4 if $\ell = 2$). All modular representation theoretic notions are considered with respect to the prime ℓ . For an e -cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$, we denote by $\mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ the e -Harish-Chandra series associated to $(\mathbf{L}, \lambda)$ and by $W_{\mathbf{G}}(\mathbf{L}, \lambda)^F := \mathbf{N}_{\mathbf{G}}(\mathbf{L})_{\lambda}^F / \mathbf{L}^F$ the corresponding relative Weyl group. The parametrisation of e -Harish-Chandra series associated to unipotent e -cuspidal pairs $(\mathbf{L}, \lambda)$ given in [BMM93, Theorem 3.2] shows the existence of a bijection

$$I_{(\mathbf{L}, \lambda)}^{\mathbf{G}} : \text{Irr}\left(W_{\mathbf{G}}(\mathbf{L}, \lambda)^F\right) \rightarrow \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)). \quad (1.1)$$

Our first result extends this parametrisation to nonunipotent e -Harish-Chandra series in groups with connected centre. Moreover, we show that these bijections can be chosen to satisfy certain additional properties. In what follows, we denote by $\text{Aut}_{\mathbb{F}}(\mathbf{G}^F)$ the set of those automorphisms of $\mathbf{G}^F$ which are obtained by restriction from bijective endomorphisms of $\mathbf{G}$ commuting with F (see Section 2.2).

Theorem A. *If $\mathbf{G}$ has connected centre and $[\mathbf{G}, \mathbf{G}]$ is simple not of type $\mathbf{E}_6$, $\mathbf{E}_7$ or $\mathbf{E}_8$, then there exists a collection of bijections*

$$I_{(\mathbf{L}, \lambda)}^{\mathbf{K}} : \text{Irr}\left(W_{\mathbf{K}}(\mathbf{L}, \lambda)^F\right) \rightarrow \mathcal{E}(\mathbf{K}^F, (\mathbf{L}, \lambda))$$

where $\mathbf{K}$ runs over the set of F -stable Levi subgroups of $\mathbf{G}$ and $(\mathbf{L}, \lambda)$ over the set of e -cuspidal pairs of $\mathbf{K}$, such that:

- (i) $I_{(\mathbf{L}, \lambda)}^{\mathbf{K}}$ is $\text{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K}, (\mathbf{L}, \lambda)}$ -equivariant;
- (ii) $I_{(\mathbf{L}, \lambda)}^{\mathbf{K}}(\eta)(1)_{\ell} = |\mathbf{K}^F : \mathbf{N}_{\mathbf{K}}(\mathbf{L}, \lambda)^F|_{\ell} \cdot \lambda(1)_{\ell} \cdot \eta(1)_{\ell}$ for every $\eta \in \text{Irr}(W_{\mathbf{K}}(\mathbf{L}, \lambda))^F$; and
- (iii) if $z \in \mathbf{Z}(\mathbf{K}^{*F})$ corresponds to characters $\widehat{z}_{\mathbf{L}} \in \text{Irr}(\mathbf{L}^F / [\mathbf{L}, \mathbf{L}]^F)$ and $\widehat{z}_{\mathbf{K}} \in \text{Irr}(\mathbf{K}^F / [\mathbf{K}, \mathbf{K}]^F)$ (see Section 2.1), then $\lambda \cdot \widehat{z}_{\mathbf{L}}$ is e -cuspidal, $W_{\mathbf{K}}(\mathbf{L}, \lambda)^F = W_{\mathbf{K}}(\mathbf{L}, \lambda \cdot \widehat{z}_{\mathbf{L}})^F$ and

$$I_{(\mathbf{L}, \lambda)}^{\mathbf{K}}(\eta) \cdot \widehat{z}_{\mathbf{K}} = I_{(\mathbf{L}, \lambda \cdot \widehat{z}_{\mathbf{L}})}^{\mathbf{K}}(\eta)$$

for every $\eta \in \text{Irr}(W_{\mathbf{K}}(\mathbf{L}, \lambda))^F$.

In Theorem 3.8 we consider similar bijections $\mathcal{I}_{(\mathbf{L}, \lambda)}^{\mathbf{K}}$ and obtain *e-Harish-Chandra theory* (as defined in [KM13, Definition 2.9]) above any e -cuspidal pair in groups with connected centre. Notice that the restrictions on the type of $\mathbf{G}$ are mainly due to the fact that the Mackey formula is not known to hold in full generality. In addition for types $\mathbf{E}_6$, $\mathbf{E}_7$ and $\mathbf{E}_8$ it is not known whether there exists a Jordan decomposition map which commutes with Deligne–Lusztig induction. If these two properties were to be established for types $\mathbf{E}_6$, $\mathbf{E}_7$ or $\mathbf{E}_8$, then Theorem A and the other results obtained in this paper would hold without restriction on the type.

Before proceeding further, we make a remark inspired by [Mal07] and [Mal14]. If λ has an extension $\widehat{\lambda}$ to $\mathbf{N}_{\mathbf{G}}(\mathbf{L})_{\lambda}^F$, then Gallagher's theorem and the Clifford correspondence imply that there is a bijection

$$\begin{aligned} \text{Irr}(W_{\mathbf{G}}(\mathbf{L}, \lambda)^F) &\rightarrow \text{Irr}(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \mid \lambda) \\ \eta &\mapsto (\widehat{\lambda}\eta)^{\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F}. \end{aligned}$$

In this case, (1.1) is equivalent to the existence of the following bijection

$$\Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}} : \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) \rightarrow \text{Irr}(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \mid \lambda). \quad (1.2)$$

Observe that the extendibility of the character λ to its stabiliser $\mathbf{N}_{\mathbf{G}}(\mathbf{L})_{\lambda}^F$ is expected to hold in general and is known in a plethora of cases.

Working with the formulation given in (1.2) and using Theorem A, we obtain a parametrisation of *e-Harish-Chandra series* for groups with nonconnected centre by assuming maximal extendibility for certain characters of e -split Levi subgroups. Recall that if $i : \mathbf{G} \rightarrow \widetilde{\mathbf{G}}$ is a regular embedding and $\mathbf{L}$ is a Levi subgroup of $\mathbf{G}$, then $\widetilde{\mathbf{L}} := \mathbf{L}\mathbf{Z}(\mathbf{G})$ is a Levi subgroup of $\widetilde{\mathbf{G}}$. Moreover, for any connected reductive group $\mathbf{H}$ with Frobenius endomorphism F , we denote by $\text{Cusp}_e(\mathbf{H}, F)$ the set of (irreducible) e -cuspidal characters of $\mathbf{H}^F$. Below we will consider some restrictions on the choice of the prime ℓ . We refer the reader to Section 2.3 for the definition of the set of primes $\Gamma(\mathbf{G}, F)$.

Theorem B. *Let $\mathbf{G}$ be simple, simply connected and not of type $\mathbf{E}_6$, $\mathbf{E}_7$ or $\mathbf{E}_8$ and consider $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$. Let $\mathbf{K}$ be an F -stable Levi subgroup of $\mathbf{G}$, $(\mathbf{L}, \lambda)$ an e -cuspidal pair of $\mathbf{K}$ and suppose there exists an $(\text{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K}, \mathbf{L}} \ltimes \text{Irr}(\widetilde{\mathbf{G}}^F / \mathbf{G}^F))$ -equivariant extension map for $\text{Cusp}_e(\widetilde{\mathbf{L}}, F)$ with respect to $\widetilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F$ (see the discussion following Definition 3.9). Then, there exists an $\text{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K}, (\mathbf{L}, \lambda)}$ -equivariant bijection*

$$\Omega_{(\mathbf{L}, \lambda)}^{\mathbf{K}} : \mathcal{E}(\mathbf{K}^F, (\mathbf{L}, \lambda)) \rightarrow \text{Irr}(\mathbf{N}_{\mathbf{K}}(\mathbf{L})^F \mid \lambda)$$

that preserves the ℓ -defect of characters.

The extendibility condition considered in Theorem B should be compared with condition B(d) of [CS19, Definition 2.2]. In particular, this has been shown to hold for groups of type **A** and **C** (see [BS20] and [Bro22]) and is expected to hold in all cases. Due to these results, in Section 5 we obtain consequences for groups of type **A** and **C** (see Corollary 5.6).

Most importantly, by working with the formulation given in (1.2) we are able to compare the Clifford theory of corresponding characters via equivalence relations on character triples as defined in [NS14] and [Spä17]. This idea has been introduced in [Ros24a, Conjectured Parametrisation B] and provides an adaptation of generalised Harish-Chandra theory to the framework of the inductive conditions for the so-called Local–Global conjectures in representation theory of finite groups.

Parametrisation C. Let $\mathbf{G}$, F , ℓ , q and e be as above. For every e -cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$ there exists a defect preserving $\text{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{(\mathbf{L}, \lambda)}$ -equivariant bijection

$$\Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}} : \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) \rightarrow \text{Irr}(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \mid \lambda)$$

such that

$$(X_{\vartheta}, \mathbf{G}^F, \vartheta) \sim_{\mathbf{G}^F} (\mathbf{N}_{X_{\vartheta}}(\mathbf{L}), \mathbf{N}_{\mathbf{G}^F}(\mathbf{L}), \Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}}(\vartheta))$$

in the sense of [Spä17, Definition 3.6] for every $\vartheta \in \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ and where $X := \mathbf{G}^F \rtimes \text{Aut}_{\mathbb{F}}(\mathbf{G}^F)$.

The bijections described in Parametrisation C play a central role in the verification of the inductive conditions for the local-global conjectures. In fact, similar bijections have been used in [MS16] and [Ruh22a] to prove the McKay Conjecture, the Alperin–McKay Conjecture and Brauer’s Height Zero Conjecture for the prime $\ell = 2$. Furthermore, in [Ros24a, Theorem F] the author shows that Parametrisation C implies certain conjectures (see [Ros24a, Conjecture C and Conjecture D]) that can be seen as geometric analogues of the local-global conjectures and their inductive conditions.

In section 4, we prove a criterion for Parametrisation C (see Theorem 4.8) and show that its validation reduces to the verification of certain requirements related to the extendibility of characters of e -split Levi subgroups. These requirements (see Definition 5.1) are analogous to the one considered in [CS19, Definition 2.2] and have already been studied when verifying the inductive McKay condition (recently settled in [CS]) as well as the inductive condition for the Alperin–McKay and the Alperin Weight conjectures (see [Spä13], [Mal14], [SF14], [CS15], [KS16a], [KS16b], [BS20], [BS22], [Bro22]). Thanks to the results obtained in [Ros24a, Section 7], our approach also suggests a way to tackle Dade’s Conjecture and its inductive condition by utilising the theory that has already been developed to verify the inductive conditions for the other counting conjectures.

If $(\mathbf{L}, \lambda)$ is an e -cuspidal pair of $\mathbf{G}$, then we say that $(\mathbf{L}, \lambda)$ is *e -Brauer–Lusztig-cuspidal* in $\mathbf{G}$ if the associated e -Harish-Chandra series $\mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ coincides with a Brauer–Lusztig block $\mathcal{E}(\mathbf{G}^F, B, [s])$ as defined in [Ros24a, Definition 4.15].

Theorem D. Let $\mathbf{G}$ be simple, simply connected and not of type $\mathbf{E}_6$, $\mathbf{E}_7$ or $\mathbf{E}_8$ and consider $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$. Let $\mathbf{L}$ be an e -split Levi subgroup of $\mathbf{G}$ and suppose that the following conditions hold:

- (i) maximal extendibility holds with respect to $\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F$ in the sense of Definition 3.9;
- (ii) there exists an $(\text{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{L}} \ltimes \text{Irr}(\tilde{\mathbf{G}}^F/\mathbf{G}^F))$ -equivariant extension map for $\text{Cusp}_e(\tilde{\mathbf{L}}, F)$ with respect to $\tilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F$ (see the discussion following Definition 3.9);
- (iii) the requirements from Definition 5.1 hold for $\mathbf{L} \leq \mathbf{G}$;

Then Parametrisation C holds for every e -Brauer–Lusztig-cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$ with abelian $\text{Out}(\mathbf{G}^F)_{\mathcal{B}}$ and where $\mathcal{B}$ is the $\tilde{\mathbf{G}}^F$ -orbit of $\text{bl}(\lambda)^{\mathbf{G}^F}$.

Assumptions (i), (ii) and (iii) of Theorem D are part of an important ongoing project in representation theory of finite reductive groups and have been verified for groups of type $\mathbf{A}$ (under certain block theoretic restrictions) and $\mathbf{C}$ in [BS20] and [Bro22] respectively (see Remark 5.2 and Lemma 5.7). Assumption (i) holds in almost all cases since $\tilde{\mathbf{G}}^F/\mathbf{G}^F$ is cyclic for groups not of type $\mathbf{D}$ (see [GM20, Proposition 1.7.5]). Next, we observe that every e -cuspidal pair associated to an ℓ -regular semisimple element is e -Brauer–Lusztig-cuspidal by [Ros24a, Theorem A] and [CE99, Theorem 4.1]. Finally, the block theoretic condition formulated in the last part of Theorem D holds, for instance, whenever $\mathbf{G}$ is of

type **B**, **C** or **E**₇. In Theorem 5.5 we prove a slightly more general result by considering a larger class of blocks. We plan to circumvent the obstructions appearing in the remaining cases by applying techniques developed by Bonnafé–Dat–Rouquier [BDR17] and Ruhstorfer [Ruh22b] on quasi-isolated blocks.

In Section 5, as a corollary of Theorem D and by using [CS17a, Theorem 4.1] and [BS20, Theorem 1.2 and Corollary 4.7], we obtain Parametrisation C for some cases in type A (see Corollary 5.10). Similarly, by using [CS17b, Theorem 3.1] and [Bro22, Theorem 1.1 and Theorem 1.2], we obtain Parametrisation C for all e -Brauer–Lusztig–cuspidal pairs of groups of type C whenever $\ell \geq 5$ (see Corollary 5.11).

The paper is organised as follows. In Section 2 we introduce our notation and recall some preliminary results. We also introduce the notion of e -Brauer–Lusztig–cuspidality in Definition 2.5 and state a weaker version of Parametrisation C by replacing $\mathbf{G}^F$ -block isomorphisms of character triples with $\mathbf{G}^F$ -central isomorphisms of character triples (see Parametrisation 2.7). In Section 3, we prove Theorem A which provides an extension of [BMM93, Theorem 3.2] to nonunipotent e -cuspidal pairs in groups with connected centre and type different from **E**₆, **E**₇ or **E**₈. Then, assuming maximal extendibility, we develop a Clifford theory for e -Harish–Chandra series with respect to regular embeddings. This is done by applying the results obtained in [Ros24a, Section 4]. As a consequence, we construct certain bijections needed in the criteria proved in the subsequent section (see Theorem 3.19). We conclude Section 3 with a proof of Theorem B. In Section 4, we prove Theorem 4.8 which provides a criterion for Parametrisation C. On the way to prove this result we also consider the weaker Parametrisation 2.7 and prove a criterion for its validity in Theorem 4.3. Finally, in Section 5 we combine the results obtained in Section 3 and Section 4 in order to obtain Theorem D. In Definition 5.1 we give a definition of certain requirements on stabilisers and extendibility of characters that should be compared with [CS19, Definition 2.2]. Then, applying the main results of [BS20] and [Bro22], we obtain consequences for groups of type A and C and prove Corollary 5.6, Corollary 5.10 and Corollary 5.11. We also prove similar results for Parametrisation 2.7 (see Corollary 5.8 and Corollary 5.9).

2. Preliminaries

In this paper, $\mathbf{G}$ is a connected reductive group defined over an algebraic closure $\mathbb{F}$ of a finite field of characteristic p and $F : \mathbf{G} \rightarrow \mathbf{G}$ is a Frobenius endomorphism endowing $\mathbf{G}$ with an $\mathbb{F}_q$ -structure for a power q of p . Let $(\mathbf{G}^*, F^*)$ be a group in duality with $(\mathbf{G}, F)$ with respect to a choice of an F -stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ and an F^* -stable maximal torus $\mathbf{T}^*$ of $\mathbf{G}^*$. Then, there is a bijection $\mathbf{L} \mapsto \mathbf{L}^*$ between the set of Levi subgroups of $\mathbf{G}$ containing $\mathbf{T}$ and the set of Levi subgroups of $\mathbf{G}^*$ containing $\mathbf{T}^*$ (see [CE04, p.123]). This bijection induces a correspondence between the set of F -stable Levi subgroups of $\mathbf{G}$ and the set of F^* -stable Levi subgroups of $\mathbf{G}^*$.

2.1. Regular embeddings

Let $\mathbf{G}, \tilde{\mathbf{G}}$ be connected reductive groups with Frobenius endomorphisms $F : \mathbf{G} \rightarrow \mathbf{G}$ and $\tilde{F} : \tilde{\mathbf{G}} \rightarrow \tilde{\mathbf{G}}$. A morphism of algebraic groups $i : \mathbf{G} \rightarrow \tilde{\mathbf{G}}$ is a regular embedding if $\tilde{F} \circ i = i \circ F$ and i induces an isomorphism of $\mathbf{G}$ with a closed subgroup $i(\mathbf{G})$ of $\tilde{\mathbf{G}}$, the centre $\mathbf{Z}(\tilde{\mathbf{G}})$ of $\tilde{\mathbf{G}}$ is connected and $[i(\mathbf{G}), i(\mathbf{G})] = [\tilde{\mathbf{G}}, \tilde{\mathbf{G}}]$. In this case we can identify $\mathbf{G}$ with its image $i(\mathbf{G})$ and $\tilde{F}$ with an extension of F to $\tilde{\mathbf{G}}$ which, by abuse of notation, we denote again by F .

Since $[\tilde{\mathbf{G}}, \mathbf{G}]$ is contained in $\mathbf{G}$, we deduce that $\mathbf{G}$ is normal in $\tilde{\mathbf{G}}$ and that $\tilde{\mathbf{G}}/\mathbf{G}$ is abelian. Moreover, as $\tilde{\mathbf{G}}$ is connected and reductive, we have $\tilde{\mathbf{G}} = \mathbf{Z}(\tilde{\mathbf{G}})[\tilde{\mathbf{G}}, \tilde{\mathbf{G}}] = \mathbf{Z}(\tilde{\mathbf{G}})\mathbf{G}$. In particular, it follows that $\mathbf{Z}(\mathbf{G}) = \mathbf{Z}(\tilde{\mathbf{G}}) \cap \mathbf{G}$. Similarly, $[\tilde{\mathbf{G}}^F, \tilde{\mathbf{G}}^F] \leq \mathbf{G}^F$ and hence $\mathbf{G}^F$ is a normal subgroup of $\tilde{\mathbf{G}}^F$ with abelian quotient $\tilde{\mathbf{G}}^F/\mathbf{G}^F$. Notice, however, that $\tilde{\mathbf{G}}^F$ might be larger than $\mathbf{Z}(\tilde{\mathbf{G}}^F)\mathbf{G}^F$.

Let $\mathbf{L}$ be an F -stable Levi subgroup of $\mathbf{G}$. Then, the group $\tilde{\mathbf{L}} := \mathbf{Z}(\tilde{\mathbf{G}})\mathbf{L}$ is an F -stable Levi subgroup of $\tilde{\mathbf{G}}$. In fact, if $\mathbf{L} = \mathbf{C}_{\mathbf{G}}(\mathbf{S})$ with $\mathbf{S} := \mathbf{Z}^\circ(\mathbf{L})$, then $\tilde{\mathbf{L}} := \mathbf{LZ}(\tilde{\mathbf{G}}) = \mathbf{C}_{\mathbf{G}}(\mathbf{S})\mathbf{Z}(\tilde{\mathbf{G}}) \leq \mathbf{C}_{\tilde{\mathbf{G}}}(\mathbf{S}) = \mathbf{C}_{\mathbf{GZ}(\tilde{\mathbf{G}})}(\mathbf{S}) \leq \mathbf{C}_{\mathbf{G}}(\mathbf{S})\mathbf{Z}(\tilde{\mathbf{G}}) = \mathbf{LZ}(\tilde{\mathbf{G}}) = \tilde{\mathbf{L}}$. Then, it is clear that $\mathbf{L} = \tilde{\mathbf{L}} \cap \mathbf{G}$ and therefore $\mathbf{N}_{\mathbf{G}}(\mathbf{L}) = \mathbf{N}_{\mathbf{G}}(\mathbf{S})$ and $\mathbf{N}_{\tilde{\mathbf{G}}}(\tilde{\mathbf{L}}) = \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L}) = \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{S})$. In addition, as $\mathbf{Z}(\tilde{\mathbf{G}})$ is contained in $\tilde{\mathbf{L}}$, observe that $\tilde{\mathbf{G}} = \tilde{\mathbf{L}}\mathbf{G}$ which implies

$\widetilde{\mathbf{G}}/\mathbf{G} \simeq \widetilde{\mathbf{L}}/\mathbf{L}$. Similarly, we have $\widetilde{\mathbf{G}}^F = \widetilde{\mathbf{L}}^F \mathbf{G}^F$ and $\widetilde{\mathbf{G}}^F/\mathbf{G}^F \simeq \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F/\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \simeq \widetilde{\mathbf{L}}^F/\mathbf{L}^F$. Observe that, since $\widetilde{\mathbf{L}}$ has connected centre by [DM91, Lemma 13.14] and $[\widetilde{\mathbf{L}}, \widetilde{\mathbf{L}}] = [\mathbf{LZ}(\widetilde{\mathbf{G}}), \mathbf{LZ}(\widetilde{\mathbf{G}})] = [\mathbf{L}, \mathbf{L}]$, the map $i_{\mathbf{L}} : \mathbf{L} \rightarrow \widetilde{\mathbf{L}}$ is a regular embedding.

Next, consider pairs $(\mathbf{G}^*, F^*)$ and $(\widetilde{\mathbf{G}}^*, F^*)$ dual to $(\mathbf{G}, F)$ and $(\widetilde{\mathbf{G}}, F)$ respectively. The map $i : \mathbf{G} \rightarrow \widetilde{\mathbf{G}}$ induces a surjective morphism $i^* : \mathbf{G}^* \rightarrow \widetilde{\mathbf{G}}^*$ such that $\text{Ker}(i^*)$ is a connected subgroup of $\mathbf{Z}(\widetilde{\mathbf{G}}^*)$ (see [CE04, Section 15.1]). When $\mathbf{G}$ is simply connected, we have $\text{Ker}(i^*) = \mathbf{Z}(\widetilde{\mathbf{G}}^*)$: observe that $\mathbf{Z}(\mathbf{G}^*)$ is trivial since $\mathbf{G}^*$ is adjoint and therefore, using the isomorphism $\widetilde{\mathbf{G}}^*/\text{Ker}(i^*) \simeq \mathbf{G}^*$, we deduce that $\mathbf{Z}(\widetilde{\mathbf{G}}^*) \leq \text{Ker}(i^*)$. As shown in [CE04, (15.2)], there exists an isomorphism

$$\begin{aligned} \text{Ker}(i^*)^F &\rightarrow \text{Irr}\left(\widetilde{\mathbf{G}}^F/\mathbf{G}^F\right) \\ z &\mapsto \widehat{z}_{\widetilde{\mathbf{G}}} \end{aligned} \tag{2.1}$$

If $\mathbf{L}$ is an F -stable Levi subgroup of $\mathbf{G}$, noticing that $\text{Ker}(i^*) \leq \mathbf{Z}(\widetilde{\mathbf{G}}^*) \leq \widetilde{\mathbf{L}}^*$, it follows that $\text{Ker}(i^*) = \text{Ker}(i_{\mathbf{L}}^*)$. As before we obtain a map $\text{Ker}(i_{\mathbf{L}}^*)^F \rightarrow \text{Irr}(\widetilde{\mathbf{L}}^F/\mathbf{L}^F)$, $z \mapsto \widehat{z}_{\widetilde{\mathbf{L}}}$ which coincides with the restriction of the map defined above, that is, $\widehat{z}_{\widetilde{\mathbf{L}}} = (\widehat{z}_{\widetilde{\mathbf{G}}})_{\widetilde{\mathbf{L}}^F}$. If no confusion arises, we denote $\mathcal{K} := \text{Ker}(i^*)^F = \text{Ker}(i_{\mathbf{L}}^*)^F$ and obtain bijections

$$\begin{aligned} \mathcal{K} &\rightarrow \text{Irr}\left(\widetilde{\mathbf{L}}^F/\mathbf{L}^F\right) \\ z &\mapsto \widehat{z}_{\widetilde{\mathbf{L}}} \end{aligned}$$

for every F -stable Levi subgroup $\mathbf{L} \leq \mathbf{G}$. To conclude this section, we define an action of the group $\mathcal{K}$ on the set of irreducible characters.

Definition 2.1. Let $\mathcal{K}$ be the set defined above. For $z \in \mathcal{K}$ and $\chi \in \text{Irr}(\widetilde{\mathbf{G}}^F)$, let

$$\chi^z := \chi \cdot \widehat{z}_{\widetilde{\mathbf{G}}},$$

where $\widehat{z}_{\widetilde{\mathbf{G}}} \in \text{Irr}(\widetilde{\mathbf{G}}^F/\mathbf{G}^F)$ corresponds to z via the isomorphism (2.1). Similarly, for an F -stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$, the group $\mathcal{K}$ acts on $\text{Irr}(\widetilde{\mathbf{L}}^F)$. Moreover, since $\widetilde{\mathbf{G}}^F/\mathbf{G}^F \simeq \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F/\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F$, we deduce that $z \in \mathcal{K}$ also acts on the characters $\psi \in \text{Irr}(\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F)$ via

$$\psi^z := \psi \cdot \widehat{z}_{\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})},$$

where $\widehat{z}_{\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})}$ denotes the restriction of $\widehat{z}_{\widetilde{\mathbf{G}}}$ to $\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F$. In the same way, we can define an action of $\mathcal{K}$ on $\text{Irr}(\widetilde{\mathbf{K}}^F)$ and on $\text{Irr}(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F)$ for every F -stable Levi subgroups $\mathbf{L}$ and $\mathbf{K}$ of $\mathbf{G}$ satisfying $\mathbf{L} \leq \mathbf{K}$.

2.2. Automorphisms

For every bijective morphism of algebraic groups $\sigma : \mathbf{G} \rightarrow \mathbf{G}$ satisfying $\sigma \circ F = F \circ \sigma$, the restriction of σ to $\mathbf{G}^F$, which by abuse of notation we denote again by σ , is an automorphism of the finite group $\mathbf{G}^F$. Let $\text{Aut}_{\mathbb{F}}(\mathbf{G}^F)$ be the set of automorphisms of $\mathbf{G}^F$ obtained in this way. As mentioned in [CS13, Section 2.4], a morphism σ of $\mathbf{G}$ as above is determined by its restriction to $\mathbf{G}^F$ up to a power of F . In particular $\text{Aut}_{\mathbb{F}}(\mathbf{G}^F)$ acts on the set of F -stable closed connected subgroups $\mathbf{H}$ of $\mathbf{G}$ and we can define the set $\text{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{H}}$ whose elements are the restrictions to $\mathbf{G}^F$ of those bijective morphisms σ considered above that stabilise $\mathbf{H}$. Observe that $\text{Aut}_{\mathbb{F}}(\mathbf{G}^F) = \text{Aut}(\mathbf{G}^F)$ whenever $\mathbf{G}$ is a simple algebraic group of simply connected type such that $\mathbf{G}^F/\mathbf{Z}(\mathbf{G}^F)$ is a nonabelian simple group (see [GLS98, Section 1.15] and the remarks in [CS13, Section 2.4]).

Next, we consider the relation between the automorphisms of $\mathbf{G}^F$ and those of its dual $\mathbf{G}^{*F^*}$. Consider a pair $(\mathbf{G}^*, F^*)$ dual to $(\mathbf{G}, F)$. According to [CS13, Section 2.4], there exists an isomorphism

$$\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)/\mathrm{Inn}(\mathbf{G}_{\mathrm{ad}}^F) \simeq \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^{*F^*})/\mathrm{Inn}(\mathbf{G}_{\mathrm{ad}}^{*F^*}).$$

If the coset of σ corresponds to the coset of σ^* via the above isomorphism, then we write $\sigma \sim \sigma^*$ (see [CS13, Definition 2.1]).

Lemma 2.2. *Let $\mathbf{L} \leq \mathbf{K}$ be F -stable Levi subgroups of $\mathbf{G}$ in duality with the Levi subgroups $\mathbf{L}^* \leq \mathbf{K}^*$ of $\mathbf{G}^*$. Then, for every $\sigma \in \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{L}, \mathbf{K}}$ there exists $\sigma^* \in \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^{*F^*})_{\mathbf{L}^*, \mathbf{K}^*}$ such that $\sigma \sim \sigma^*$.*

Proof. Define the groups $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{L}, \mathbf{K}} := \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{L}} \cap \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K}}$ and $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^{*F^*})_{\mathbf{L}^*, \mathbf{K}^*} := \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^{*F^*})_{\mathbf{L}^*} \cap \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^{*F^*})_{\mathbf{K}^*}$. If $\mathbf{L} = \mathbf{K}$ the result follows from [CS13, Proposition 2.2] while a similar argument applies in the general case. $\square$

Assume now that $\mathbf{G}$ is simple of simply connected type. Fix a maximally split torus $\mathbf{T}_0$ contained in an F -stable Borel subgroup $\mathbf{B}_0$ of $\mathbf{G}$. This choice corresponds to a set of simple roots $\Delta \subseteq \Phi := \Phi(\mathbf{G}, \mathbf{T}_0)$. For every $\alpha \in \Phi$ consider a one-parameter subgroup $x_\alpha : \mathbb{G}_a \rightarrow \mathbf{G}$. Then $\mathbf{G}$ is generated by the elements $x_\alpha(t)$, where $t \in \mathbb{G}_a$ and $\alpha \in \pm\Delta$. Consider the field endomorphism $F_0 : \mathbf{G} \rightarrow \mathbf{G}$ given by $F_0(x_\alpha(t)) := x_\alpha(t^p)$ for every $t \in \mathbb{G}_a$ and $\alpha \in \Phi$. Moreover, for every symmetry γ of the Dynkin diagram of Δ , we have a graph automorphism $\gamma : \mathbf{G} \rightarrow \mathbf{G}$ given by $\gamma(x_\alpha(t)) := x_{\gamma(\alpha)}(t)$ for every $t \in \mathbb{G}_a$ and $\alpha \in \pm\Delta$. Then, up to inner automorphisms of $\mathbf{G}$, any Frobenius endomorphism F defining an $\mathbb{F}_q$ -structure on $\mathbf{G}$ can be written as $F = F_0^m \gamma$, for some symmetry γ and $m \in \mathbb{Z}$ with $q = p^m$ (see [MT11, Theorem 22.5]). One can construct a regular embedding $\mathbf{G} \leq \tilde{\mathbf{G}}$ in such a way that the Frobenius endomorphism F_0 extends to an algebraic group endomorphism $F_0 : \tilde{\mathbf{G}} \rightarrow \tilde{\mathbf{G}}$ defining an $\mathbb{F}_p$ -structure on $\tilde{\mathbf{G}}$. Moreover, every graph automorphism γ can be extended to an algebraic group automorphism of $\tilde{\mathbf{G}}$ commuting with F_0 (see [MS16, Section 2B]). If we denote by $\mathcal{A}$ the group generated by γ and F_0 , then we can construct the semidirect product $\tilde{\mathbf{G}}^F \rtimes \mathcal{A}$. Finally, we define the set of diagonal automorphisms of $\mathbf{G}^F$ to be the set of those automorphisms induced by the action of $\tilde{\mathbf{G}}^F$ on $\mathbf{G}^F$. If $\mathbf{G}^F/\mathbf{Z}(\mathbf{G}^F)$ is a nonabelian simple group, then the group $\tilde{\mathbf{G}}^F \rtimes \mathcal{A}$ acts on $\mathbf{G}^F$ and induces all the automorphisms of $\mathbf{G}^F$ (see, for instance, the proof of [Spä12, Proposition 3.4] and of [CS19, Theorem 2.4]).

2.3. Restrictions on primes

For the rest of this section we consider the following setting.

Notation 2.3. Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over an algebraic closure of a finite field of characteristic p and $F : \mathbf{G} \rightarrow \mathbf{G}$ a Frobenius endomorphism defining an $\mathbb{F}_q$ -structure on $\mathbf{G}$, for a power q of p . Consider a prime ℓ different from p and denote by e the multiplicative order of q modulo ℓ (modulo 4 if $\ell = 2$). All blocks are considered with respect to the prime ℓ .

Here we recall the definition of good primes and define the set $\Gamma(\mathbf{G}, F)$ (see also [CE94, Notation 1.1]). First, recall that ℓ is a *good prime* for $\mathbf{G}$ if it is good for each simple factor of $\mathbf{G}$, while the conditions for the simple factors are

$$\begin{aligned} \mathbf{A}_n &: \text{every prime is good} \\ \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n &: \ell \neq 2 \\ \mathbf{G}_2, \mathbf{F}_4, \mathbf{E}_6, \mathbf{E}_7 &: \ell \neq 2, 3 \\ \mathbf{E}_8 &: \ell \neq 2, 3, 5. \end{aligned}$$

We say that ℓ is a *bad prime* for $\mathbf{G}$ if it is not a good prime. Then, we denote by $\gamma(\mathbf{G}, F)$ the set of primes ℓ such that: ℓ is odd, $\ell \neq p$, ℓ is good for $\mathbf{G}$ and ℓ doesn't divide $|\mathbf{Z}(\mathbf{G})^F : \mathbf{Z}^\circ(\mathbf{G})^F|$. Let $(\mathbf{G}^*, F^*)$ be in duality with $(\mathbf{G}, F)$ and set $\Gamma(\mathbf{G}, F) := (\gamma(\mathbf{G}, F) \cap \gamma(\mathbf{G}^*, F^*)) \setminus \{3\}$ if $\mathbf{G}_{\mathrm{ad}}^F$ has a component of type ${}^3\mathbf{D}_4(q^m)$ and $\Gamma(\mathbf{G}, F) := \gamma(\mathbf{G}, F) \cap \gamma(\mathbf{G}^*, F^*)$ otherwise.

2.4. *e*-Harish-Chandra theory

Let $\mathbf{G}$, F , q , ℓ and e be as in Notation 2.3 and consider an F -stable Levi complement of a (not necessarily F -stable) parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. Deligne–Lusztig [DL76] and Lusztig [Lus76] defined two $\mathbb{Z}$ -linear maps

$$\mathbf{R}_{\mathbf{L} \leq \mathbf{P}}^{\mathbf{G}} : \mathbb{Z}\mathrm{Irr}(\mathbf{L}^F) \rightarrow \mathbb{Z}\mathrm{Irr}(\mathbf{G}^F) \quad \text{and} \quad {}^*\mathbf{R}_{\mathbf{L} \leq \mathbf{P}}^{\mathbf{G}} : \mathbb{Z}\mathrm{Irr}(\mathbf{G}^F) \rightarrow \mathbb{Z}\mathrm{Irr}(\mathbf{L}^F)$$

called Deligne–Lusztig induction and restriction respectively. It is conjectured that $\mathbf{R}_{\mathbf{L} \leq \mathbf{P}}^{\mathbf{G}}$ and ${}^*\mathbf{R}_{\mathbf{L} \leq \mathbf{P}}^{\mathbf{G}}$ do not depend on the choice of $\mathbf{P}$. This would, for instance, follow by the Mackey formula which has been proved whenever $\mathbf{G}^F$ does not have components of type ${}^2\mathbf{E}_6(2)$, $\mathbf{E}_7(2)$ or $\mathbf{E}_8(2)$ (see [BM11]). In what follows, we just write $\mathbf{R}_{\mathbf{L}}^{\mathbf{G}}$ and ${}^*\mathbf{R}_{\mathbf{L}}^{\mathbf{G}}$ whenever the independence on the choice of $\mathbf{P}$ is known.

An F -stable torus $\mathbf{T}$ of $\mathbf{G}$ is called a Φ_e -torus if its order polynomial is of the form $P_{(\mathbf{T}, F)} = \Phi_e^n$ for some non-negative integer n and where Φ_e denotes the e -th cyclotomic polynomial (see [CE04, Definition 13.3]). The centralisers of Φ_e -tori are called e -split Levi subgroups. Then $(\mathbf{L}, \lambda)$ is an e -cuspidal pair of $(\mathbf{G}, F)$ (or simply of $\mathbf{G}$ when no confusion arises) if $\mathbf{L}$ is an e -split Levi subgroup of $\mathbf{G}$ and $\lambda \in \mathrm{Irr}(\mathbf{L}^F)$ satisfies ${}^*\mathbf{R}_{\mathbf{M} \leq \mathbf{Q}}^{\mathbf{L}}(\lambda) = 0$ for every e -split Levi subgroup $\mathbf{M} < \mathbf{L}$ and every parabolic subgroup $\mathbf{Q}$ of $\mathbf{L}$ containing $\mathbf{M}$ as Levi complement. An e -cuspidal pair $(\mathbf{L}, \lambda)$ is (e, ℓ') -cuspidal if λ lies in a Lusztig series associated with an ℓ' -regular semisimple element of the dual group $\mathbf{L}^{*F^*}$. To any e -cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$ we associate the e -Harish-Chandra series $\mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ consisting of the irreducible constituents of the virtual characters $\mathbf{R}_{\mathbf{L} \leq \mathbf{P}}^{\mathbf{G}}(\lambda)$ for every parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ containing $\mathbf{L}$ as a Levi subgroup.

Using Deligne–Lusztig induction, one can define a partial order relation on the set of e -pairs of $\mathbf{G}$. If $(\mathbf{L}, \lambda)$ and $(\mathbf{M}, \mu)$ are e -pairs of $\mathbf{G}$, then we write $(\mathbf{L}, \lambda) \leq_e (\mathbf{M}, \mu)$ if $\mathbf{L} \leq \mathbf{M}$ and μ is an irreducible constituent of $\mathbf{R}_{\mathbf{L} \leq \mathbf{Q}}^{\mathbf{M}}$ for some parabolic subgroup $\mathbf{Q}$ of $\mathbf{M}$ having $\mathbf{L}$ as a Levi complement. Then, $\ll_e$ denotes the transitive closure of $\leq_e$. It is conjectured that $\leq_e$ is transitive and hence coincides with $\ll_e$ (see [CE99, Notation 1.11] and [Ros24a, Proposition 3.6, Proposition 4.5 and Corollary 4.11]).

Recall by [Ros24a, Definition 4.15] that a Brauer–Lusztig block is any nonempty set of characters of $\mathbf{G}^F$ of the form $\mathcal{E}(\mathbf{G}^F, B, [s]) := \mathcal{E}(\mathbf{G}^F, [s]) \cap \mathrm{Irr}(B)$, where $\mathcal{E}(\mathbf{G}^F, [s])$ denotes the rational Lusztig series associated to the semisimple element $s \in \mathbf{G}^{*F^*}$ and B is an ℓ -block of $\mathbf{G}^F$. In [Ros24a, Theorem A] the author gives a description of the Brauer–Lusztig blocks in terms of e -Harish-Chandra series under suitable hypotheses. More precisely, we assume the following hypothesis.

Hypothesis 2.4. Let $\mathbf{G}$, $F : \mathbf{G} \rightarrow \mathbf{G}$, q , ℓ and e be as in Notation 2.3. Assume that:

- (i) $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$ and the Mackey formula holds for $(\mathbf{G}, F)$;
- (ii) If $\mathbf{K}$ is an F -stable Levi subgroup of $\mathbf{G}$, then

$$\left\{ \kappa \in \mathrm{Irr}(\mathbf{K}^F) \mid (\mathbf{L}, \lambda) \ll_e (\mathbf{K}, \kappa) \right\} = \mathrm{Irr}(\mathbf{R}_{\mathbf{L}}^{\mathbf{K}}(\lambda))$$

for every (e, ℓ') -cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{K}$.

Under Hypothesis 2.4, [Ros24a, Theorem A] shows that for every e -cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$ we have

$$\mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) \subseteq \mathcal{E}(\mathbf{G}^F, B, [s])$$

where s is a semisimple element of $\mathbf{L}^{*F^*}$ such that $\lambda \in \mathcal{E}(\mathbf{L}^F, [s])$, $B = \mathrm{bl}(\lambda)^{\mathbf{G}^F}$ and $\mathcal{E}(\mathbf{G}^F, B, [s])$ is the associated Brauer–Lusztig block. Inspired by this result, we introduce the following definition.

Definition 2.5. An e -Brauer–Lusztig-cuspidal pair of $\mathbf{G}$ is an e -cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$ such that

$$\mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) = \mathcal{E}(\mathbf{G}^F, B, [s])$$

for some semisimple element s of $\mathbf{G}^{*F^*}$ and some ℓ -block B of $\mathbf{G}^F$.

By [Ros24a, Theorem A] and [CE99, Theorem 4.1] it follows that every e -cuspidal pair $(\mathbf{L}, \lambda)$ such that λ lies in a Lusztig series associated to an ℓ -regular semisimple element is e -Brauer–Lusztig-cuspidal.

We conclude this section with a remark on the validity of Hypothesis 2.4. Observe that the Mackey formula and Hypothesis 2.4 (ii) are expected to hold for any connected reductive group.

Remark 2.6. Suppose that $[\mathbf{G}, \mathbf{G}]$ is simply connected and has no irreducible rational components of type ${}^2\mathbf{E}_6(2)$, $\mathbf{E}_7(2)$ or $\mathbf{E}_8(2)$ and consider $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$. Then Hypothesis 2.4 is satisfied (see [Ros24a, Remark 4.2]).

2.5. A nonblockwise version of Parametrisation C

By replacing $\mathbf{G}^F$ -block isomorphisms of character triples with $\mathbf{G}^F$ -central isomorphisms of character triples (see [Ros22, Definition 3.3.4]) in the statement of Parametrisation C, we obtain a nonblockwise version of Parametrisation C. For the reader's convenience and for future reference, we include this statement below.

Parametrisation 2.7. Let $\mathbf{G}, F : \mathbf{G} \rightarrow \mathbf{G}$, q , ℓ and e be as in Notation 2.3 and consider an e -cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$. Then there exists an ℓ -defect preserving $\text{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{(\mathbf{L}, \lambda)}$ -equivariant bijection

$$\Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}} : \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) \rightarrow \text{Irr}(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \mid \lambda)$$

such that

$$(X_{\vartheta}, \mathbf{G}^F, \vartheta) \sim_{\mathbf{G}^F}^c (X_{X_{\vartheta}}(\mathbf{L}), \mathbf{N}_{\mathbf{G}^F}(\mathbf{L}), \Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}}(\vartheta))$$

for every $\vartheta \in \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ and where $X := \mathbf{G}^F \rtimes \text{Aut}_{\mathbb{F}}(\mathbf{G}^F)$.

We point out that it is much easier to verify Parametrisation 2.7 than it is to verify Parametrisation C. As a hint to this fact, the reader should compare Theorem 5.4 and Theorem 5.5.

Furthermore, proceeding as in [Ros24a, Section 6] we could use Parametrisation 2.7 to obtain nonblockwise versions of [Ros24a, Conjecture C and Conjecture D]. On the way to prove our results for Parametrisation C, we also obtain similar statements for the simpler Parametrisation 2.7.

3. Parametrisation of nonunipotent e -Harish-Chandra series

In this section we start by proving Theorem A and hence extend [BMM93, Theorem 3.2] to nonunipotent e -cuspidal pairs of reductive groups with connected centre and type different from $\mathbf{E}_6$, $\mathbf{E}_7$ or $\mathbf{E}_8$. Then, assuming maximal extendibility for e -cuspidal characters of e -split Levi subgroups, we prove Theorem 3.19 and obtain certain bijections that are part of the requirements of the criteria we prove in Section 4 (see Assumption 4.1 (ii) and Assumption 4.4 (ii)).

3.1. e -Harish-Chandra theory for groups with connected centre

In what follows we make use of the fact that, under suitable hypotheses, there exists a Jordan decomposition that commutes with Deligne–Lusztig induction (see [GM20, Theorem 4.7.2 and Theorem 4.7.5]). More precisely, we consider the following hypothesis.

Hypothesis 3.1. Let $\mathbf{G}, F : \mathbf{G} \rightarrow \mathbf{G}$, q , ℓ and e be as in Notation 2.3 and suppose that $[\mathbf{G}, \mathbf{G}]$ is simple and not of type $\mathbf{E}_6$, $\mathbf{E}_7$ or $\mathbf{E}_8$.

We observe that if we assume Hypothesis 3.1 with $[\mathbf{G}, \mathbf{G}]$ simply connected and $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$, then Hypothesis 2.4 holds thanks to Remark 2.6.

Theorem 3.2. Assume Hypothesis 3.1 and suppose that $\mathbf{G}$ has connected centre. Then there exists a collection of bijections

$$J_{\mathbf{L},s} : \mathcal{E}(\mathbf{L}^F, [s]) \rightarrow \mathcal{E}(\mathbf{C}_{\mathbf{L}^*}(s)^{F^*}, [1])$$

for every F -stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ and every semisimple element $s \in \mathbf{L}^{*F^*}$, such that the following properties are satisfied:

- (i) $J_{\mathbf{L},s}(\lambda)^{\sigma^*} = J_{\mathbf{L},\sigma^*(s)}(\lambda^\sigma)$ for every $\lambda \in \mathcal{E}(\mathbf{L}^F, [s])$, $\sigma \in \text{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{L}}$ and $\sigma^* \in \text{Aut}_{\mathbb{F}}(\mathbf{G}^{*F^*})_{\mathbf{L}^*}$ with $\sigma \sim \sigma^*$ (see [CS13, Proposition 2.2]);
- (ii) $J_{\mathbf{K},s} \circ \mathbf{R}_{\mathbf{L}}^{\mathbf{K}} = \mathbf{R}_{\mathbf{C}_{\mathbf{L}^*}(s)}^{\mathbf{C}_{\mathbf{K}^*}(s)} \circ J_{\mathbf{L},s}$ for every F -stable Levi subgroup $\mathbf{K}$ of $\mathbf{G}$ containing $\mathbf{L}$;
- (iii) $\lambda(1) = |\mathbf{L}^{*F^*} : \mathbf{C}_{\mathbf{L}^*}(s)^{F^*}|_{p'} \cdot J_{\mathbf{L},s}(\lambda)(1)$ for every $\lambda \in \mathcal{E}(\mathbf{L}^F, [s])$; and
- (iv) if $z \in \mathbf{Z}(\mathbf{L}^{*F^*})$ corresponds to the character $\widehat{z}_{\mathbf{L}} \in \text{Irr}(\mathbf{L}^F)$ via [CE04, (8.19)], then

$$J_{\mathbf{L},s}(\lambda) = J_{\mathbf{L},sz}(\lambda \cdot \widehat{z}_{\mathbf{L}})$$

for every $\lambda \in \mathcal{E}(\mathbf{L}^F, [s])$, or equivalently

$$J_{\mathbf{L},s}^{-1}(\nu) \cdot \widehat{z}_{\mathbf{L}} = J_{\mathbf{L},sz}^{-1}(\nu)$$

for every $\nu \in \mathcal{E}(\mathbf{C}_{\mathbf{L}^*}(s)^{F^*}, [1]) = \mathcal{E}(\mathbf{C}_{\mathbf{L}^*}(sz)^{F^*}, [1])$

Proof. The required bijections are constructed in [DM90, Theorem 7.1] and satisfy (iv) by [DM90, Theorem 7.1 (iii)]. The properties (i) and (ii) follow from [CS13, Theorem 3.1] and [GM20, Theorem 4.7.2 and Theorem 4.7.5] respectively. For (iii) see, for instance, the description given in [Mal07, (2.1)]. $\square$

As a consequence of the equivariance of the above Jordan decomposition, we obtain an isomorphism of relative Weyl groups. This result should be compared with [CS13, Corollary 3.3]

Corollary 3.3. Assume Hypothesis 3.1, suppose that $\mathbf{G}$ has connected centre and let $\mathbf{L} \leq \mathbf{K}$ be F -stable Levi subgroups of $\mathbf{G}$. Then, there exists a collection of isomorphisms

$$i_{\mathbf{L},\lambda}^{\mathbf{K}} : W_{\mathbf{K}}(\mathbf{L}, \lambda)^F \rightarrow W_{\mathbf{C}_{\mathbf{K}^*}(s)}(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L},s}(\lambda))^{F^*}$$

for every $\lambda \in \mathcal{E}(\mathbf{L}^F, [s])$, such that

$$\sigma^* \circ i_{\mathbf{L},\lambda}^{\mathbf{K}} = i_{\mathbf{L},\lambda\sigma}^{\mathbf{K}} \circ \sigma$$

for every $\sigma \in \text{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L}}$ and $\sigma^* \in \text{Aut}_{\mathbb{F}}(\mathbf{G}^{*F^*})_{\mathbf{K}^*,\mathbf{L}^*}$ with $\sigma \sim \sigma^*$ (see Lemma 2.2). Moreover, if $z \in \mathbf{Z}(\mathbf{K}^{*F^*})$ corresponds to the character $\widehat{z}_{\mathbf{L}} \in \text{Irr}(\mathbf{L}^F)$ via [CE04, (8.19)], then

$$W_{\mathbf{K}}(\mathbf{L}, \lambda)^F = W_{\mathbf{K}}(\mathbf{L}, \lambda \cdot \widehat{z}_{\mathbf{L}})^F,$$

$$W_{\mathbf{C}_{\mathbf{K}^*}(s)}(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L},s}(\lambda))^{F^*} = W_{\mathbf{C}_{\mathbf{K}^*}(sz)}(\mathbf{C}_{\mathbf{L}^*}(sz), J_{\mathbf{L},sz}(\lambda \cdot \widehat{z}_{\mathbf{L}}))^{F^*}$$

and

$$i_{\mathbf{L},\lambda}^{\mathbf{K}} = i_{\mathbf{L},\lambda \cdot \widehat{z}_{\mathbf{L}}}^{\mathbf{K}}$$

Proof. The first statement follows from the proof of [CS13, Corollary 3.3]. The second statement follows from Theorem 3.2 (iv). $\square$

Before proving Theorem A, we state an equivariant version of [BMM93, Theorem 3.2]. The following statement is a slight improvement of [CS13, Theorem 3.4].

Theorem 3.4. *Let $\mathbf{H}$ be a connected reductive group with a Frobenius endomorphism $F : \mathbf{H} \rightarrow \mathbf{H}$ defining an $\mathbb{F}_q$ -structure on $\mathbf{H}$, ℓ a prime not dividing q and e the order of q modulo ℓ (modulo 4 if $\ell = 2$). For any e -split Levi subgroup $\mathbf{M}$ of $\mathbf{H}$ and $\mu \in \mathcal{E}(\mathbf{M}^F, [1])$ with $(\mathbf{M}, \mu)$ a unipotent e -cuspidal pair, there exists an $\text{Aut}_{\mathbb{F}}(\mathbf{H}^F)_{(\mathbf{M}, \mu)}$ -equivariant bijection*

$$I_{(\mathbf{M}, \mu)}^{\mathbf{H}} : \text{Irr}\left(W_{\mathbf{H}}(\mathbf{M}, \mu)^F\right) \rightarrow \mathcal{E}\left(\mathbf{H}^F, (\mathbf{M}, \mu)\right)$$

such that

$$I_{(\mathbf{M}, \mu)}^{\mathbf{H}}(\eta)(1)_{\ell} = |\mathbf{H}^F : \mathbf{N}_{\mathbf{H}}(\mathbf{M}, \mu)^F|_{\ell} \cdot \mu(1)_{\ell} \cdot \eta(1)_{\ell}$$

for every $\eta \in \text{Irr}(W_{\mathbf{H}}(\mathbf{M}, \mu)^F)$.

Proof. This follows from the proof of [CS13, Theorem 3.4] applied to arbitrary e -split Levi subgroups (see the comment in the proof of [BS20, Proposition 5.5]). Regarding the statement on character degrees, see [Mal07, Theorem 4.2] and the argument used to prove [BS20, Lemma 5.3]. $\square$

Let $\mathbf{K}$ be an F -stable Levi subgroup of $\mathbf{G}$ and consider an e -cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{K}$. Let s be a semisimple element of $\mathbf{L}^{*F*}$ such that $\lambda \in \mathcal{E}(\mathbf{L}^F, [s])$. By [CE99, Proposition 1.10], the unipotent character $J_{\mathbf{L}, s}(\lambda)$ is e -cuspidal. Moreover, using the fact that $\mathbf{L}$ is an e -split Levi subgroup of $\mathbf{K}$, we conclude that $\mathbf{C}_{\mathbf{L}^*}(s)$ is an e -split Levi subgroup of $\mathbf{C}_{\mathbf{K}^*}(s)$. This shows that $(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))$ is a unipotent e -cuspidal pair of $\mathbf{C}_{\mathbf{K}^*}(s)$. Now, we can define the map

$$I_{(\mathbf{L}, \lambda)}^{\mathbf{K}} : \text{Irr}\left(W_{\mathbf{K}}(\mathbf{L}, \lambda)^F\right) \rightarrow \mathcal{E}\left(\mathbf{K}^F, (\mathbf{L}, \lambda)\right) \quad (3.1)$$

given by

$$I_{(\mathbf{L}, \lambda)}^{\mathbf{K}}(\eta) := J_{\mathbf{K}, s}^{-1} \left(I_{(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))}^{\mathbf{C}_{\mathbf{K}^*}(s)} \left((\eta)^{i_{\mathbf{L}, \lambda}^{\mathbf{K}}} \right) \right)$$

for every $\eta \in \text{Irr}(W_{\mathbf{K}}(\mathbf{L}, \lambda)^F)$ and where $\eta^{i_{\mathbf{L}, \lambda}^{\mathbf{K}}} \in \text{Irr}(W_{\mathbf{C}_{\mathbf{K}^*}(s)}(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))^{F*})$ corresponds to η via the isomorphism $i_{\mathbf{L}, \lambda}^{\mathbf{K}}$ of Corollary 3.3 and $I_{(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))}^{\mathbf{C}_{\mathbf{K}^*}(s)}$ is the map constructed in Theorem 3.4.

Lemma 3.5. *Assume Hypothesis 3.1 and suppose that $\mathbf{G}$ has connected centre. Then the map $I_{(\mathbf{L}, \lambda)}^{\mathbf{K}}$ is an $\text{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K}, (\mathbf{L}, \lambda)}$ -equivariant bijection.*

Proof. First, we observe that the map $I_{(\mathbf{L}, \lambda)}^{\mathbf{K}}$ is a bijection because of Theorem 3.2 (ii), in fact

$$\begin{aligned} I_{(\mathbf{L}, \lambda)}^{\mathbf{K}} \left(\text{Irr}\left(W_{\mathbf{K}}(\mathbf{L}, \lambda)^F\right) \right) &= J_{\mathbf{K}, s}^{-1} \left(\text{Irr}\left(\mathbf{R}_{\mathbf{C}_{\mathbf{L}^*}(s)}^{\mathbf{C}_{\mathbf{K}^*}(s)}(J_{\mathbf{L}, s}(\lambda))\right) \right) \\ &= \text{Irr}\left(J_{\mathbf{K}, s}^{-1} \circ \mathbf{R}_{\mathbf{C}_{\mathbf{L}^*}(s)}^{\mathbf{C}_{\mathbf{K}^*}(s)} \circ J_{\mathbf{L}, s}(\lambda)\right) \\ &= \text{Irr}\left(\mathbf{R}_{\mathbf{L}}^{\mathbf{K}}(\lambda)\right). \end{aligned}$$

To show that the bijection is equivariant, let $\sigma \in \text{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K}, \mathbf{L}}$ and consider $\sigma^* \in \text{Aut}_{\mathbb{F}}(\mathbf{G}^{*F*})_{\mathbf{K}^*, \mathbf{L}^*}$ with $\sigma \sim \sigma^*$ (see Lemma 2.2). If $\sigma \in \text{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K}, (\mathbf{L}, \lambda)}$, then σ^* stabilises the $\mathbf{L}^{*F*}$ -orbit of s . Without loss of generality, we may assume that $\sigma^*(s) = s$. Then Theorem 3.2 (i) implies that σ^* stabilises $J_{\mathbf{L}, s}(\lambda)$. Applying Theorem 3.2 (i) and the equivariance properties of Corollary 3.3 and Theorem 3.4, we conclude that

$$\begin{aligned}
I_{(\mathbf{L}, \lambda)}^{\mathbf{K}}(\eta)^{\sigma} &= J_{\mathbf{K}, s}^{-1} \left(I_{(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))}^{\mathbf{C}_{\mathbf{K}^*}(s)} \left(\eta^{i_{\mathbf{L}, \lambda}^{\mathbf{K}}} \right) \right)^{\sigma} \\
&= J_{\mathbf{K}, s}^{-1} \left(I_{(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))}^{\mathbf{C}_{\mathbf{K}^*}(s)} \left(\left(\eta^{i_{\mathbf{L}, \lambda}^{\mathbf{K}}} \right)^{\sigma} \right) \right) \\
&= J_{\mathbf{K}, s}^{-1} \left(I_{(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))}^{\mathbf{C}_{\mathbf{K}^*}(s)} \left((\eta^{\sigma})^{i_{\mathbf{L}, \lambda}^{\mathbf{K}}} \right) \right) \\
&= I_{(\mathbf{L}, \lambda)}^{\mathbf{K}}(\eta^{\sigma})
\end{aligned}$$

for every $\eta \in \text{Irr}(W_{\mathbf{K}}(\mathbf{L}, \lambda)^F)$. $\square$

Lemma 3.6. Assume Hypothesis 3.1 and suppose that $\mathbf{G}$ has connected centre. Then $I_{(\mathbf{L}, \lambda)}^{\mathbf{K}}(\eta)(1)_{\ell} = |\mathbf{K}^F : \mathbf{N}_{\mathbf{K}}(\mathbf{L}, \lambda)^F|_{\ell} \cdot \lambda(1)_{\ell} \cdot \eta(1)_{\ell}$ for every $\eta \in \text{Irr}(W_{\mathbf{K}}(\mathbf{L}, \lambda)^F)$.

Proof. By the condition on character degrees given in Theorem 3.4 together with Theorem 3.2 (iii), we deduce that

$$\begin{aligned}
\eta(1)_{\ell} &= (\eta)^{i_{\mathbf{L}, \lambda}^{\mathbf{K}}}(1)_{\ell} = \frac{I_{(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))}^{\mathbf{C}_{\mathbf{K}^*}(s)} \left((\eta)^{i_{\mathbf{L}, \lambda}^{\mathbf{K}}} \right) (1)_{\ell}}{J_{\mathbf{L}, s}(\lambda)(1)_{\ell} \cdot |\mathbf{C}_{\mathbf{K}^*}(s)^{F^*} : \mathbf{N}_{\mathbf{C}_{\mathbf{K}^*}(s)}(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))^{F^*}|_{\ell}} \\
&= \frac{I_{(\mathbf{L}, \lambda)}^{\mathbf{K}}(\eta)(1)_{\ell} \cdot |\mathbf{C}_{\mathbf{K}^*}(s)^{F^*}|_{\ell} \cdot |\mathbf{L}^F|_{\ell}}{\lambda(1)_{\ell} \cdot |\mathbf{C}_{\mathbf{L}^*}(s)^{F^*}|_{\ell} \cdot |\mathbf{K}^F|_{\ell} \cdot |\mathbf{C}_{\mathbf{K}^*}(s)^{F^*} : \mathbf{N}_{\mathbf{C}_{\mathbf{K}^*}(s)}(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))^{F^*}|_{\ell}} \\
&= \frac{I_{(\mathbf{L}, \lambda)}^{\mathbf{K}}(\eta)(1)_{\ell}}{\lambda(1)_{\ell} \cdot |\mathbf{K}^F : \mathbf{N}_{\mathbf{K}}(\mathbf{L}, \lambda)^F|_{\ell}}
\end{aligned}$$

for every $\eta \in \text{Irr}(W_{\mathbf{K}}(\mathbf{L}, \lambda)^F)$. The result follows immediately from the above equality. $\square$

Lemma 3.7. Assume Hypothesis 3.1 and suppose that $\mathbf{G}$ has connected centre. If $z \in \mathbf{Z}(\mathbf{K}^{*F^*})$ corresponds to the characters $\widehat{z}_{\mathbf{L}} \in \text{Irr}(\mathbf{L}^F)$ and $\widehat{z}_{\mathbf{K}} \in \text{Irr}(\mathbf{K}^F)$ via [CE04, (8.19)], then $\lambda \cdot \widehat{z}_{\mathbf{L}}$ is e -cuspidal, $W_{\mathbf{K}}(\mathbf{L}, \lambda)^F = W_{\mathbf{K}}(\mathbf{L}, \lambda \cdot \widehat{z}_{\mathbf{L}})^F$ and

$$I_{(\mathbf{L}, \lambda)}^{\mathbf{K}}(\eta) \cdot \widehat{z}_{\mathbf{K}} = I_{(\mathbf{L}, \lambda \cdot \widehat{z}_{\mathbf{L}})}^{\mathbf{K}}(\eta)$$

for every $\eta \in \text{Irr}(W_{\mathbf{K}}(\mathbf{L}, \lambda)^F)$.

Proof. According to [Bon06, Proposition 12.1] the character $\lambda \cdot \widehat{z}_{\mathbf{L}}$ is e -cuspidal, while Corollary 3.3 shows that $W_{\mathbf{K}}(\mathbf{L}, \lambda)^F = W_{\mathbf{K}}(\mathbf{L}, \lambda \cdot \widehat{z}_{\mathbf{L}})^F$ and that $i_{\mathbf{L}, \lambda}^{\mathbf{K}} = i_{\mathbf{L}, \lambda \cdot \widehat{z}_{\mathbf{L}}}^{\mathbf{K}}$. Using Theorem 3.2 (iv) we obtain

$$I_{(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))}^{\mathbf{C}_{\mathbf{K}^*}(s)} = I_{(\mathbf{C}_{\mathbf{L}^*}(sz), J_{\mathbf{L}, sz}(\lambda \cdot \widehat{z}_{\mathbf{L}}))}^{\mathbf{C}_{\mathbf{K}^*}(sz)}$$

and

$$\begin{aligned}
I_{(\mathbf{L}, \lambda)}^{\mathbf{K}}(\eta) \cdot \widehat{z}_{\mathbf{K}} &= J_{\mathbf{K}, s}^{-1} \left(I_{(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))}^{\mathbf{C}_{\mathbf{K}^*}(s)} \left((\eta)^{i_{\mathbf{L}, \lambda}^{\mathbf{K}}} \right) \right) \cdot \widehat{z}_{\mathbf{K}} \\
&= J_{\mathbf{K}, s}^{-1} \left(I_{(\mathbf{C}_{\mathbf{L}^*}(sz), J_{\mathbf{L}, sz}(\lambda \cdot \widehat{z}_{\mathbf{L}}))}^{\mathbf{C}_{\mathbf{K}^*}(sz)} \left((\eta)^{i_{\mathbf{L}, \lambda \cdot \widehat{z}_{\mathbf{L}}}^{\mathbf{K}}} \right) \right) \cdot \widehat{z}_{\mathbf{K}} \\
&= J_{\mathbf{K}, sz}^{-1} \left(I_{(\mathbf{C}_{\mathbf{L}^*}(sz), J_{\mathbf{L}, sz}(\lambda \cdot \widehat{z}_{\mathbf{L}}))}^{\mathbf{C}_{\mathbf{K}^*}(sz)} \left((\eta)^{i_{\mathbf{L}, \lambda \cdot \widehat{z}_{\mathbf{L}}}^{\mathbf{K}}} \right) \right) \\
&= I_{(\mathbf{L}, \lambda \cdot \widehat{z}_{\mathbf{L}})}^{\mathbf{K}}(\eta)
\end{aligned}$$

for every $\eta \in \text{Irr}(W_{\mathbf{K}}(\mathbf{L}, \lambda)^F)$. $\square$

Now, combining Lemma 3.5, Lemma 3.6 and Lemma 3.7, we obtain Theorem A.

To conclude, we prove one final result which, although not used directly in the subsequent sections, might be of independent interest. Under the Hypothesis 3.1, the bijections $I_{(\mathbf{L}, \lambda)}^{\mathbf{K}}$ from (3.1) extend by linearity to $\mathbb{Z}$ -linear bijections

$$I_{(\mathbf{L}, \lambda)}^{\mathbf{K}} : \mathbb{Z}\text{Irr}\left(W_{\mathbf{K}}(\mathbf{L}, \lambda)^F\right) \rightarrow \mathbb{Z}\mathcal{E}\left(\mathbf{K}^F, (\mathbf{L}, \lambda)\right). \quad (3.2)$$

If we consider the definition given in (3.1) and replace the maps $I_{(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))}^{\mathbf{C}_{\mathbf{K}^*}(s)}$ given by Theorem 3.4 with those given by [BMM93, Theorem 3.2 (2)], then we obtain a collection of isometries

$$\mathcal{I}_{(\mathbf{L}, \lambda)}^{\mathbf{K}} : \mathbb{Z}\text{Irr}\left(W_{\mathbf{K}}(\mathbf{L}, \lambda)^F\right) \rightarrow \mathbb{Z}\mathcal{E}\left(\mathbf{K}^F, (\mathbf{L}, \lambda)\right)$$

that satisfy certain important properties. However, notice that the maps given in (3.2) agree with the new maps $\mathcal{I}_{(\mathbf{L}, \lambda)}^{\mathbf{K}}$ only up to a choice of signs.

The next result should be compared to [BMM93, Theorem 3.2] and [KM13, Theorem 1.4 (b)] (see also [KM13, Definition 2.9]).

Theorem 3.8. *Assume Hypothesis 3.1 and suppose that $\mathbf{G}$ has connected centre. Then there exists a collection of isometries*

$$\mathcal{I}_{(\mathbf{L}, \lambda)}^{\mathbf{K}} : \mathbb{Z}\text{Irr}\left(W_{\mathbf{K}}(\mathbf{L}, \lambda)^F\right) \rightarrow \mathbb{Z}\mathcal{E}\left(\mathbf{K}^F, (\mathbf{L}, \lambda)\right)$$

where $\mathbf{K}$ runs over the set of e -split Levi subgroups of $\mathbf{G}$ and $(\mathbf{L}, \lambda)$ over the set of e -cuspidal pairs of $\mathbf{K}$ such that:

(i) for all $\mathbf{K}$ and all $(\mathbf{L}, \lambda)$ we have

$$\mathbf{R}_{\mathbf{K}}^{\mathbf{G}} \circ \mathcal{I}_{(\mathbf{L}, \lambda)}^{\mathbf{K}} = \mathcal{I}_{(\mathbf{L}, \lambda)}^{\mathbf{G}} \circ \text{Ind}_{W_{\mathbf{K}}(\mathbf{L}, \lambda)^F}^{W_{\mathbf{G}}(\mathbf{L}, \lambda)^F};$$

(ii) the collection $(\mathcal{I}_{(\mathbf{L}, \lambda)}^{\mathbf{K}})_{\mathbf{K}, (\mathbf{L}, \lambda)}$ is stable under the action of the Weyl group $W_{\mathbf{G}^F}$;

(iii) $\mathcal{I}_{(\mathbf{L}, \lambda)}^{\mathbf{K}}$ maps the trivial character of the trivial group $W_{\mathbf{L}}(\mathbf{L}, \lambda)^F$ to λ .

Proof. As explained above, the maps are constructed as in (3.1) by replacing the maps $I_{(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))}^{\mathbf{C}_{\mathbf{K}^*}(s)}$ given by Theorem 3.4 with those given by [BMM93, Theorem 3.2 (2)]. Consider $\mathbf{K}$ and $(\mathbf{L}, \lambda)$ as above and fix $\eta \in \text{Irr}(W_{\mathbf{K}}(\mathbf{L}, \lambda)^F)$. Since $W_{\mathbf{K}}(\mathbf{L}, \lambda)^F \leq W_{\mathbf{G}}(\mathbf{L}, \lambda)^F$, the construction given in the proof of [CS13, Corollary 3.3] shows that the map $i_{\mathbf{L}, \lambda}^{\mathbf{K}}$ given by Corollary 3.3 coincides with the restriction of $i_{\mathbf{L}, \lambda}^{\mathbf{G}}$ to $W_{\mathbf{K}}(\mathbf{L}, \lambda)^F$. In particular, if we write $\rho_{\mathbf{L}, \lambda}^{\mathbf{K}}$ to denote the element of $\mathbb{Z}\text{Irr}(W_{\mathbf{C}_{\mathbf{K}^*}(s)}(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))^{F^*})$ corresponding to $\rho \in \mathbb{Z}\text{Irr}(W_{\mathbf{K}}(\mathbf{L}, \lambda)^F)$ via the isomorphism $i_{\mathbf{L}, \lambda}^{\mathbf{K}}$, it follows by elementary character theory that

$$\left(\text{Ind}_{W_{\mathbf{K}}(\mathbf{L}, \lambda)^F}^{W_{\mathbf{G}}(\mathbf{L}, \lambda)^F}(\eta)\right)^{i_{\mathbf{L}, \lambda}^{\mathbf{G}}} = \text{Ind}_{W_{\mathbf{C}_{\mathbf{K}^*}(s)}(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))^{F^*}}^{W_{\mathbf{C}_{\mathbf{G}^*}(s)}(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))^{F^*}}\left(\eta^{i_{\mathbf{L}, \lambda}^{\mathbf{K}}}\right). \quad (3.3)$$

By the definition given in (3.1) and applying Theorem 3.2 (ii) and [BMM93, Theorem 3.2 (2.a)] we conclude that

$$\begin{aligned}
\mathbf{R}_{\mathbf{K}}^{\mathbf{G}} \circ \mathcal{I}_{(\mathbf{L}, \lambda)}^{\mathbf{K}}(\eta) &= \mathbf{R}_{\mathbf{K}}^{\mathbf{G}} \left(J_{\mathbf{K}, s}^{-1} \left(\mathcal{I}_{(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))}^{\mathbf{C}_{\mathbf{K}^*}(s)} \left(\eta^{i_{\mathbf{L}, \lambda}^{\mathbf{K}}} \right) \right) \right) \\
&= J_{\mathbf{G}, s}^{-1} \circ \mathbf{R}_{\mathbf{C}_{\mathbf{K}^*}(s)}^{\mathbf{C}_{\mathbf{G}^*}(s)} \left(\mathcal{I}_{(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))}^{\mathbf{C}_{\mathbf{K}^*}(s)} \left(\eta^{i_{\mathbf{L}, \lambda}^{\mathbf{K}}} \right) \right) \\
&= J_{\mathbf{G}, s}^{-1} \circ \mathcal{I}_{(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))}^{\mathbf{C}_{\mathbf{G}^*}(s)} \left(\text{Ind}_{W_{\mathbf{C}_{\mathbf{K}^*}(s)}(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))^{F^*}}^{W_{\mathbf{C}_{\mathbf{G}^*}(s)}(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))^{F^*}} \left(\eta^{i_{\mathbf{L}, \lambda}^{\mathbf{K}}} \right) \right) \\
&= J_{\mathbf{G}, s}^{-1} \circ \mathcal{I}_{(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))}^{\mathbf{C}_{\mathbf{G}^*}(s)} \left(\left(\text{Ind}_{W_{\mathbf{K}}(\mathbf{L}, \lambda)^F}^{W_{\mathbf{G}}(\mathbf{L}, \lambda)^F}(\eta) \right)^{i_{\mathbf{L}, \lambda}^{\mathbf{G}}} \right) \\
&= \mathcal{I}_{(\mathbf{L}, \lambda)}^{\mathbf{G}} \circ \text{Ind}_{W_{\mathbf{K}}(\mathbf{L}, \lambda)^F}^{W_{\mathbf{G}}(\mathbf{L}, \lambda)^F}(\eta)
\end{aligned}$$

where the penultimate equality holds because of (3.3). This proves (i).

The other properties follow by a similar argument. First, (ii) follows from [BMM93, Theorem 3.2 (2.b)] together with Theorem 3.2 (i) and recalling the compatibility with automorphisms obtained in Corollary 3.3. Secondly, to prove (iii) we observe that the trivial character of $W_{\mathbf{L}}(\mathbf{L}, \lambda)^F$ maps to the trivial character of $W_{\mathbf{C}_{\mathbf{L}^*}(s)}(\mathbf{C}_{\mathbf{L}^*}(s), J_{\mathbf{L}, s}(\lambda))^F$ via the isomorphism $i_{\mathbf{L}, \lambda}^{\mathbf{L}}$, while the character $J_{\mathbf{L}, s}(\lambda)$ is mapped to λ via $J_{\mathbf{L}, s}^{-1}$. Then (iii) follows from [BMM93, Theorem 3.2 (2.c)]. $\square$

3.2. Consequences of equivariant maximal extendibility

We start by recalling the definition of maximal extendibility (see [MS16, Definition 3.5]).

Definition 3.9. Let $Y \trianglelefteq X$ be finite groups and consider $\mathcal{Y} \subseteq \text{Irr}(Y)$. Then, we say that *maximal extendibility* holds for $\mathcal{Y}$ with respect to $Y \trianglelefteq X$ if every $\vartheta \in \mathcal{Y}$ extends to X_{ϑ} . In this case, an *extension map* is any map

$$\Lambda : \mathcal{Y} \rightarrow \coprod_{Y \leq X' \leq X} \text{Irr}(X')$$

such that for every $\vartheta \in \mathcal{Y}$, the character $\Lambda(\vartheta) \in \text{Irr}(X_{\vartheta})$ is an extension of ϑ . If $\mathcal{Y} = \text{Irr}(Y)$, then we just say that maximal extendibility holds with respect to $Y \trianglelefteq X$.

As in Section 2.2, consider $\mathbf{G}$ simple of simply connected type, let $i : \mathbf{G} \rightarrow \tilde{\mathbf{G}}$ be a regular embedding compatible with F and consider the group $\mathcal{A}$ generated by field and graph automorphisms of $\mathbf{G}$ in such a way that $\mathcal{A}$ acts on $\tilde{\mathbf{G}}^F$. Then we can define the semidirect product $\tilde{\mathbf{G}}^F \rtimes \mathcal{A}$. For every F -stable closed connected subgroup $\mathbf{H}$ of $\mathbf{G}$ we denote by $(\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{H}}$ the stabiliser of $\mathbf{H}$ under the action of $\tilde{\mathbf{G}}^F \mathcal{A}$.

Consider $\mathcal{K}$ as in Section 2.1. We form the external semidirect product $(\tilde{\mathbf{G}}^F \mathcal{A}) \rtimes \mathcal{K}$ where, for $x \in \tilde{\mathbf{G}}^F \mathcal{A}$ and $z \in \mathcal{K}$, the element z^x is defined as the unique element of $\mathcal{K}$ corresponding to $(\widehat{z}_{\tilde{\mathbf{G}}})^x \in \text{Irr}(\tilde{\mathbf{G}}^F / \mathbf{G}^F)$ via (2.1). For every F -stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$, notice that $(\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}} \rtimes \mathcal{K}$ acts on $\text{Irr}(\tilde{\mathbf{L}}^F)$ via

$$\widetilde{\lambda}^{xz} := \widetilde{\lambda}^x \cdot \widehat{z}_{\mathbf{L}}$$

for every $\widetilde{\lambda} \in \text{Irr}(\tilde{\mathbf{L}}^F)$, $x \in (\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}}$ and $z \in \mathcal{K}$. We denote by $((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}} \rtimes \mathcal{K})_{\widetilde{\lambda}}$ the stabiliser of $\widetilde{\lambda}$ under this action.

Let $\tilde{\mathbf{L}}$ and $\tilde{\mathbf{K}}$ be F -stable Levi subgroups of $\tilde{\mathbf{G}}$ with $\tilde{\mathbf{L}} \leq \tilde{\mathbf{K}}$ and consider an extension map $\tilde{\Lambda}$ with respect to $\tilde{\mathbf{L}} \trianglelefteq \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F$. In this case notice that

$$\tilde{\Lambda}(\widetilde{\lambda})^{xz} := \tilde{\Lambda}(\widetilde{\lambda})^x \cdot \widehat{z}_{\mathbf{N}_{\tilde{\mathbf{K}}}(\tilde{\mathbf{L}}, \widetilde{\lambda}^x)^F}$$

is an extension of $\widetilde{\lambda}^{xz}$ to $\mathbf{N}_{\tilde{\mathbf{K}}}(\tilde{\mathbf{L}}, \widetilde{\lambda}^{xz})^F = \mathbf{N}_{\tilde{\mathbf{K}}}(\tilde{\mathbf{L}}, \widetilde{\lambda}^x)^F$, where $\widehat{z}_{\mathbf{N}_{\tilde{\mathbf{K}}}(\tilde{\mathbf{L}}, \widetilde{\lambda}^x)^F}$ denotes the restriction of $\widehat{z}_{\tilde{\mathbf{K}}}$ to $\mathbf{N}_{\tilde{\mathbf{K}}}(\tilde{\mathbf{L}}, \widetilde{\lambda}^x)^F$.

The next definition should be compared with condition B(d) of [CS19, Definition 2.2] with $d = e$.

Definition 3.10. We say that an extension map $\tilde{\Lambda}$ with respect to $\tilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F$ is $((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, \mathbf{L}} \ltimes \mathcal{K})$ -equivariant if $\tilde{\Lambda}(\tilde{\lambda}^{xz}) = \tilde{\Lambda}(\tilde{\lambda})^{xz}$ for every $\tilde{\lambda} \in \text{Irr}(\tilde{\mathbf{L}}^F)$, $x \in (\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, \mathbf{L}}$ and $z \in \mathcal{K}$. Moreover, if $\text{Cusp}_e(\tilde{\mathbf{L}}, F)$ denotes the set of (irreducible) e -cuspidal characters of $\tilde{\mathbf{L}}^F$, then $((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, \mathbf{L}} \ltimes \mathcal{K})$ acts on $\text{Cusp}_e(\tilde{\mathbf{L}}, F)$ (see [Bon06, Proposition 12.1]) and therefore we can also consider a $((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, \mathbf{L}} \ltimes \mathcal{K})$ -equivariant extension map $\tilde{\Lambda}$ for $\text{Cusp}_e(\tilde{\mathbf{L}}, F)$ with respect to $\tilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F$.

Let $\tilde{\mathbf{K}}$ be an F -stable Levi subgroup of $\tilde{\mathbf{G}}$ and consider an e -cuspidal pair $(\tilde{\mathbf{L}}, \tilde{\lambda})$ of $\tilde{\mathbf{K}}$. Using the bijection $I_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}}$ from (3.1) and assuming the existence of an extension map $\tilde{\Lambda}$ for $\text{Cusp}_e(\tilde{\mathbf{L}}, F)$ with respect to $\tilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F$, we can define the map

$$\begin{aligned} \Upsilon_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}} : \mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda})) &\rightarrow \text{Irr}(\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F \mid \tilde{\lambda}) \\ I_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}}(\tilde{\eta}) &\mapsto \left(\tilde{\Lambda}(\tilde{\lambda}) \cdot \tilde{\eta} \right)^{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F} \end{aligned} \quad (3.4)$$

for every $\tilde{\eta} \in \text{Irr}(W_{\tilde{\mathbf{K}}}(\tilde{\mathbf{L}}, \tilde{\lambda})^F)$. Notice that $\Upsilon_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}}$ is a bijection by the Clifford correspondence and Gallagher's theorem (see [Isa76, Theorem 6.11 and Corollary 6.17]).

First we show that the bijection $\Upsilon_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}}$ from (3.4) preserves the ℓ -defect of characters. Recall that for any finite group X and any $\chi \in \text{Irr}(X)$, the ℓ -defect of χ is the non-negative integer $d(\chi)$ such that $\ell^{d(\chi)} \chi(1)_{\ell} = |X|_{\ell}$. Moreover, observe that if Hypothesis 3.1 holds for $\mathbf{G}$ then it holds for any regular embedding $\tilde{\mathbf{G}}$ of $\mathbf{G}$ since $[\tilde{\mathbf{G}}, \tilde{\mathbf{G}}] = [\mathbf{G}, \mathbf{G}]$.

Lemma 3.11. Assume Hypothesis 3.1 and suppose there exists an extension map $\tilde{\Lambda}$ for $\text{Cusp}_e(\tilde{\mathbf{L}}, F)$ with respect to $\tilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F$. For every $\tilde{\eta} \in \text{Irr}(W_{\tilde{\mathbf{K}}}(\tilde{\mathbf{L}}, \tilde{\lambda})^F)$ we have

$$d\left(I_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}}(\tilde{\eta})\right) = d\left(\left(\tilde{\Lambda}(\tilde{\lambda}) \cdot \tilde{\eta}\right)^{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F}\right).$$

Proof. This follows immediately from Lemma 3.6 applied to $\tilde{\mathbf{G}}$ after noticing that induction of characters preserves the defect (this follows from the degree formula for induced characters). $\square$

The bijection $\Upsilon_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}}$ from (3.4) also preserves central characters.

Lemma 3.12. Assume Hypothesis 3.1 and suppose there exists an extension map $\tilde{\Lambda}$ for $\text{Cusp}_e(\tilde{\mathbf{L}}, F)$ with respect to $\tilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F$. For every $\tilde{\eta} \in \text{Irr}(W_{\tilde{\mathbf{K}}}(\tilde{\mathbf{L}}, \tilde{\lambda})^F)$ we have

$$\text{Irr}\left(I_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}}(\tilde{\eta})_{\mathbf{Z}(\tilde{\mathbf{K}}^F)}\right) = \text{Irr}\left(\left(\left(\tilde{\Lambda}(\tilde{\lambda}) \cdot \tilde{\eta}\right)^{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F}\right)_{\mathbf{Z}(\tilde{\mathbf{K}}^F)}\right).$$

Proof. First, by Clifford theory we deduce that

$$\text{Irr}\left(\left(\left(\tilde{\Lambda}(\tilde{\lambda}) \cdot \tilde{\eta}\right)^{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F}\right)_{\mathbf{Z}(\tilde{\mathbf{G}}^F)}\right) = \text{Irr}\left(\tilde{\lambda}_{\mathbf{Z}(\tilde{\mathbf{G}}^F)}\right). \quad (3.5)$$

On the other hand, by using the character formula [DM91, Proposition 12.2 (i)], we obtain

$$\mathbf{R}_{\tilde{\mathbf{L}}}^{\tilde{\mathbf{K}}}(\tilde{\lambda})_{\mathbf{Z}(\tilde{\mathbf{K}}^F)} = \mathbf{R}_{\tilde{\mathbf{L}}}^{\tilde{\mathbf{K}}}(\tilde{\lambda})(1) \cdot \tilde{\lambda}_{\mathbf{Z}(\tilde{\mathbf{K}}^F)}$$

and hence

$$\mathrm{Irr}\left(I_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}}(\tilde{\eta})_{\mathbf{Z}(\tilde{\mathbf{K}}^F)}\right) = \mathrm{Irr}\left(\tilde{\lambda}_{\mathbf{Z}(\tilde{\mathbf{K}}^F)}\right). \quad (3.6)$$

Now the result follows by combining (3.5) with (3.6). $\square$

Next, we show that the bijection $\Upsilon_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}}$ from (3.4) is compatible with block induction.

Lemma 3.13. *Assume Hypothesis 3.1 and consider $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$. Suppose there exists an extension map $\tilde{\Lambda}$ for $\mathrm{Cusp}_e(\tilde{\mathbf{L}}, F)$ with respect to $\tilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F$. Then*

$$\mathrm{bl}(\tilde{\chi}) = \mathrm{bl}\left(\Upsilon_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}}(\tilde{\chi})\right)^{\tilde{\mathbf{K}}^F}$$

for every $\tilde{\chi} \in \mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda}))$.

Proof. Notice that Hypothesis 2.4 holds for $\tilde{\mathbf{G}}$ under our assumptions and so $\mathrm{bl}(\tilde{\lambda})^{\tilde{\mathbf{K}}^F} = \mathrm{bl}(\tilde{\chi})$ by [Ros24a, Proposition 4.8] and $\mathrm{bl}(\Upsilon_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}}(\tilde{\chi})) = \mathrm{bl}(\tilde{\lambda})^{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F}$ by [Ros24a, Lemma 5.5]. Then the result follows from the transitivity of block induction. $\square$

Finally, we show that the bijection $\Upsilon_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}}$ from (3.4) is equivariant.

Lemma 3.14. *Assume Hypothesis 3.1 with $\mathbf{G}$ simple and simply connected and suppose there exists a $(\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, \mathbf{L}} \ltimes \mathcal{K}$ -equivariant extension map $\tilde{\Lambda}$ for $\mathrm{Cusp}_e(\tilde{\mathbf{L}}, F)$ with respect to $\tilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F$. Then $\Upsilon_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}}$ is $((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, \mathbf{L}} \ltimes \mathcal{K})_{\tilde{\lambda}}$ -equivariant.*

Proof. Let $(x, z) \in ((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, \mathbf{L}} \ltimes \mathcal{K})_{\tilde{\lambda}}$. Since $\tilde{\lambda} = \tilde{\lambda}^x \cdot \tilde{z}_{\tilde{\mathbf{L}}}$, we have

$$\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L}, \tilde{\lambda})^F = \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L}, \tilde{\lambda}^x \cdot \tilde{z}_{\tilde{\mathbf{L}}})^F = \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L}, \tilde{\lambda}^x)^F.$$

By using the equivariance properties of $\tilde{\Lambda}$, we obtain

$$\begin{aligned} \left(\left(\tilde{\Lambda}(\tilde{\lambda}) \cdot \tilde{\eta} \right)^{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F} \right)^{(x, z)} &= \left(\left(\tilde{\Lambda}(\tilde{\lambda}) \cdot \tilde{\eta} \right)^x \right)^{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F} \cdot \tilde{z}_{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})} \\ &= \left(\left(\tilde{\Lambda}(\tilde{\lambda}) \cdot \tilde{\eta} \right)^x \cdot \tilde{z}_{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L}, \tilde{\lambda})^F} \right)^{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F} \\ &= \left(\tilde{\Lambda}(\tilde{\lambda}^x \cdot \tilde{z}_{\tilde{\mathbf{L}}}) \cdot \tilde{\eta}^x \right)^{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F} \\ &= \left(\tilde{\Lambda}(\tilde{\lambda}) \cdot \tilde{\eta}^x \right)^{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F}. \end{aligned} \quad (3.7)$$

On the other hand, considering Lemma 3.5 and Lemma 3.7 with respect to $\tilde{\mathbf{G}}$ it follows that

$$\begin{aligned} I_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}}(\tilde{\eta})^{(x, z)} &= I_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}}(\tilde{\eta})^x \cdot \tilde{z}_{\tilde{\mathbf{K}}} \\ &= I_{(\tilde{\mathbf{L}}, \tilde{\lambda}^x \cdot \tilde{z}_{\tilde{\mathbf{L}}})}^{\tilde{\mathbf{K}}}(\tilde{\eta}^x) \\ &= I_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}}(\tilde{\eta}^x). \end{aligned} \quad (3.8)$$

Now, the result follows immediately from (3.7) and (3.8). $\square$

3.3. e -Harish-Chandra series and regular embeddings

We use the results obtained in the previous two subsections in order to obtain the bijections needed in the criteria we prove in Section 4 (see Theorem 4.3 and Theorem 4.8).

To start, we study the behaviour of e -Harish-Chandra series with respect to the regular embedding $i: \mathbf{G} \rightarrow \tilde{\mathbf{G}}$. Fix an F -stable Levi subgroup $\mathbf{K}$ of $\mathbf{G}$ and an e -cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{K}$. Observe that $\tilde{\mathbf{K}} := \mathbf{KZ}(\tilde{\mathbf{G}})$ is an F -stable Levi subgroup of $\tilde{\mathbf{G}}$ and that $(\tilde{\mathbf{L}}, \tilde{\lambda})$ is an e -cuspidal pair of $\tilde{\mathbf{K}}$ for every $\tilde{\lambda} \in \text{Irr}(\tilde{\mathbf{L}}^F \mid \lambda)$ and where $\tilde{\mathbf{L}} := \mathbf{LZ}(\tilde{\mathbf{G}})$ (see [Bon06, Proposition 10.10]).

Definition 3.15. Let $\mathcal{HC}(\tilde{\mathbf{K}}^F, (\mathbf{L}, \lambda))$ be the set of e -Harish-Chandra series $\mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda}))$ with $\tilde{\lambda} \in \text{Irr}(\tilde{\mathbf{L}}^F \mid \lambda)$. The group $\mathcal{K}$ from Section 2.1 acts on the set $\mathcal{HC}(\tilde{\mathbf{K}}^F, (\mathbf{L}, \lambda))$ via

$$\mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda}))^z := \mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda} \cdot \hat{z}_{\tilde{\mathbf{L}}}))$$

for every $\mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda})) \in \mathcal{HC}(\tilde{\mathbf{K}}^F, (\mathbf{L}, \lambda))$, $z \in \mathcal{K}$ and where $\hat{z}_{\tilde{\mathbf{L}}}$ corresponds to z via (2.1). Here notice that, as λ is e -cuspidal, then so are $\tilde{\lambda}$ and $\tilde{\lambda} \cdot \hat{z}_{\tilde{\mathbf{L}}}$ (see [Bon06, Proposition 10.10 and Proposition 10.11]). Moreover, if we define $\mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda})) \cdot \hat{z}_{\tilde{\mathbf{K}}}$ to be the set of characters $\tilde{\chi} \cdot \hat{z}_{\tilde{\mathbf{K}}}$ for $\tilde{\chi} \in \mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda}))$, then

$$\mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda}))^z = \mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda})) \cdot \hat{z}_{\tilde{\mathbf{K}}}$$

by [Bon06, Proposition 10.11].

We want to compare the action of $\mathcal{K}$ on $\mathcal{HC}(\tilde{\mathbf{K}}^F, (\mathbf{L}, \lambda))$ with the action of $\mathcal{K}$ on the set of characters $\text{Irr}(\tilde{\mathbf{L}}^F \mid \lambda)$. First, observe that [Isa76, Problem 6.2] implies that both actions are transitive.

Lemma 3.16. Assume Hypothesis 2.4 for $(\tilde{\mathbf{G}}, F)$ and let $\tilde{\lambda}_i \in \text{Irr}(\tilde{\mathbf{L}}^F \mid \lambda)$ for $i = 1, 2$. Let $z \in \mathcal{K}$, then

$$\mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda}_1)) = \mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda}_2))^z$$

if and only if

$$\tilde{\lambda}_1 = \tilde{\lambda}_2^x \cdot \hat{z}_{\tilde{\mathbf{L}}}$$

for some $x \in \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L}, \lambda)^F$.

Proof. First, assume $\mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda}_1)) = \mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda}_2))^z$. By [Ros24a, Proposition 4.10], there exists $u \in \tilde{\mathbf{K}}^F$ such that $(\tilde{\mathbf{L}}, \tilde{\lambda}_1) = (\tilde{\mathbf{L}}, \tilde{\lambda}_2 \cdot \hat{z}_{\tilde{\mathbf{L}}})^u$. This implies that $u \in \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L}, \lambda)^F$ and that $\tilde{\lambda}_1 = \tilde{\lambda}_2^u \cdot \hat{z}_{\tilde{\mathbf{L}}}$. Moreover, since $\tilde{\lambda}_1$ lies over both λ and λ^u , it follows from Clifford's theorem that $\lambda = \lambda^{uv}$, for some $v \in \tilde{\mathbf{L}}^F$. Then $x := uv \in \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L}, \lambda)^F$ and $\tilde{\lambda}_1 = \tilde{\lambda}_2^x \cdot \hat{z}_{\tilde{\mathbf{L}}}$. Conversely, if $\tilde{\lambda}_1 = \tilde{\lambda}_2^x \cdot \hat{z}_{\tilde{\mathbf{L}}}$ for some $x \in \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L}, \lambda)^F$, then [Bon06, Proposition 10.11] yields the desired equality. $\square$

Corollary 3.17. Assume Hypothesis 2.4 for $(\tilde{\mathbf{G}}, F)$ and consider $\tilde{\lambda} \in \text{Irr}(\tilde{\mathbf{L}}^F \mid \lambda)$. Then

$$\mathcal{K}_{\mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda}))} \leq \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L}, \lambda)^F (\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L}, \lambda)^F \ltimes \mathcal{K})_{\tilde{\lambda}}$$

where $\mathcal{K}_{\mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda}))}$ denotes the stabiliser of $\mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda}))$ under the action of $\mathcal{K}$ on $\mathcal{HC}(\tilde{\mathbf{K}}^F, (\mathbf{L}, \lambda))$.

Proof. Let $z \in \mathcal{K}$ stabilise $\mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda}))$. By Lemma 3.16 there exists $x \in \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L}, \lambda)^F$ such that $\tilde{\lambda} = \tilde{\lambda}^x \cdot \hat{z}_{\tilde{\mathbf{L}}}$ and hence $z = x^{-1}xz \in \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L}, \lambda)^F (\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L}, \lambda)^F \ltimes \mathcal{K})_{\tilde{\lambda}}$. $\square$

For every finite group X with subgroup $Y \leq X$ and every subset of characters $\mathcal{Y} \subseteq \text{Irr}(Y)$, we denote by $\text{Irr}(X \mid \mathcal{Y})$ the set of characters $\chi \in \text{Irr}(X)$ lying over some character $\psi \in \mathcal{Y}$.

Proposition 3.18. Assume Hypothesis 2.4 for $(\tilde{\mathbf{G}}, F)$ and let $\tilde{\lambda} \in \text{Irr}(\tilde{\mathbf{L}}^F \mid \lambda)$. If $\mathcal{T}$ is a transversal for the cosets of $\mathcal{K}_{\mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda}))}$ in $\mathcal{K}$, then

$$\text{Irr}\left(\tilde{\mathbf{K}}^F \mid \mathcal{E}\left(\mathbf{K}^F, (\mathbf{L}, \lambda)\right)\right) = \coprod_{z \in \mathcal{T}} \mathcal{E}\left(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda})\right) \cdot \widehat{z}_{\tilde{\mathbf{K}}} \quad (3.9)$$

and

$$\text{Irr}\left(\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F \mid \lambda\right) = \coprod_{z \in \mathcal{T}} \text{Irr}\left(\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F \mid \tilde{\lambda}\right) \cdot \widehat{z}_{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})}, \quad (3.10)$$

where $\text{Irr}(\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F \mid \tilde{\lambda}) \cdot \widehat{z}_{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})}$ is the set of characters $\tilde{\psi} \cdot \widehat{z}_{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})}$ for $\tilde{\psi} \in \text{Irr}(\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F \mid \tilde{\lambda})$.

Proof. Set $\tilde{\mathcal{G}} := \text{Irr}(\tilde{\mathbf{K}}^F \mid \mathcal{E}(\mathbf{K}^F, (\mathbf{L}, \lambda)))$ and $\tilde{\mathcal{N}} := \text{Irr}(\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F \mid \lambda)$. First, we claim that $\tilde{\mathcal{G}}$ is the union of the e -Harish-Chandra series in the set $\mathcal{HC}(\tilde{\mathbf{K}}^F, (\mathbf{L}, \lambda))$. In fact, if $\tilde{\chi} \in \tilde{\mathcal{G}}$, then there exists $\chi \in \mathcal{E}(\mathbf{K}^F, (\mathbf{L}, \lambda))$ lying below $\tilde{\chi}$. By [GM20, Corollary 3.3.25], it follows that $\tilde{\chi}$ is an irreducible constituent of $\mathbf{R}_{\tilde{\mathbf{L}}}^{\tilde{\mathbf{K}}}(\lambda^{\tilde{\mathbf{L}}^F})$ and therefore there exists $\tilde{\nu} \in \text{Irr}(\tilde{\mathbf{L}}^F \mid \lambda)$ such that $\tilde{\chi} \in \mathcal{E}(\tilde{\mathbf{K}}, (\tilde{\mathbf{L}}, \tilde{\nu}))$. On the other hand, if $\tilde{\nu} \in \text{Irr}(\tilde{\mathbf{L}}^F \mid \lambda)$ and $\tilde{\chi} \in \mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\nu}))$, then [GM20, Corollary 3.3.25] implies that $\tilde{\chi}$ lies over some character $\chi \in \mathcal{E}(\mathbf{K}^F, (\mathbf{L}, \lambda))$. Since the action of $\mathcal{K}$ on $\mathcal{HC}(\tilde{\mathbf{K}}^F, (\mathbf{L}, \lambda))$ is transitive and recalling the definition of $\mathcal{T}$, we hence obtain (3.9).

Now we prove (3.10). By Clifford theory, we know that every element of $\mathcal{G}$ lies above some character $\tilde{\nu} \in \text{Irr}(\tilde{\mathbf{L}} \mid \lambda)$. Since $\mathcal{K}$ is transitive on $\text{Irr}(\tilde{\mathbf{L}}^F \mid \lambda)$, we deduce that $\tilde{\mathcal{N}}$ is contained in the union

$$\bigcup_{z \in \mathcal{K}} \text{Irr}\left(\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F \mid \tilde{\lambda}\right) \cdot \widehat{z}_{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})}.$$

Moreover, we claim that the above union coincides with

$$\bigcup_{z \in \mathcal{T}} \text{Irr}\left(\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F \mid \tilde{\lambda}\right) \cdot \widehat{z}_{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})}. \quad (3.11)$$

To see this, let $z \in \mathcal{K}$ and write $z = z_0 t$, for some $z_0 \in \mathcal{K}_{\mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda}))}$ and $t \in \mathcal{T}$. By Corollary 3.17 we obtain $z_0 \in \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L}, \lambda)^F (\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L}, \lambda)^F \rtimes \mathcal{K})_{\tilde{\lambda}}$ and therefore

$$\text{Irr}\left(\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F \mid \tilde{\lambda}\right) \cdot \widehat{z}_{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})} = \text{Irr}\left(\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F \mid \tilde{\lambda}\right) \cdot \widehat{t}_{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})}.$$

This proves our claim and it remains to show that the union in (3.11) is disjoint. Assume that, for some $z \in \mathcal{T}$, there exists a character $\tilde{\psi}$ inside both $\text{Irr}(\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F \mid \tilde{\lambda})$ and $\text{Irr}(\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F \mid \tilde{\lambda}) \cdot \widehat{z}_{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})}$. By [Isa76, Problem 5.3] we deduce that $\text{Irr}(\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F \mid \tilde{\lambda}) \cdot \widehat{z}_{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})} = \text{Irr}(\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F \mid \tilde{\lambda} \cdot \widehat{z}_{\tilde{\mathbf{L}}})$ and hence $\tilde{\psi}$ lies above $\tilde{\lambda}$ and $\tilde{\lambda} \cdot \widehat{z}_{\tilde{\mathbf{L}}}$. By Clifford's theorem $\tilde{\lambda} = (\tilde{\lambda} \cdot \widehat{z}_{\tilde{\mathbf{L}}})^u = \tilde{\lambda}^u \cdot \widehat{z}_{\tilde{\mathbf{L}}}$, for some $u \in \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F$ and now Lemma 3.16 implies $\mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda})) = \mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda}))^z$. By the definition of $\mathcal{T}$ it follows that the union in (3.11) is disjoint. $\square$

As a corollary of Proposition 3.18 and using the bijection $\Upsilon_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}}$ from (3.4), we are finally able to prove the main result of this subsection. The bijection described in the following theorem is part of the requirements of the criteria we prove in Section 4 (see Assumption 4.1 (ii) and Assumption 4.4 (ii)).

Theorem 3.19. Assume Hypothesis 3.1 with $\mathbf{G}$ simple and simply connected and consider $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$. Suppose there exists a $((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, \mathbf{L}} \rtimes \mathcal{K})$ -equivariant extension map for $\text{Cusp}_e(\tilde{\mathbf{L}}, F)$ with respect to $\tilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F$. Then, there exists a defect preserving $\left((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, (\mathbf{L}, \lambda)} \rtimes \mathcal{K}\right)$ -equivariant bijection

$$\tilde{\Omega}_{(\mathbf{L}, \lambda)}^{\mathbf{K}} : \text{Irr}(\tilde{\mathbf{K}}^F \mid \mathcal{E}(\mathbf{K}^F, (\mathbf{L}, \lambda))) \rightarrow \text{Irr}(\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F \mid \lambda)$$

such that, for every $\tilde{\chi} \in \text{Irr}(\tilde{\mathbf{K}}^F \mid \mathcal{E}(\mathbf{K}^F, (\mathbf{L}, \lambda)))$, the following conditions hold:

- (i) $\text{Irr}(\tilde{\chi}_{\mathbf{Z}(\tilde{\mathbf{K}}^F)}) = \text{Irr}(\tilde{\Psi}(\tilde{\chi})_{\mathbf{Z}(\tilde{\mathbf{K}}^F)});$
- (ii) $\text{bl}(\tilde{\chi}) = \text{bl}(\tilde{\Psi}(\tilde{\chi}))^{\tilde{\mathbf{K}}^F}.$

Proof. First observe that under our assumptions Hypothesis 2.4 holds for $\mathbf{G}$ and for $\tilde{\mathbf{G}}$. Set $\tilde{\mathcal{G}} := \text{Irr}(\tilde{\mathbf{K}}^F \mid \mathcal{E}(\mathbf{K}^F, (\mathbf{L}, \lambda)))$ and $\tilde{\mathcal{N}} := \text{Irr}(\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F \mid \lambda)$ and fix $\tilde{\lambda} \in \text{Irr}(\tilde{\mathbf{L}}^F \mid \lambda)$. Let

$$\Upsilon_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}} : \mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda})) \rightarrow \text{Irr}(\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F \mid \tilde{\lambda})$$

be the bijection constructed in (3.4). Let $\tilde{\mathbb{T}}_{\text{glo}}$ be a $((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, \mathbf{L}} \rtimes \mathcal{K})_{\tilde{\lambda}}$ -transversal in $\mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda}))$ and observe that, by Lemma 3.14, the set $\tilde{\mathbb{T}}_{\text{loc}} := \{\Upsilon_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}}(\tilde{\chi}) \mid \tilde{\chi} \in \tilde{\mathbb{T}}_{\text{glo}}\}$ is a $((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, \mathbf{L}} \rtimes \mathcal{K})_{\tilde{\lambda}}$ -transversal in $\text{Irr}(\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F \mid \lambda)$.

Next, we fix a transversal $\mathcal{T}$ for $\mathcal{K}_{\mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda}))}$ in $\mathcal{K}$ and we claim that

$$\tilde{\mathbf{K}}_{\mathbf{L}}^F \left(\left((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, \mathbf{L}} \rtimes \mathcal{K} \right)_{\tilde{\lambda}} \cdot \mathcal{T} = \tilde{\mathbf{K}}_{\mathbf{L}}^F \left((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, (\mathbf{L}, \lambda)} \rtimes \mathcal{K} \right). \quad (3.12)$$

To prove this equality, consider $xz \in ((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, (\mathbf{L}, \lambda)} \rtimes \mathcal{K})$. Then both $\tilde{\lambda}$ and $\tilde{\lambda}^x \cdot \tilde{z}_{\tilde{\mathbf{L}}}$ lie over λ and by [Isa76, Problem 6.2] there exists $u \in \mathcal{K}$ such that $\tilde{\lambda} = \tilde{\lambda}^x \cdot \tilde{z}_{\tilde{\mathbf{L}}} \cdot \tilde{u}_{\tilde{\mathbf{L}}}$. Therefore $xz \in ((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, \mathbf{L}} \rtimes \mathcal{K})_{\tilde{\lambda}} \cdot \mathcal{K}$. On the other hand, applying Corollary 3.17, we obtain $\mathcal{K}_{\mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda}))} \leq \tilde{\mathbf{K}}_{\mathbf{L}}^F (\tilde{\mathbf{K}}_{\mathbf{L}}^F \rtimes \mathcal{K})_{\tilde{\lambda}}$ and by the definition of $\mathcal{T}$, we conclude that

$$\tilde{\mathbf{K}}_{\mathbf{L}}^F \left(\left((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, \mathbf{L}} \rtimes \mathcal{K} \right)_{\tilde{\lambda}} \cdot \mathcal{T} \geq \tilde{\mathbf{K}}_{\mathbf{L}}^F \left((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, (\mathbf{L}, \lambda)} \rtimes \mathcal{K} \right).$$

To prove the remaining inclusion it's enough to show that

$$\tilde{\mathbf{L}}^F \left(\left((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, (\mathbf{L}, \lambda)} \rtimes \mathcal{K} \right)_{\tilde{\lambda}} = \left((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, \mathbf{L}} \rtimes \mathcal{K} \right)_{\tilde{\lambda}}.$$

Since $\tilde{\lambda}$ is $\tilde{\mathbf{L}}^F$ -invariant, one inclusion is trivial. So let $xz \in ((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, \mathbf{L}} \rtimes \mathcal{K})_{\tilde{\lambda}}$ and observe that $\tilde{\lambda} = \tilde{\lambda}^x \cdot \tilde{z}_{\tilde{\mathbf{L}}}$ lies both over λ and over λ^x . By Clifford's theorem there exists $y \in \tilde{\mathbf{L}}^F$ such that $\lambda = \lambda^{xy}$ and hence $xz \in \tilde{\mathbf{L}}^F ((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, (\mathbf{L}, \lambda)} \rtimes \mathcal{K})_{\tilde{\lambda}}$. This proves the claim.

Now, using (3.12), we show that $\tilde{\mathbb{T}}_{\text{glo}}$ is a $((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, (\mathbf{L}, \lambda)} \rtimes \mathcal{K})$ -transversal in $\tilde{\mathcal{G}}$. Consider $\tilde{\chi} \in \tilde{\mathcal{G}}$. By Proposition 3.18 there exist unique $z \in \mathcal{T}$ and $\tilde{\chi}'_0 \in \mathcal{E}(\tilde{\mathbf{K}}^F, (\tilde{\mathbf{L}}, \tilde{\lambda}))$ such that $\tilde{\chi} = \tilde{\chi}'_0 \cdot \tilde{z}_{\tilde{\mathbf{K}}}$. Let $\tilde{\chi}_0$ be the unique element in $\tilde{\mathbb{T}}_{\text{glo}}$ such that $\tilde{\chi}'_0 = \tilde{\chi}_0^x \cdot \tilde{u}_{\tilde{\mathbf{K}}}$, for some $xu \in ((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, \mathbf{L}} \rtimes \mathcal{K})_{\tilde{\lambda}}$. Then $\tilde{\chi} = \tilde{\chi}_0^x \cdot \tilde{u}_{\tilde{\mathbf{K}}} \cdot \tilde{z}_{\tilde{\mathbf{K}}}$, for $xuz \in ((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, \mathbf{L}} \rtimes \mathcal{K})_{\tilde{\lambda}} \cdot \mathcal{T}$. But using (3.12) and since $\tilde{\chi}$ and $\tilde{\chi}_0$ are $\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F$ -invariant, we conclude that $\tilde{\chi} = \tilde{\chi}_0^y \cdot \tilde{v}_{\tilde{\mathbf{K}}}$, for some $y \in (\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, (\mathbf{L}, \lambda)}$ and $v \in \mathcal{K}$. This argument also shows that $\tilde{\chi}_0$ is the unique element of $\tilde{\mathbb{T}}_{\text{glo}}$ with this property.

Similarly, using (3.12), we deduce that the set $\tilde{\mathbb{T}}_{\text{loc}}$ is a $((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, (\mathbf{L}, \lambda)} \rtimes \mathcal{K})$ -transversal in $\tilde{\mathcal{N}}$. Now, the map

$$\tilde{\Omega}_{(\mathbf{L}, \lambda)}^{\mathbf{K}} : \tilde{\mathcal{G}} \rightarrow \tilde{\mathcal{N}}$$

defined by

$$\tilde{\Omega}_{(\mathbf{L}, \lambda)}^{\mathbf{K}}(\tilde{\chi}^x \cdot \widehat{z}_{\tilde{\mathbf{K}}}) := Y_{(\tilde{\mathbf{L}}, \tilde{\lambda})}^{\tilde{\mathbf{K}}}(\tilde{\chi})^x \cdot \widehat{z}_{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})},$$

for every $\tilde{\chi} \in \tilde{\mathbb{T}}_{\text{glo}}$, $x \in (\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, (\mathbf{L}, \lambda)}$ and $z \in \mathcal{K}$, is a $((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, (\mathbf{L}, \lambda)} \ltimes \mathcal{K})$ -equivariant bijection. The remaining properties follow from Lemma 3.11, Lemma 3.12 and Lemma 3.13 after noticing that $\widehat{z}_{\tilde{\mathbf{K}}}$ and $\widehat{z}_{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})}$ are linear characters and that

$$\text{bl}\left(\tilde{\psi} \cdot \widehat{z}_{\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})}\right)^{\tilde{\mathbf{K}}^F} = \text{bl}\left(\tilde{\psi}\right)^{\tilde{\mathbf{K}}^F} \cdot \widehat{z}_{\tilde{\mathbf{K}}}$$

for every $\tilde{\psi} \in \text{Irr}(\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F)$ and $z \in \mathcal{K}$. □

Thanks to Remark 2.6, Theorem B follows from our next theorem.

Theorem 3.20. *Assume Hypothesis 3.1 with $\mathbf{G}$ simple and simply connected and let $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$. Consider an F -stable Levi subgroup $\mathbf{K}$ of $\mathbf{G}$ and an e -cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{K}$. Suppose there exists a $((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, \mathbf{L}} \ltimes \mathcal{K})$ -equivariant extension map for $\text{Cusp}_e(\tilde{\mathbf{L}}, F)$ with respect to $\tilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F$. Then, there exists a defect preserving $(\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, (\mathbf{L}, \lambda)}$ -equivariant bijection*

$$\Omega_{(\mathbf{L}, \lambda)}^{\mathbf{K}} : \mathcal{E}(\mathbf{K}^F, (\mathbf{L}, \lambda)) \rightarrow \text{Irr}(\mathbf{N}_{\mathbf{K}}(\mathbf{L})^F \mid \lambda).$$

Proof. Fix a $((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, (\mathbf{L}, \lambda)} \ltimes \mathcal{K})$ -transversal $\tilde{\mathbb{T}}_{\text{glo}}$ in $\text{Irr}(\tilde{\mathbf{K}}^F \mid \mathcal{E}(\mathbf{K}^F, (\mathbf{L}, \lambda)))$. By Theorem 3.19 the set $\tilde{\mathbb{T}}_{\text{loc}} := \{\tilde{\Omega}_{(\mathbf{L}, \lambda)}^{\mathbf{K}}(\tilde{\chi}) \mid \tilde{\chi} \in \tilde{\mathbb{T}}_{\text{glo}}\}$ is a $((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, (\mathbf{L}, \lambda)} \ltimes \mathcal{K})$ -transversal in $\text{Irr}(\mathbf{N}_{\tilde{\mathbf{K}}}(\mathbf{L})^F \mid \lambda)$. For every $\tilde{\chi} \in \tilde{\mathbb{T}}_{\text{glo}}$ fix an irreducible constituent $\chi \in \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ of $\tilde{\chi}_{\mathbf{G}^F}$ and define the set $\mathbb{T}_{\text{glo}}$ consisting of such characters χ , while $\tilde{\chi}$ runs over the elements of $\tilde{\mathbb{T}}_{\text{glo}}$. Similarly, for every $\tilde{\psi} \in \tilde{\mathbb{T}}_{\text{loc}}$, fix an irreducible constituent $\psi \in \text{Irr}(\mathbf{N}_{\mathbf{K}}(\mathbf{L})^F \mid \lambda)$ of $\tilde{\psi}_{\mathbf{N}_{\mathbf{K}}(\mathbf{L})^F}$ and define the set $\mathbb{T}_{\text{loc}}$ consisting of such characters ψ , while $\tilde{\psi}$ runs over the elements of $\tilde{\mathbb{T}}_{\text{loc}}$. Then $\mathbb{T}_{\text{glo}}$ and $\mathbb{T}_{\text{loc}}$ are $(\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, (\mathbf{L}, \lambda)}$ -transversals in $\mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ and $\text{Irr}(\mathbf{N}_{\mathbf{K}}(\mathbf{L})^F \mid \lambda)$ respectively. Fix $\chi \in \mathbb{T}_{\text{glo}}$ and let $\tilde{\chi}$ be the unique element of $\tilde{\mathbb{T}}_{\text{glo}}$ lying above χ . Let $\tilde{\psi} := \tilde{\Omega}_{(\mathbf{L}, \lambda)}^{\mathbf{K}}(\tilde{\chi}) \in \tilde{\mathbb{T}}_{\text{loc}}$ and consider the unique element ψ of $\mathbb{T}_{\text{loc}}$ lying below $\tilde{\psi}$. This defines a bijection

$$\mathbb{T}_{\text{glo}} \rightarrow \mathbb{T}_{\text{loc}}. \tag{3.13}$$

Then, defining

$$\Omega_{(\mathbf{L}, \lambda)}^{\mathbf{K}}(\chi^x) := \psi^x$$

for every $x \in (\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{K}, (\mathbf{L}, \lambda)}$ and every $\chi \in \mathbb{T}_{\text{glo}}$ corresponding to $\psi \in \mathbb{T}_{\text{loc}}$ via (3.13) we obtain the wanted bijection. □

The above result provides a way to extend [BMM93, Theorem 3.2 (ii)] and obtain a parametrisation of e -Harish-Chandra series for nonunipotent e -cuspidal pairs of simple algebraic groups with (possibly) disconnected centre.

4. The criteria

In this section we prove Theorem 4.3 and Theorem 4.8 which serve as criteria for Parametrisation 2.7 and Parametrisation C respectively. The assumptions of these criteria consist of two main parts: first we assume the existence of certain bijections (see Assumption 4.1 (ii) and Assumption 4.4 (ii)). These bijections have been constructed in Theorem 3.19 under suitable hypotheses. Secondly, we need to control

the action of automorphisms on irreducible characters (see Assumption 4.1 (iii)-(iv) and Assumption 4.4 (iii)-(iv)) in order to construct projective representations via an application of [Spä12, Lemma 2.11]. This second problem is part of an important ongoing project in representation theory of finite reductive groups. Moreover, in order to obtain $\mathbf{G}^F$ -block isomorphisms of character triples, in the assumption of the criterion for Parametrisation C we need to include certain block theoretic requirements (see Assumption 4.4 (v)-(vi)). These restrictions are analogous to those introduced in [CS15, Theorem 4.1], [BS20, Theorem 2.4] and [BS22, Theorem 4.5].

The results presented in this section should be compared to [Spä12, Theorem 2.12], [CS15, Theorem 4.1], [BS22, Theorem 4.5], [Ruh22b, Theorem 2.1] and [Ruh22c, Theorem 9.2].

4.1. The criterion for Parametrisation 2.7

We start by dealing with Parametrisation 2.7. The results obtained in this subsection are then used in the next one to prove the criterion for Parametrisation C under additional restrictions.

Consider $\mathbf{G}, F : \mathbf{G} \rightarrow \mathbf{G}, q, \ell$ and e as in Notation 2.3 and let $i : \mathbf{G} \rightarrow \tilde{\mathbf{G}}$ be a regular embedding. We recall once more that the group $\mathcal{K}$ introduced in Section 2.1 acts on the sets of irreducible characters of $\mathbf{G}^F$ and $\mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F$ (see Definition 2.1).

Assumption 4.1. Let $(\mathbf{L}, \lambda)$ be an e -cuspidal pair of $\mathbf{G}$, set

$$\mathcal{G} := \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) \quad \text{and} \quad \mathcal{N} := \text{Irr}(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \mid \lambda)$$

and consider

$$\tilde{\mathcal{G}} := \text{Irr}(\tilde{\mathbf{G}}^F \mid \mathcal{G}) \quad \text{and} \quad \tilde{\mathcal{N}} := \text{Irr}(\mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F \mid \mathcal{N}).$$

Assume that:

- (i) (a) There is a semidirect decomposition $\tilde{\mathbf{G}}^F \rtimes \mathcal{A}$, with $\mathcal{A}$ a finite abelian group, such that

$$\mathbf{C}_{\tilde{\mathbf{G}}^F \mathcal{A}}(\mathbf{G}^F) = \mathbf{Z}(\tilde{\mathbf{G}}^F) \quad \text{and} \quad \tilde{\mathbf{G}}^F \mathcal{A} / \mathbf{Z}(\tilde{\mathbf{G}}^F) \simeq \text{Aut}_{\mathbb{F}}(\mathbf{G}^F);$$

- (b) Maximal extendibility holds with respect to $\mathbf{G}^F \trianglelefteq \tilde{\mathbf{G}}^F$;

- (c) Maximal extendibility holds with respect to $\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F$.

- (ii) For $A := (\tilde{\mathbf{G}}^F \mathcal{A})_{(\mathbf{L}, \lambda)}$ there exists a defect preserving $(A \rtimes \mathcal{K})$ -equivariant bijection

$$\tilde{\Omega}_{(\mathbf{L}, \lambda)}^{\mathbf{G}} : \tilde{\mathcal{G}} \rightarrow \tilde{\mathcal{N}}$$

such that $\text{Irr}(\tilde{\chi}_{\mathbf{Z}(\tilde{\mathbf{G}}^F)}) = \text{Irr}(\tilde{\Omega}_{(\mathbf{L}, \lambda)}^{\mathbf{G}}(\tilde{\chi})_{\mathbf{Z}(\tilde{\mathbf{G}}^F)})$ for every $\tilde{\chi} \in \tilde{\mathcal{G}}$.

- (iii) For every $\tilde{\chi} \in \tilde{\mathcal{G}}$ there exists $\chi \in \mathcal{G} \cap \text{Irr}(\tilde{\chi}_{\mathbf{G}^F})$ such that:

- (a) $(\tilde{\mathbf{G}}^F \mathcal{A})_{\chi} = \tilde{\mathbf{G}}_{\chi}^F \mathcal{A}_{\chi}$;

- (b) χ extends to $\chi' \in \text{Irr}(\mathbf{G}^F \mathcal{A}_{\chi})$.

- (iv) For every $\tilde{\psi} \in \tilde{\mathcal{N}}$ there exists $\psi \in \mathcal{N} \cap \text{Irr}(\tilde{\psi}_{\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F})$ such that:

- (a) $(\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}, \psi} = \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})_{\psi}^F (\mathbf{G}^F \mathcal{A})_{\mathbf{L}, \psi}$;

- (b) ψ extends to $\psi' \in \text{Irr}((\mathbf{G}^F \mathcal{A})_{\mathbf{L}, \psi})$.

Our aim is to show that Assumption 4.1 implies Parametrisation 2.7. Before giving a proof of this result, we show that Assumption 4.1 (iii.a) and Assumption 4.1 (iv.a) are equivalent in the presence of an equivariant bijection $\Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}} : \mathcal{G} \rightarrow \mathcal{N}$.

Lemma 4.2. Assume Hypothesis 2.4. Let $(\mathbf{L}, \lambda)$ be an e -cuspidal pair of $\mathbf{G}$ and suppose that there exists a $(\tilde{\mathbf{G}}^F \mathcal{A})_{(\mathbf{L}, \lambda)}$ -equivariant bijection

$$\Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}} : \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) \rightarrow \text{Irr}(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \mid \lambda).$$

If $\chi \in \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ and $\psi := \Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}}(\chi)$, then

$$(\tilde{\mathbf{G}}^F \mathcal{A})_{\chi} = \tilde{\mathbf{G}}_{\chi}^F \mathcal{A}_{\chi} \quad (4.1)$$

if and only if

$$(\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}, \psi} = \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})_{\psi}^F (\mathbf{G}^F \mathcal{A})_{\mathbf{L}, \psi}. \quad (4.2)$$

Proof. As the two implications can be shown by similar arguments, we only show that (4.1) implies (4.2). To start, consider the subgroups

$$T := \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \left(\tilde{\mathbf{G}}_{(\mathbf{L}, \lambda), \chi}^F \cdot (\mathbf{G}^F \mathcal{A})_{(\mathbf{L}, \lambda), \chi} \right) = \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \left(\tilde{\mathbf{G}}_{(\mathbf{L}, \lambda), \psi}^F \cdot (\mathbf{G}^F \mathcal{A})_{(\mathbf{L}, \lambda), \psi} \right)$$

and

$$V := \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F (\tilde{\mathbf{G}}^F \mathcal{A})_{(\mathbf{L}, \lambda), \chi} = \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F (\tilde{\mathbf{G}}^F \mathcal{A})_{(\mathbf{L}, \lambda), \psi},$$

where the equalities follow since $\Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}}$ is equivariant by assumption.

Define $U(\chi) := (\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}, \chi}$ and $U(\psi) := (\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}, \psi}$. We claim that $U(\chi) = U(\psi)$. To prove this fact, notice that it is enough to show that $U(\chi)$ and $U(\psi)$ are contained in V , in fact this would imply $U(\chi) = U(\chi) \cap V = (\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}} \cap V = U(\psi) \cap V = U(\psi)$. If $x \in U(\chi)$, then $\chi \in \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) \cap \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)^x)$ and, by [Ros24a, Proposition 4.10], there exists $y \in \mathbf{G}^F$ such that $(\mathbf{L}, \lambda) = (\mathbf{L}, \lambda)^{xy}$. Notice that $y \in \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F$ and hence $x \in V$. On the other hand, if $x \in U(\psi)$, then ψ lies over λ^x and by Clifford's theorem $\lambda^{xy} = \lambda$, for some $y \in \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F$. Also in this case $x \in V$. Now $U(\chi) = U(\psi)$ and we denote this group by U .

Next, we claim that $T = U$. If this is true, then we deduce that $T \leq \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})_{\psi}^F (\mathbf{G}^F \mathcal{A})_{\mathbf{L}, \psi} \leq U = T$ and therefore (4.2) holds. First, observe that $T \leq U$. As $T \cap \mathbf{G}^F = \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F = U \cap \mathbf{G}^F$ and $T \leq U \leq (\tilde{\mathbf{G}}^F \mathcal{A})_{\chi}$, it is enough to show that $T \mathbf{G}^F = (\tilde{\mathbf{G}}^F \mathcal{A})_{\chi}$. First, repeating the same argument as before, a Frattini argument shows that

$$(\tilde{\mathbf{G}}^F \mathcal{A})_{\chi} = \mathbf{G}^F (\tilde{\mathbf{G}}^F \mathcal{A})_{(\mathbf{L}, \lambda), \chi} \quad (4.3)$$

and

$$\tilde{\mathbf{G}}_{\chi}^F = \mathbf{G}^F \tilde{\mathbf{G}}_{(\mathbf{L}, \lambda), \chi}^F. \quad (4.4)$$

Then using the hypothesis we finally obtain

$$\begin{aligned}
 \left(\widetilde{\mathbf{G}}^F \mathcal{A} \right)_\chi &\stackrel{(4.3)}{=} \mathbf{G}^F \left(\widetilde{\mathbf{G}}^F \mathcal{A} \right)_{(\mathbf{L}, \lambda), \chi} \\
 &\stackrel{(4.1)}{=} \mathbf{G}^F \left(\widetilde{\mathbf{G}}_\chi^F \left(\mathbf{G}^F \mathcal{A} \right)_\chi \right)_{(\mathbf{L}, \lambda)} \\
 &\stackrel{(4.4)}{=} \mathbf{G}^F \left(\mathbf{G}^F \widetilde{\mathbf{G}}_{(\mathbf{L}, \lambda), \chi}^F \left(\mathbf{G}^F \mathcal{A} \right)_\chi \right)_{(\mathbf{L}, \lambda)} \\
 &= \mathbf{G}^F \widetilde{\mathbf{G}}_{(\mathbf{L}, \lambda), \chi}^F \left(\mathbf{G}^F \mathcal{A} \right)_{(\mathbf{L}, \lambda), \chi} \\
 &= \mathbf{G}^F T.
 \end{aligned}$$

This concludes the proof. $\square$

We are now ready to prove the criterion for Parametrisation 2.7. It should be clear from the proof of this result that, by using Lemma 4.2, only one amongst Assumption 4.1 (iii.a) and Assumption 4.1 (iv.a) is actually necessary. In fact, the equivariant map required in Lemma 4.2 is constructed in the following proof independently from the choices of characters satisfying Assumption 4.1 (iii.a) and Assumption 4.1 (iv.a).

Theorem 4.3. *Assume Hypothesis 2.4 and Assumption 4.1 with respect to an e -cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$. Then Parametrisation 2.7 holds for $(\mathbf{L}, \lambda)$ and $\mathbf{G}$.*

Proof. We start by fixing an $(A \ltimes \mathcal{K})$ -transversal $\widetilde{\mathbb{T}}_{\text{glo}}$ in $\widetilde{\mathcal{G}}$. As $\widetilde{\Omega}_{(\mathbf{L}, \lambda)}^{\mathbf{G}}$ is $(A \ltimes \mathcal{K})$ -equivariant, we deduce that the set $\widetilde{\mathbb{T}}_{\text{loc}} := \{ \widetilde{\Omega}_{(\mathbf{L}, \lambda)}^{\mathbf{G}}(\widetilde{\chi}) \mid \widetilde{\chi} \in \widetilde{\mathbb{T}}_{\text{glo}} \}$ is an $(A \ltimes \mathcal{K})$ -transversal in $\widetilde{\mathcal{N}}$. For every $\widetilde{\chi} \in \widetilde{\mathbb{T}}_{\text{glo}}$, we choose a character $\chi \in \mathcal{G} \cap \text{Irr}(\widetilde{\chi}_{\mathbf{G}^F})$ satisfying Assumption 4.1 (iii). Denote by $\mathbb{T}_{\text{glo}}$ the set of such characters χ , where $\widetilde{\chi}$ runs over $\widetilde{\mathbb{T}}_{\text{glo}}$. Similarly, for every $\widetilde{\psi} \in \widetilde{\mathbb{T}}_{\text{loc}}$, fix a character $\psi \in \mathcal{N} \cap \text{Irr}(\widetilde{\psi}_{\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F})$ satisfying Assumption 4.1 (iv) and denote by $\mathbb{T}_{\text{loc}}$ the set of such characters ψ . Now, arguing as in the proof of Theorem 3.20, we obtain an A -equivariant bijection between $\mathcal{G}$ and $\mathcal{N}$ by setting

$$\Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}}(\chi^x) := \psi^x$$

for every $x \in A$ and $\chi \in \mathbb{T}_{\text{glo}}$, where ψ is the unique character in $\mathbb{T}_{\text{loc}}$ lying below $\widetilde{\psi} := \widetilde{\Omega}_{(\mathbf{L}, \lambda)}^{\mathbf{G}}(\widetilde{\chi})$ and $\widetilde{\chi}$ is the unique character in $\widetilde{\mathbb{T}}_{\text{glo}}$ lying over χ . By Assumption 4.1 (i.a) this means that $\Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}}$ is $\text{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{(\mathbf{L}, \lambda)}$ -equivariant.

To show that $\Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}}$ preserves the defect, we use Assumption 4.1 (i.b) and (i.c). Clearly it's enough to show that $d(\chi) = d(\psi)$, for $\chi \in \mathbb{T}_{\text{glo}}$ and $\psi := \Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}}(\chi) \in \mathbb{T}_{\text{loc}}$. Let $\widetilde{\chi}$ (resp. $\widetilde{\psi}$) be the unique element of $\widetilde{\mathbb{T}}_{\text{glo}}$ (resp. $\widetilde{\mathbb{T}}_{\text{loc}}$) lying over χ (resp. ψ). Then $\widetilde{\Omega}_{(\mathbf{L}, \lambda)}^{\mathbf{G}}(\widetilde{\chi}) = \widetilde{\psi}$ and $d(\widetilde{\chi}) = d(\widetilde{\psi})$ by Assumption 4.1 (ii). Moreover, since $\widetilde{\mathbf{G}}^F / \mathbf{G}^F \simeq \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F / \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F$ is abelian and using Assumption 4.1 (i.b) and (i.c), we deduce that the Clifford correspondent $\widehat{\chi} \in \text{Irr}(\widetilde{\mathbf{G}}_\chi^F)$ of $\widetilde{\chi}$ over χ is an extension of χ and, similarly, that the Clifford correspondent $\widehat{\psi} \in \text{Irr}(\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})_\psi^F)$ of $\widetilde{\psi}$ over ψ is an extension of ψ . As a consequence

$$\ell^{d(\chi)} = \ell^{d(\widehat{\chi})} \cdot \left| \widetilde{\mathbf{G}}_\chi^F : \mathbf{G}^F \right|_\ell$$

and

$$\ell^{d(\psi)} = \ell^{d(\widehat{\psi})} \cdot \left| \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})_\psi^F : \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \right|_\ell.$$

Therefore, as the defect is preserved by induction of characters, we obtain $d(\widehat{\chi}) = d(\widetilde{\chi}) = d(\widetilde{\psi}) = d(\widehat{\psi})$ and it remains to show that $|\widetilde{\mathbf{G}}_{\chi}^F : \mathbf{G}^F|_{\ell} = |\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})_{\psi}^F : \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F|_{\ell}$. This follows from the proof of Lemma 4.2: in fact there it is shown that $\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})_{\psi}^F = \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})_{\chi}^F$ and therefore $\widetilde{\mathbf{G}}_{\chi}^F / \mathbf{G}^F \simeq \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})_{\chi}^F / \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F = \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})_{\psi}^F / \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F$.

Next, we prove the condition on character triples. Applying a simplified version of [Spä17, Theorem 5.3] adapted to $\mathbf{G}^F$ -central isomorphic character triples (this immediately follows by the argument used in the proof of [Spä17, Theorem 5.3]), it is enough to show that

$$\left((\widetilde{\mathbf{G}}^F \mathcal{A})_{\chi}, \mathbf{G}^F, \chi \right) \sim_{\mathbf{G}^F}^c \left((\widetilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}, \psi}, \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F, \Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}}(\chi) \right). \quad (4.5)$$

Moreover, as the equivalence relation $\sim_{\mathbf{G}^F}^c$ is compatible with conjugation, it's enough to prove this condition for a fixed $\chi \in \mathbb{T}_{\text{glo}}$ and $\psi := \Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}}(\chi) \in \mathbb{T}_{\text{loc}}$.

First of all, notice that the required group theoretical properties are satisfied by the proof of Lemma 4.2. In fact, there we have shown that $(\widetilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}, \chi} = (\widetilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}, \psi}$ and that $(\widetilde{\mathbf{G}}^F \mathcal{A})_{\chi} = \mathbf{G}^F (\widetilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}, \chi}$, while

$$\mathbf{C}_{(\widetilde{\mathbf{G}}^F \mathcal{A})_{\chi}}(\mathbf{G}^F) \leq \mathbf{C}_{(\widetilde{\mathbf{G}}^F \mathcal{A})_{\chi}}(\mathbf{L}^F) \leq (\widetilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}, \chi} = (\widetilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}, \psi}.$$

To construct the relevant projective representations, we make use of [Spä12, Lemma 2.11]. As before, consider the corresponding $\widetilde{\chi} \in \mathbb{T}_{\text{glo}}$ and $\widetilde{\psi} \in \mathbb{T}_{\text{loc}}$ with $\widetilde{\Omega}_{(\mathbf{L}, \lambda)}^{\mathbf{G}}(\widetilde{\chi}) = \widetilde{\psi}$, $\widetilde{\chi}$ lying over χ and $\widetilde{\psi}$ lying over ψ . Furthermore, consider the Clifford correspondent $\widehat{\chi} \in \text{Irr}(\widetilde{\mathbf{G}}_{\chi}^F | \chi)$ of $\widetilde{\chi}$ and the Clifford correspondent $\widehat{\psi} \in \text{Irr}(\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})_{\psi}^F | \psi)$ of $\widetilde{\psi}$. Let $\widehat{\mathcal{D}}_{\text{glo}}$ be a representation affording $\widehat{\chi}$ and notice that, by the choice of χ and using Assumption 4.1 (iii.b), there exists a representation $\mathcal{D}'_{\text{glo}}$ affording an extension $\chi' \in \text{Irr}(\mathbf{G}^F \mathcal{A}_{\chi})$ of χ . Additionally, we may choose $\mathcal{D}'_{\text{glo}}$ such that $\mathcal{D}'_{\text{glo}}(x) = \widehat{\mathcal{D}}_{\text{glo}}(x)$ for all $x \in \mathbf{G}^F$. Similarly, let $\widehat{\mathcal{D}}_{\text{loc}}$ be a representation affording $\widehat{\psi}$ and observe that, by the choice of ψ , there is a representation $\mathcal{D}'_{\text{loc}}$ affording an extension $\psi' \in \text{Irr}((\mathbf{G}^F \mathcal{A})_{\mathbf{L}, \psi})$ of ψ . Also here, we may assume that $\widehat{\mathcal{D}}_{\text{loc}}(x) = \mathcal{D}'_{\text{loc}}(x)$ for all $x \in \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F$. Applying [Spä12, Lemma 2.11] with $L := \mathbf{G}^F$, $\widetilde{L} := \widetilde{\mathbf{G}}_{\chi}^F$, $C := \mathbf{G}^F \mathcal{A}_{\chi}$, $X := (\widetilde{\mathbf{G}}^F \mathcal{A})_{\chi}$ and recalling that $X = \widetilde{L}C$ because Assumption 4.1 (iii.a) holds for χ , we deduce that the map

$$\mathcal{P}_{\text{glo}} : (\widetilde{\mathbf{G}}^F \mathcal{A})_{\chi} \rightarrow \text{GL}_{\chi(1)}(\mathbb{C})$$

given by $\mathcal{P}_{\text{glo}}(x_1 x_2) := \widehat{\mathcal{D}}_{\text{glo}}(x_1) \mathcal{D}'_{\text{glo}}(x_2)$, for every $x_1 \in \widetilde{\mathbf{G}}_{\chi}^F$ and $x_2 \in \mathbf{G}^F \mathcal{A}_{\chi}$, is a projective representation associated with χ whose factor set α_{glo} satisfies

$$\alpha_{\text{glo}}(x_1 x_2, y_1 y_2) = \mu_{x_2}^{\text{glo}}(y_1) \quad (4.6)$$

for every $x_1, y_1 \in \widetilde{\mathbf{G}}_{\chi}^F$ and $x_2, y_2 \in \mathbf{G}^F \mathcal{A}_{\chi}$, where $\mu_{x_2}^{\text{glo}} \in \text{Irr}(\widetilde{\mathbf{G}}_{\chi}^F / \mathbf{G}^F)$ is determined by the equality $\widehat{\chi} = \mu_{x_2}^{\text{glo}} \widehat{\chi}^{x_2}$ via Gallagher's theorem. In a similar way, considering $L := \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F$, $\widetilde{L} := \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})_{\chi}^F$, $C := (\mathbf{G}^F \mathcal{A})_{\mathbf{L}, \chi}$, $X := (\widetilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}, \chi}$ and noticing that $X = \widetilde{L}C$ because Assumption 4.1 (iv.a) holds for ψ , we deduce that the map

$$\mathcal{P}_{\text{loc}} : (\widetilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}, \chi} \rightarrow \text{GL}_{\psi(1)}(\mathbb{C})$$

given by $\mathcal{P}_{\text{loc}}(x_1 x_2) := \widehat{\mathcal{D}}_{\text{loc}}(x_1) \mathcal{D}'_{\text{loc}}(x_2)$, for every $x_1 \in \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})_{\chi}^F$ and $x_2 \in (\mathbf{G}^F \mathcal{A})_{\mathbf{L}, \chi}$, is a projective representation associated with ψ whose factor set α_{loc} satisfies

$$\alpha_{\text{loc}}(x_1 x_2, y_1 y_2) = \mu_{x_2}^{\text{loc}}(y_1) \quad (4.7)$$

for every $x_1, y_1 \in \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})_{\chi}^F$ and $x_2, y_2 \in (\mathbf{G}^F \mathcal{A})_{\mathbf{L}, \chi}$, where $\mu_{x_2}^{\text{loc}} \in \text{Irr}(\mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})_{\chi}^F / \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F)$ is determined by $\widehat{\psi} = \mu_{x_2}^{\text{loc}} \widehat{\psi}^{x_2}$. In order to obtain the condition on factor sets required to prove (4.5) we have to show that the restriction of α_{glo} to $(\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}, \chi} \times (\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}, \chi}$ coincides with α_{loc} . Using (4.6) and (4.7), it is enough to show that

$$\left(\mu_x^{\text{glo}} \right)_{\mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})_{\chi}^F} = \mu_x^{\text{loc}}$$

for every $x \in (\mathbf{G}^F \mathcal{A})_{\mathbf{L}, \chi}$ and where $\widehat{\chi} = \mu_x^{\text{glo}} \widehat{\chi}^x$ and $\widehat{\psi} = \mu_x^{\text{loc}} \widehat{\psi}^x$. To prove this equality, since $(\mathbf{G}^F \mathcal{A})_{\mathbf{L}, \chi} = \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F A_{\chi}$ (see the proof of Lemma 4.2), we may assume $x \in A_{\chi}$. Then, we conclude since $\tilde{\Omega}_{(\mathbf{L}, \lambda)}^{\mathbf{G}}$ is $(A \ltimes \mathcal{K})$ -equivariant.

To conclude we need to check one of the equivalent conditions of [Spä17, Lemma 3.4]. Recalling that $\mathbf{C}_{(\tilde{\mathbf{G}}^F \mathcal{A})_{\chi}}(\mathbf{G}^F) = \mathbf{Z}(\tilde{\mathbf{G}}^F)$ by Assumption 4.1 (i.a), if ζ_{glo} and ζ_{loc} are the scalar functions of $\mathcal{P}_{\text{glo}}$ and $\mathcal{P}_{\text{loc}}$ respectively, we have to show that ζ_{glo} and ζ_{loc} coincide as characters of $\mathbf{Z}(\tilde{\mathbf{G}}^F)$. By the definition of $\mathcal{P}_{\text{glo}}$, it follows that ζ_{glo} coincides with the unique irreducible constituent ν of $\widehat{\chi}_{\mathbf{Z}(\tilde{\mathbf{G}}^F)}$. Moreover, by Clifford theory we know that ν is also the unique irreducible constituent of $\widehat{\chi}_{\mathbf{Z}(\tilde{\mathbf{G}}^F)}$. Therefore, we conclude that $\{\zeta_{\text{glo}}\} = \text{Irr}(\widehat{\chi}_{\mathbf{Z}(\tilde{\mathbf{G}}^F)})$ and a similar argument shows that $\{\zeta_{\text{loc}}\} = \text{Irr}(\widehat{\psi}_{\mathbf{Z}(\tilde{\mathbf{G}}^F)})$. Then, Assumption 4.1 (ii) implies that $\zeta_{\text{glo}} = \zeta_{\text{loc}}$. This completes the proof. $\square$

4.2. The criterion for Parametrisation C

We now prove a criterion for Parametrisation C. To do so, we sharpen the argument used in the proof of Theorem 4.3. As mentioned at the beginning of this section, some additional restrictions are required in order to deal with the necessary block theoretic demands. Recall that whenever $\ell \in \Gamma(\mathbf{G}, F)$, $\mathbf{L}$ is an e -split Levi subgroup of $\mathbf{G}$ and $\lambda \in \text{Irr}(\mathbf{L}^F)$, the induced block $\text{bl}(\lambda)^{\mathbf{G}^F}$ is defined (see the comment preceding [Ros24a, Lemma 4.6]). The following assumption is obtained by adding extra conditions to those considered in Assumption 4.1, namely, (ii.b), (v) and (vi) below. For the reader's convenience, we have decided to rewrite below the full set of conditions required to prove our criterion for Parametrization C. This would also be useful for future reference.

Assumption 4.4. Let $(\mathbf{L}, \lambda)$ be an e -cuspidal pair of $\mathbf{G}$ and suppose that $B := \text{bl}(\lambda)^{\mathbf{G}^F}$ is defined. Set

$$\mathcal{G} := \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) \quad \text{and} \quad \mathcal{N} := \text{Irr}(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \mid \lambda)$$

and consider

$$\tilde{\mathcal{G}} := \text{Irr}(\tilde{\mathbf{G}}^F \mid \mathcal{G}) \quad \text{and} \quad \tilde{\mathcal{N}} := \text{Irr}(\mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F \mid \mathcal{N}).$$

Assume that:

- (i) (a) There is a semidirect decomposition $\tilde{\mathbf{G}}^F \rtimes \mathcal{A}$, with $\mathcal{A}$ a finite abelian group, such that

$$\mathbf{C}_{(\tilde{\mathbf{G}}^F \mathcal{A})}(\mathbf{G}^F) = \mathbf{Z}(\tilde{\mathbf{G}}^F) \quad \text{and} \quad (\tilde{\mathbf{G}}^F \mathcal{A}) / \mathbf{Z}(\tilde{\mathbf{G}}^F) \simeq \text{Aut}_{\mathbb{F}}(\mathbf{G}^F);$$

- (b) Maximal extendibility holds with respect to $\mathbf{G}^F \trianglelefteq \tilde{\mathbf{G}}^F$;
 (c) Maximal extendibility holds with respect to $\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F$.

(ii) For $A := (\tilde{\mathbf{G}}^F \mathcal{A})_{(\mathbf{L}, \lambda)}$ there exists a defect preserving $(A \rtimes \mathcal{K})$ -equivariant bijection

$$\tilde{\Omega}_{(\mathbf{L}, \lambda)}^{\mathbf{G}} : \tilde{\mathcal{G}} \rightarrow \tilde{\mathcal{N}}$$

such that, for every $\tilde{\chi} \in \tilde{\mathcal{G}}$, the following conditions hold:

$$(a) \operatorname{Irr}(\tilde{\chi}_{\mathbf{Z}(\tilde{\mathbf{G}}^F)}) = \operatorname{Irr}(\tilde{\Omega}_{(\mathbf{L}, \lambda)}^{\mathbf{G}}(\tilde{\chi})_{\mathbf{Z}(\tilde{\mathbf{G}}^F)});$$

$$(b) \operatorname{bl}(\tilde{\chi}) = \operatorname{bl}(\tilde{\Omega}_{(\mathbf{L}, \lambda)}^{\mathbf{G}}(\tilde{\chi}))^{\tilde{\mathbf{G}}^F}.$$

(iii) For every $\tilde{\chi} \in \tilde{\mathcal{G}}$ there exists $\chi \in \operatorname{Irr}(\tilde{\chi}_{\mathbf{G}^F})$ such that:

$$(a) (\tilde{\mathbf{G}}^F \mathcal{A})_{\chi} = \tilde{\mathbf{G}}_{\chi}^F \mathcal{A}_{\chi};$$

$$(b) \chi \text{ extends to } \chi' \in \operatorname{Irr}(\mathbf{G}^F \mathcal{A}_{\chi}).$$

(iv) For every $\tilde{\psi} \in \tilde{\mathcal{N}}$ there exists $\psi \in \mathcal{N} \cap \operatorname{Irr}(\tilde{\psi}_{\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F})$ such that:

$$(a) (\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}, \psi} = \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})_{\psi}^F (\mathbf{G}^F \mathcal{A})_{\mathbf{L}, \psi};$$

$$(b) \psi \text{ extends to } \psi' \in \operatorname{Irr}((\mathbf{G}^F \mathcal{A})_{\mathbf{L}, \psi}).$$

(v) Assume one of the following conditions:

(a) $\operatorname{Out}(\mathbf{G}^F)_{\mathcal{B}}$ is abelian, where $\mathcal{B}$ is the $\tilde{\mathbf{G}}^F$ -orbit of B . In particular (iii) holds for every $\tilde{\mathbf{G}}^F$ -conjugate of χ (see the proof of [BS22, Lemma 4.7]); or

(b) for every subgroup $\mathbf{G}^F \leq J \leq \tilde{\mathbf{G}}^F$ we have that every block $C \in \operatorname{Bl}(J \mid B)$ is $\tilde{\mathbf{G}}^F$ -invariant.

(vi) The pair $(\mathbf{L}, \lambda)$ is e -Brauer–Lusztig–cuspidal in the sense of Definition 2.5.

Remark 4.5. Here we comment on Assumption 4.4. First, observe that (v.a) holds for every block of $\mathbf{G}^F$ whenever $\mathbf{G}$ is a simple algebraic group of type $\mathbf{B}$, $\mathbf{C}$ or $\mathbf{E}_7$. Next, notice that condition (v.b) holds for blocks of maximal defect (see [CS15, Proposition 5.4] and observe that the proof of this result holds in general in our situation by [CE04, Proposition 13.19]) and for unipotent blocks: if B is a unipotent block of $\mathbf{G}^F$, then there exists a unipotent character $\chi \in \operatorname{Irr}(B)$. By [DM91, Proposition 13.20] we deduce that χ extends to a character $\tilde{\chi} \in \operatorname{Irr}(\tilde{\mathbf{G}}^F)$. If $\mathbf{G}^F \leq J \leq \tilde{\mathbf{G}}^F$ and C is a block of J that covers B , then we can find a character $\psi \in \operatorname{Irr}(C)$ that lies above χ . Since $\tilde{\chi}_J$ is an irreducible character of J lying above χ , we deduce that $\psi = \tilde{\chi}_J \tilde{z}_J$ for some $z \in \mathcal{K}$ corresponding to $\tilde{z}_{\tilde{\mathbf{G}}} \in \operatorname{Irr}(\tilde{\mathbf{G}}^F / \mathbf{G}^F)$ and where $\tilde{z}_J$ is the restriction of $\tilde{z}_{\tilde{\mathbf{G}}}$ to J . Then ψ is $\tilde{\mathbf{G}}^F$ -invariant and therefore C is $\tilde{\mathbf{G}}^F$ -invariant. This proves that (v.b) holds for unipotent blocks.

Next, we point out that the character χ from Assumption 4.4 (iii) is not required to lie in $\mathcal{G}$. In fact, if such a character χ exists, then a character with the same properties and lying in $\mathcal{G}$ can always be found under Assumption 4.4 (v)–(vi). To see this, fix $\tilde{\chi} \in \tilde{\mathcal{G}}$ and $\chi \in \operatorname{Irr}(\tilde{\chi}_{\mathbf{G}^F})$ satisfying Assumption 4.4 (iii). By the definition of $\tilde{\mathcal{G}}$ there exists $\chi_0 \in \operatorname{Irr}(\tilde{\chi}_{\mathbf{G}^F}) \cap \mathcal{G}$. In particular χ and χ_0 are $\mathbf{G}^F$ -conjugate. Now, if (v.a) holds, then all $\tilde{\mathbf{G}}^F$ -conjugates of χ satisfy Assumption 4.4 (iii.a) and (iii.b) according to the proof of [BS22, Lemma 4.7]. Then χ_0 is the character we were looking for. If (v.b) holds, then B is $\tilde{\mathbf{G}}^F$ -invariant and, since $\operatorname{bl}(\chi_0) = B$, we deduce that $\operatorname{bl}(\chi) = B$. On the other hand, if s is a semisimple element of $\mathbf{L}^{*F^*}$ such that $\lambda \in \mathcal{E}(\mathbf{L}^F, [s])$, then $\chi_0 \in \mathcal{G} \subseteq \mathcal{E}(\mathbf{G}^F, [s])$ by [CE04, Proposition 15.7]. Thus $\chi \in \mathcal{E}(\mathbf{G}^F, [s])$ by [CE04, Proposition 15.6] and we conclude that $\chi \in \operatorname{Irr}(B) \cap \mathcal{E}(\mathbf{G}^F, [s]) = \mathcal{G}$ by applying Assumption 4.4 (vi).

Finally, as discussed in Section 2.4, it is expected that Assumption 4.4 (vi) holds for every e -cuspidal pair under suitable hypotheses on the prime ℓ .

We now prove the criterion for Parametrisation C. Our argument makes use of the notion of Dade’s *ramification group*. For every block b of a normal subgroup N of G , Dade introduced a normal subgroup $G[b]$ of the subgroup G_b such that $G[b] \leq G_{\chi}$ for every $\chi \in \operatorname{Irr}(b)$. Here we use the following equivalent definition given by Murai in [Mur13] (see also [CS15, Definition 3.1]).

Definition 4.6. For every $N \trianglelefteq G$ and $b \in \text{Bl}(G)$ define

$$G[b] := \{g \in G_b \mid \lambda_{b^{(g)}}(\mathfrak{C}\mathfrak{I}_{\langle N, g \rangle}(h)^+) \neq 0, \text{ for some } h \in Ng\}$$

where $b^{(g)}$ is any block of $\langle N, g \rangle$ covering b , $\lambda_{b^{(g)}}$ is the central character associated to $b^{(g)}$ and $\mathfrak{C}\mathfrak{I}_{\langle N, g \rangle}(h)^+$ is the conjugacy class sum of h in $\langle N, g \rangle$. It can be shown that this definition does not depend on the choices of the blocks $b^{(g)}$ covering b .

See [Dad73], [Mur13] and [KS15] for further details on ramification groups.

Before proving the criterion for Parametrisation C, we need the following result in which we show how to choose transversals with good properties.

Proposition 4.7. Assume Hypothesis 2.4 and Assumption 4.4. Let $\widetilde{\mathbb{T}}_{\text{glo}}$ be any $(A \ltimes \mathcal{K})$ -transversal in $\widetilde{\mathcal{G}}$ and consider the $(A \ltimes \mathcal{K})$ -transversal $\widetilde{\mathbb{T}}_{\text{loc}} := \{\widetilde{\Omega}_{(\mathbf{L}, \chi)}^{\mathbf{G}}(\widetilde{\chi}) \mid \widetilde{\chi} \in \widetilde{\mathbb{T}}_{\text{glo}}\}$ in $\widetilde{\mathcal{N}}$. Then there exist A -transversals $\mathbb{T}_{\text{glo}}$ in $\mathcal{G}$ and $\mathbb{T}_{\text{loc}}$ in $\mathcal{N}$ with the following properties:

- (i) Every $\chi \in \mathbb{T}_{\text{glo}}$ satisfies Assumption 4.4 (iii.a) and (iii.b);
- (ii) Every $\psi \in \mathbb{T}_{\text{loc}}$ satisfies Assumption 4.4 (iv.a) and (iv.b);
- (iii) For every $\chi \in \mathbb{T}_{\text{glo}}$ there exists a unique $\widetilde{\chi} \in \widetilde{\mathbb{T}}_{\text{glo}}$ lying over χ . Conversely χ is the only character of $\mathbb{T}_{\text{glo}}$ lying under $\widetilde{\chi}$;
- (iv) For every $\psi \in \mathbb{T}_{\text{loc}}$ there exists a unique $\widetilde{\psi} \in \widetilde{\mathbb{T}}_{\text{loc}}$ lying over ψ . Conversely ψ is the only character of $\mathbb{T}_{\text{loc}}$ lying under $\widetilde{\psi}$;
- (v) Let $\chi \in \mathbb{T}_{\text{glo}}$ and $\psi \in \mathbb{T}_{\text{loc}}$ such that $\widetilde{\Omega}_{(\mathbf{L}, \chi)}^{\mathbf{G}}(\widetilde{\chi}) = \widetilde{\psi}$, where $\widetilde{\chi}$ is the unique character of $\widetilde{\mathbb{T}}_{\text{glo}}$ lying above χ and $\widetilde{\psi}$ is the unique character of $\widetilde{\mathbb{T}}_{\text{loc}}$ lying above ψ . Then

$$\text{bl}(\widehat{\chi}_J) = \text{bl}\left(\widehat{\psi}_{\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})_{\chi}^F \cap J}\right)^J$$

for every $\mathbf{G}^F \leq J \leq \widetilde{\mathbf{G}}^F$, where $\widehat{\chi} \in \text{Irr}(\widetilde{\mathbf{G}}_{\chi}^F)$ is the Clifford correspondent of $\widetilde{\chi}$ over χ and $\widehat{\psi} \in \text{Irr}(\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})_{\psi}^F)$ is the Clifford correspondent of $\widetilde{\psi}$ over ψ .

Proof. For every $\widetilde{\psi} \in \widetilde{\mathbb{T}}_{\text{loc}}$ fix a character $\psi \in \mathcal{N} \cap \text{Irr}(\widetilde{\psi}_{\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F})$ satisfying Assumption 4.4 (iv) and denote by $\mathbb{T}_{\text{loc}}$ the set of such characters ψ , while $\widetilde{\psi}$ runs over $\widetilde{\mathbb{T}}_{\text{loc}}$. As proved in Theorem 4.3, the set $\mathbb{T}_{\text{loc}}$ is an A -transversal in $\mathcal{N}$ satisfying (iv) above. Next, for every $\widetilde{\chi} \in \widetilde{\mathbb{T}}_{\text{glo}}$, we are going to find a character $\chi \in \mathcal{G} \cap \text{Irr}(\widetilde{\chi}_{\mathbf{G}^F})$ satisfying Assumption 4.4 (iii.a) and (iii.b) and such that

$$\text{bl}(\widehat{\chi}_J) = \text{bl}\left(\widehat{\psi}_{\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})_{\chi}^F \cap J}\right)^J \quad (4.8)$$

for every $\mathbf{G}^F \leq J \leq \widetilde{\mathbf{G}}_{\chi}^F$ and where $\widehat{\chi} \in \text{Irr}(\widetilde{\mathbf{G}}_{\chi}^F \mid \chi)$ is the Clifford correspondent of $\widetilde{\chi}$ over χ and $\widehat{\psi} \in \text{Irr}(\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})_{\psi}^F \mid \psi)$ is the Clifford correspondent of $\widetilde{\psi}$ over ψ with $\widetilde{\psi} := \widetilde{\Omega}_{(\mathbf{L}, \chi)}^{\mathbf{G}}(\widetilde{\chi})$ and $\psi \in \mathbb{T}_{\text{loc}}$ corresponding to $\widetilde{\psi}$. Then, as shown in the proof of Theorem 4.3, the set $\mathbb{T}_{\text{glo}}$ of such characters χ while $\widetilde{\chi}$ runs over $\widetilde{\mathbb{T}}_{\text{glo}}$ will be an A -transversal in $\mathcal{G}$ satisfying (iii) above. Moreover (v) will be satisfied by our choice.

We first prove the claim assuming Assumption 4.4 (v.a). We start by showing that, for every $\widetilde{\chi} \in \widetilde{\mathbb{T}}_{\text{glo}}$, there exists a character $\chi \in \mathcal{G} \cap \text{Irr}(\widetilde{\chi}_{\mathbf{G}^F})$ such that

$$\text{bl}\left(\widehat{\chi}_{\widetilde{\mathbf{G}}^F[B]}\right) = \text{bl}\left(\widehat{\psi}_{\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F[C]}\right)^{\widetilde{\mathbf{G}}^F[B]}, \quad (4.9)$$

where $\widehat{\chi} \in \text{Irr}(\widetilde{\mathbf{G}}_{\chi}^F \mid \chi)$ is the Clifford correspondent of $\widetilde{\chi}$ over χ and $\widehat{\psi} \in \text{Irr}(\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})_{\psi}^F \mid \psi)$ is the Clifford correspondent of $\widetilde{\psi}$ over ψ with $\widetilde{\psi} := \widetilde{\Omega}_{(\mathbf{L}, \lambda)}^{\mathbf{G}}(\widetilde{\chi})$ and $\psi \in \mathbb{T}_{\text{loc}}$ corresponding to $\widetilde{\psi}$ and $C := \text{bl}(\psi)$. Notice that, as pointed out in Remark 4.5, under Assumption 4.4 (v.a) such a character χ automatically satisfies Assumption 4.4 (iii.a) and (iii.b).

Set $b := \text{bl}(\lambda)$ and recall that, as every block of $\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F$ is $\mathbf{L}^F$ -regular (see [Ros24a, Lemma 5.5]), C must coincide with $b^{\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F}$ and therefore $C^{\mathbf{G}^F} = b^{\mathbf{G}^F} = B$. Moreover, for $E := \mathbf{Z}^{\circ}(\mathbf{L})_{\ell}^F$, we have $\mathbf{N}_X(\mathbf{L}) = \mathbf{N}_X(E)$ for every $\mathbf{G}^F \leq X \leq \widetilde{\mathbf{G}}^F$ (see [Ros24a, Lemma 2.5]). Then, for every $\mathbf{G}^F \leq Y \leq X \leq \widetilde{\mathbf{G}}^F$, every $C_0 \in \text{Bl}(\mathbf{N}_Y(\mathbf{L}))$ and $C_1 \in \text{Bl}(\mathbf{N}_X(\mathbf{L}) \mid C_0)$, the induced block $B_1 := C_1^X$ is well defined and covers C_0^Y (see [KS15, Theorem B]): in fact for a defect group $D \in \delta(C_0)$ we have $E \leq \mathbf{O}_{\ell}(\mathbf{N}_Y(\mathbf{L})^F) \leq D$ and hence $C_X(D) \leq \mathbf{N}_X(E) = \mathbf{N}_X(\mathbf{L})$.

Consider $\widetilde{C} := \text{bl}(\widetilde{\psi})$, $\widetilde{B} := \text{bl}(\widetilde{\chi})$ and recall that $\widetilde{B} = (\widetilde{C})^{\widetilde{\mathbf{G}}^F}$ by Assumption 4.4 (ii.b). Notice that $\widetilde{\mathbf{G}}^F[B] = \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F[C] \cdot \mathbf{G}^F$ (see [KS15, Lemma 3.2 (c) and Lemma 3.6]) and set $C_1 := \text{bl}(\widehat{\psi}_{\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F[C]})$ and $B_1 := C_1^{\widetilde{\mathbf{G}}^F[B]}$. By the previous paragraph (applied with $Y = \mathbf{G}^F$ and $X = \widetilde{\mathbf{G}}^F[B]$) the block B_1 covers B and the exact same argument (applied with $Y = \widetilde{\mathbf{G}}^F[B]$ and $X = \widetilde{\mathbf{G}}^F$) can be used to show that $\widetilde{B}$ covers B_1 . In particular there exists $\chi_1 \in \text{Irr}(B_1)$ lying under $\widetilde{\chi}$. We claim that $\chi_1, \mathbf{G}^F$ is irreducible and lies in $\mathcal{G}$. If χ is an irreducible constituent of $\chi_1, \mathbf{G}^F$, then B_1 covers $\text{bl}(\chi)$. As B is $\widetilde{\mathbf{G}}^F[B]$ -invariant, we conclude that $\text{bl}(\chi) = B$. Then $\widetilde{\mathbf{G}}^F[B] \leq \widetilde{\mathbf{G}}_{\chi}^F$ and Assumption 4.4 (i.b) implies that $\chi_1, \mathbf{G}^F = \chi$. Furthermore, since for every $\mathbf{G}^F \leq J \leq \widetilde{\mathbf{G}}_{\chi}^F$ there exists a unique irreducible character of J lying over χ and under $\widetilde{\chi}$, we conclude that $\chi_1 = \widehat{\chi}_{\widetilde{\mathbf{G}}^F[B]}$, where $\widehat{\chi} \in \text{Irr}(\widetilde{\mathbf{G}}_{\chi}^F)$ is the Clifford correspondent of $\widetilde{\chi}$ over χ . To conclude, since $\widetilde{\chi} \in \widetilde{\mathcal{G}}$ covers χ_1 and hence χ , [CE04, Proposition 15.6] implies that $\chi \in \mathcal{E}(\mathbf{G}^F, [s])$ where $s \in \mathbf{L}_{\text{ss}}^{*F}$ such that $\lambda \in \mathcal{E}(\mathbf{L}^F, [s])$. By Assumption 4.4 (vi) we conclude that $\chi \in \mathcal{G} \cap \text{Irr}(\widetilde{\chi}_{\mathbf{G}^F})$ and satisfies (4.9).

Next, we deduce (4.8) from (4.9). First, since $\text{bl}(\widehat{\psi}_{\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F[C]})$ is covered by $\text{bl}(\widehat{\psi})$, by the same argument used before (applied with $Y = \widetilde{\mathbf{G}}^F[B]$ and $X = \widetilde{\mathbf{G}}_{\chi}^F$) we deduce that $\text{bl}(\widehat{\psi}_{\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F[C]})^{\widetilde{\mathbf{G}}^F[B]} = \text{bl}(\widehat{\chi}_{\widetilde{\mathbf{G}}^F[B]})$ is covered by $\text{bl}(\widehat{\psi})^{\widetilde{\mathbf{G}}_{\chi}^F}$. Since $\widetilde{\mathbf{G}}_{\chi}^F$ has a unique block that covers $\text{bl}(\widehat{\chi}_{\widetilde{\mathbf{G}}^F[B]})$ (see [Mur13, Theorem 3.5]), we conclude that $\text{bl}(\widehat{\psi})^{\widetilde{\mathbf{G}}_{\chi}^F} = \text{bl}(\widehat{\chi})$. Finally, for $\mathbf{G}^F \leq J \leq \widetilde{\mathbf{G}}_{\chi}^F$, observe that $\text{bl}(\widehat{\chi}_J)$ is $\widetilde{\mathbf{G}}_{\chi}^F$ -stable and therefore it is the unique block of J covered by $\text{bl}(\widehat{\chi})$. Since, again by using the previous argument (applied with $Y = J$ and $X = \widetilde{\mathbf{G}}_{\chi}^F$), $\text{bl}(\widehat{\psi}_{\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})_{\chi}^F \cap J})^J$ is covered by $\text{bl}(\widehat{\psi})^{\widetilde{\mathbf{G}}_{\chi}^F} = \text{bl}(\widehat{\chi})$ we conclude that χ is a character of $\text{Irr}(\widetilde{\chi}_{\mathbf{G}^F}) \cap \mathcal{G}$ satisfying Assumption 4.4 (iii.a) and (iii.b) and such that (4.8) holds. This proves the claim under Assumption 4.4 (v.a).

We now prove the claim under Assumption 4.4 (v.b). Consider $\chi \in \text{Irr}(\widetilde{\chi}_{\mathbf{G}^F})$ satisfying Assumption 4.4 (iii) and notice that, as shown in Remark 4.5, under Assumption 4.4 (v.b) we automatically have $\chi \in \mathcal{G}$. As shown in the previous part, the block $\widehat{B} := \text{bl}(\widehat{\psi})^{\widetilde{\mathbf{G}}_{\chi}^F}$ is covered by $\widetilde{B} := \text{bl}(\widetilde{\chi})$ and covers B . Since $\widetilde{B}$ covers $\widehat{B}$, we deduce that $\widehat{B}$ and $\text{bl}(\widehat{\chi})$ are $\widetilde{\mathbf{G}}^F$ -conjugate. On the other hand our assumption implies that $\widehat{B}$ is $\widetilde{\mathbf{G}}^F$ -stable and therefore coincides with $\text{bl}(\widehat{\chi})$. This shows that $\text{bl}(\widehat{\chi}) = \text{bl}(\widehat{\psi})^{\widetilde{\mathbf{G}}_{\chi}^F}$ and, arguing as in the final part of the previous paragraph, we conclude that (4.8) holds. This completes the proof. $\square$

We can finally prove the criterion for Parametrisation C.

Theorem 4.8. *Assume Hypothesis 2.4 and Assumption 4.4 with respect to the e -cuspidal pair $(\mathbf{L}, \lambda)$. Then Parametrisation C holds for $(\mathbf{L}, \lambda)$ and $\mathbf{G}$.*

Proof. Choose transversals $\widetilde{\mathbb{T}}_{\text{glo}}, \widetilde{\mathbb{T}}_{\text{loc}}, \mathbb{T}_{\text{glo}}$ and $\mathbb{T}_{\text{loc}}$ as in Proposition 4.7. As in the proof of Theorem 4.3, setting

$$\Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}}(\chi^x) := \psi^x$$

for every $x \in A$ and $\chi \in \mathbb{T}_{\text{glo}}$, where ψ is the unique character in $\mathbb{T}_{\text{loc}}$ lying below $\tilde{\psi} := \tilde{\Omega}_{(\mathbf{L}, \lambda)}^{\mathbf{G}}(\tilde{\chi})$ and $\tilde{\chi}$ is the unique character in $\tilde{\mathbb{T}}_{\text{glo}}$ lying over χ , defines an A -equivariant bijection between $\mathcal{G}$ and $\mathcal{N}$. By Assumption 4.4 (i.a.) this means that $\Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}}$ is $\text{Aut}(\mathbf{G}^F)_{(\mathbf{L}, \lambda)}$ -equivariant.

The argument used in the proof of Theorem 4.3 shows that $\Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}}$ is defect preserving. By [Spä17, Theorem 5.3], we deduce that to conclude the proof it's enough to show that

$$\left((\tilde{\mathbf{G}}^F \mathcal{A})_{\chi}, \mathbf{G}^F, \chi \right) \sim_{\mathbf{G}^F} \left((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}, \chi}, \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F, \psi \right), \quad (4.10)$$

for every $\chi \in \mathcal{G}$, $\psi := \Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}}(\chi)$. Moreover, as the equivalence relation $\sim_{\mathbf{G}^F}$ is compatible with conjugation, it is enough to prove (4.10) for a fixed $\chi \in \mathbb{T}_{\text{glo}}$ and $\psi := \Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}}(\chi) \in \mathbb{T}_{\text{loc}}$. As before, consider the corresponding $\tilde{\chi} \in \tilde{\mathbb{T}}_{\text{glo}}$ and $\tilde{\psi} \in \tilde{\mathbb{T}}_{\text{loc}}$ with $\tilde{\Omega}_{(\mathbf{L}, \lambda)}^{\mathbf{G}}(\tilde{\chi}) = \tilde{\psi}$, $\tilde{\chi}$ lying over χ and $\tilde{\psi}$ lying over ψ . Furthermore, consider the Clifford correspondent $\hat{\chi} \in \text{Irr}(\tilde{\mathbf{G}}_{\chi}^F \mid \chi)$ of $\tilde{\chi}$ and the Clifford correspondent $\hat{\psi} \in \text{Irr}(\mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})_{\psi}^F \mid \psi)$ of $\tilde{\psi}$.

Proceeding as in the proof of Theorem 4.3, we can construct a projective representation associated with χ

$$\mathcal{P}_{\text{glo}} : \left(\tilde{\mathbf{G}}^F \mathcal{A} \right)_{\chi} \rightarrow \text{GL}_{\chi(1)}(\mathbb{C})$$

given by $\mathcal{P}_{\text{glo}}(Zx_1x_2) := \widehat{\mathcal{D}}_{\text{glo}}(x_1)\mathcal{D}'_{\text{glo}}(x_2)$ for every $x_1 \in \tilde{\mathbf{G}}_{\chi}^F$ and $x_2 \in (\mathbf{G}^F \mathcal{A})_{\chi}$. Similarly, we obtain a projective representation associated with ψ

$$\mathcal{P}_{\text{loc}} : \left(\tilde{\mathbf{G}}^F \mathcal{A} \right)_{\mathbf{L}, \chi} \rightarrow \text{GL}_{\psi(1)}(\mathbb{C})$$

given by $\mathcal{P}_{\text{loc}}(Zx_1x_2) := \widehat{\mathcal{D}}_{\text{loc}}(x_1)\mathcal{D}'_{\text{loc}}(x_2)$ for every $x_1 \in \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})_{\psi}^F$ and $x_2 \in (\mathbf{G}^F \mathcal{A})_{\mathbf{L}, \psi}$. Moreover, by the proof of Theorem 4.3, we know that

$$\left((\tilde{\mathbf{G}}^F \mathcal{A})_{\chi}, \mathbf{G}^F, \chi \right) \sim_{\mathbf{G}^F}^c \left((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}, \psi}, \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F, \psi \right)$$

via the projective representations $(\mathcal{P}_{\text{glo}}, \mathcal{P}_{\text{loc}})$. Consider the factor sets α_{glo} of $\mathcal{P}_{\text{glo}}$ and α_{loc} of $\mathcal{P}_{\text{loc}}$. Let S be the group generated by the values of α_{glo} and denote by A_{glo} the central extension of $(\tilde{\mathbf{G}}^F \mathcal{A})_{\chi}$ by S induced by α_{glo} . Let $\epsilon : A_{\text{glo}} \rightarrow (\tilde{\mathbf{G}}^F \mathcal{A})_{\chi}$ be the canonical morphism with kernel S . As α_{glo} is trivial on $\tilde{\mathbf{G}}_{\chi}^F \times \tilde{\mathbf{G}}_{\chi}^F$, every subgroup $X \leq \tilde{\mathbf{G}}_{\chi}^F$ is isomorphic to the subgroup $X_0 := \{(x, 1) \mid x \in X\}$ of A_{glo} and $\epsilon^{-1}(X) = X_0 \times S$. In particular, we have $H_{\text{glo}} := \epsilon^{-1}(\tilde{\mathbf{G}}_{\chi}^F) = (\tilde{\mathbf{G}}_{\chi}^F)_0 \times S$. The map given by

$$\mathcal{Q}_{\text{glo}}(x, s) := s\mathcal{P}_{\text{glo}}(x),$$

for every $s \in S$ and $x \in (\tilde{\mathbf{G}}^F \mathcal{A})_{\chi}$, is an irreducible representation of A_{glo} affording an extension χ_1 of the character χ_0 of $(\mathbf{G}^F)_0$ corresponding to χ . Notice that

$$\chi_1, H_{\text{glo}} = \widehat{\chi}_0 \times \iota, \quad (4.11)$$

where $\iota(s) := s$ and $\widehat{\chi}_0$ is the character of $(\tilde{\mathbf{G}}_{\chi}^F)_0$ corresponding to $\hat{\chi} \in \text{Irr}(\tilde{\mathbf{G}}_{\chi}^F)$. Next, set $A_{\text{loc}} := \epsilon^{-1}((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}, \chi}^F)$ and notice that, because the factor set α_{loc} of $\mathcal{P}_{\text{loc}}$ is the restriction of the factor set α_{glo} of $\mathcal{P}_{\text{glo}}$, the map given by

$$\mathcal{Q}_{\text{loc}}(x, s) := s\mathcal{P}_{\text{loc}}(x),$$

for every $s \in S$ and $x \in (\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}, \chi}^F$, is an irreducible representation of A_{loc} affording an extension ψ_1 of the character ψ_0 of $(\mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F)_0$ corresponding to ψ . As before, we have

$$\psi_{1, H_{\text{loc}}} = \widehat{\psi}_0 \times \iota, \quad (4.12)$$

where $H_{\text{loc}} := \epsilon^{-1}(\mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F) = (\mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F)_0 \times S$ and $\widehat{\psi}_0$ is the character of $(\mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F)_0$ corresponding to $\widehat{\psi} \in \text{Irr}(\mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F)$. Now, (4.11), (4.12) and (4.8) imply that

$$\text{bl}(\chi_{1, J}) = \text{bl}(\psi_{1, J \cap H_{\text{glo}}})^J \quad (4.13)$$

for every $(\mathbf{G}^F)_0 \leq J \leq H_{\text{glo}}$ (see the argument at the end of the proof of [CS15, Proposition 4.2]). By [KS15, Theorem C] there exists $\varphi_1 \in \text{Irr}(A_{\text{glo}}[B_0])$ such that $\varphi_{1, (\mathbf{G}^F)_0}$ is irreducible and lies in the block B_0 and

$$\text{bl}(\varphi_{1, J}) = \text{bl}(\psi_{1, J \cap A_{\text{loc}}})^J \quad (4.14)$$

for every $(\mathbf{G}^F)_0 \leq J \leq A_{\text{glo}}[B_0]$. It follows from (4.13) and (4.14) that

$$\text{bl}(\varphi_{1, J}) = \text{bl}(\psi_{1, J \cap H_{\text{loc}}})^J = \text{bl}(\chi_{1, J})$$

for every $(\mathbf{G}^F)_0 \leq J \leq H_{\text{glo}}[B_0] = H_{\text{glo}} \cap A_{\text{glo}}[B_0]$. In particular $B_0 = \text{bl}(\chi_{1, (\mathbf{G}^F)_0}) = \text{bl}(\chi_0)$. Therefore the conditions of [CS15, Lemma 3.2] are satisfied and we obtain an extension $\chi_2 \in \text{Irr}(A_{\text{glo}})$ of $\chi_{1, H_{\text{glo}}}$ satisfying

$$\text{bl}(\varphi_{1, J}) = \text{bl}(\chi_{2, J}) \quad (4.15)$$

for every $(\mathbf{G}^F)_0 \leq J \leq A_{\text{glo}}[B_0]$. From (4.14) and (4.15) we obtain

$$\text{bl}(\psi_{1, J \cap A_{\text{loc}}})^J = \text{bl}(\chi_{2, J})$$

for every $(\mathbf{G}^F)_0 \leq J \leq A_{\text{glo}}[B_0]$. The latter equation, together with [Mur13, Theorem 3.5], yields

$$\begin{aligned} \text{bl}(\psi_{1, J \cap A_{\text{loc}}})^J &= \left(\text{bl}(\psi_{1, J \cap A_{\text{loc}} \cap A_{\text{glo}}[B_0]})^{J \cap A_{\text{loc}}} \right)^J \\ &= \left(\text{bl}(\chi_{2, J \cap A_{\text{glo}}[B_0]}) \right)^J \\ &= \text{bl}(\chi_{2, J}) \end{aligned} \quad (4.16)$$

for every $(\mathbf{G}^F)_0 \leq J \leq A_{\text{glo}}$. Finally, observe that using Assumption 4.4 (i.a) and [Spä17, Theorem 4.1 (d)] we obtain

$$\begin{aligned} \mathbf{C}_{A_{\text{glo}}}((\mathbf{G}^F)_0) &= \mathbf{C}_{A_{\text{glo}}}((\mathbf{G}^F)_0 \times S) \\ &\leq \epsilon^{-1} \left(\mathbf{C}_{(\tilde{\mathbf{G}}^F \mathcal{A})_{\chi}}(\mathbf{G}^F) \right) \\ &= \epsilon^{-1} \left(\mathbf{Z}(\tilde{\mathbf{G}}^F) \right) \\ &= \mathbf{Z}(\tilde{\mathbf{G}}^F)_0 \times S. \end{aligned}$$

Recalling that $\text{Irr}(\chi_{\mathbf{Z}(\mathbf{G}^F)}) = \text{Irr}(\psi_{\mathbf{Z}(\mathbf{G}^F)})$, we obtain $\text{Irr}(\widehat{\chi}_{\mathbf{Z}(\widetilde{\mathbf{G}}^F)}) = \text{Irr}(\widehat{\psi}_{\mathbf{Z}(\widetilde{\mathbf{G}}^F)})$ and hence

$$\begin{aligned} \text{Irr}\left(\chi_{2,\mathbf{Z}(\widetilde{\mathbf{G}}^F)_0 \times S}\right) &= \text{Irr}\left(\chi_{1,\mathbf{Z}(\widetilde{\mathbf{G}}^F)_0 \times S}\right) \\ &= \text{Irr}\left(\widehat{\chi}_{0,\mathbf{Z}(\widetilde{\mathbf{G}}^F)_0} \times \iota\right) \\ &= \text{Irr}\left(\widehat{\psi}_{0,\mathbf{Z}(\widetilde{\mathbf{G}}^F)_0} \times \iota\right) \\ &= \text{Irr}\left(\psi_{1,\mathbf{Z}(\widetilde{\mathbf{G}}^F)_0 \times S}\right). \end{aligned} \quad (4.17)$$

Thanks to (4.16) and (4.17), we can apply [Spä17, Lemma 3.10] which implies

$$\left(A_{\text{glo}}, \left(\mathbf{G}^F\right)_0, \chi_0\right) \sim_{(\mathbf{G}^F)_0} \left(A_{\text{loc}}, \left(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F\right)_0, \psi_0\right).$$

Then (4.10) follows by using [Spä17, Theorem 4.1 (i)]. This completes the proof. $\square$

5. Stabilisers, extendibility and consequences

In this section, we combine the results obtained in Section 3 and Section 4 and show how to obtain the parametrisation of e -Harish-Chandra series introduced in (1.2) and its compatibility with automorphisms and Clifford theory, considered in Parametrisation C and Parametrisation 2.7, by assuming certain requirements on stabilisers and extendibility of characters. This proves Theorem D and, in particular provides a strategy to extend the parametrisation given in [BMM93, Theorem 3.2] to nonunipotent e -cuspidal pairs in groups with disconnected centre. At the end of this section, we apply the main results of [BS20] and [Bro22] and obtain some consequences for groups of type A and C.

5.1. Proof of Theorem D

Let $\mathbf{G}, F : \mathbf{G} \rightarrow \mathbf{G}$, q, ℓ and e as in Notation 2.3 with $\mathbf{G}$ simple of simply connected type. Let $i : \mathbf{G} \rightarrow \widetilde{\mathbf{G}}$ be a regular embedding compatible with the action of F and consider the group $\mathcal{A}$ generated by field and graph automorphisms of $\mathbf{G}$ in such a way that $\mathcal{A}$ acts on $\widetilde{\mathbf{G}}^F$ (see Section 2.2). Let $\mathcal{K}$ be the group introduced in Section 2.1 and define the semidirect product $(\widetilde{\mathbf{G}}^F \mathcal{A}) \ltimes \mathcal{K}$ as discussed at the beginning of Section 3.2.

We now come to the proof of Theorem D. As said before, this reduces the verification of Parametrisation C to questions on stabilisers and extendibility of characters. Before proceeding further, we give an exact definition of these conditions. The following should be compared to [CS19, Definition 2.2].

Definition 5.1. For every e -split Levi subgroup $\mathbf{L}$ of $\mathbf{G}$, we define the following condition.

There exists a $\widetilde{\mathbf{L}}^F$ -transversal $\mathcal{T}$ in $\text{Cusp}_e(\mathbf{L}, F)$ such that:

- (G) For every $\lambda \in \mathcal{T}$ and every $\chi \in \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ there exists an $\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})_{\lambda}^F$ -conjugate χ_0 of χ such that:
 - (i) $\left(\widetilde{\mathbf{G}}^F \mathcal{A}\right)_{\chi_0} = \widetilde{\mathbf{G}}_{\chi_0}^F \mathcal{A}_{\chi_0}$, and
 - (ii) χ_0 extends to $\mathbf{G}^F \mathcal{A}_{\chi_0}$.
- (L) For every $\lambda \in \mathcal{T}$ and every $\psi \in \text{Irr}(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \mid \lambda)$ there exists an $\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})_{\lambda}^F$ -conjugate ψ_0 of ψ such that:
 - (i) $\left(\widetilde{\mathbf{G}}^F \mathcal{A}\right)_{\mathbf{L}, \psi_0} = \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})_{\psi_0}^F (\mathbf{G}^F \mathcal{A})_{\mathbf{L}, \psi_0}$, and
 - (ii) ψ_0 extends to $(\mathbf{G}^F \mathcal{A})_{\mathbf{L}, \psi_0}$.

We make a remark on the global condition of Definition 5.1. In fact, this condition is slightly stronger than condition A(∞) of [CS19, Definition 2.2]. However, these two conditions are equivalent under additional assumptions.

Remark 5.2. Assume that Hypothesis 2.4 holds for $(\mathbf{G}, F)$ and let $(\mathbf{L}, \lambda)$ be an e -cuspidal pair of $\mathbf{G}$. Set $B := \text{bl}(\lambda)^{\mathbf{G}^F}$ and suppose that either:

- (i) $\text{Out}(\mathbf{G}^F)_B$ is abelian, where B denotes the $\tilde{\mathbf{G}}^F$ -orbit of B ; or
- (ii) B is $\tilde{\mathbf{G}}^F$ -invariant and $(\mathbf{L}, \lambda)$ is e -Brauer–Lusztig-cuspidal.

Then Definition 5.1 (G) is equivalent to the following:

(G') For every $\lambda \in \mathcal{T}$ and every $\chi \in \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ there exists a $\tilde{\mathbf{G}}^F$ -conjugate χ_0 of χ such that:

- (i) $(\tilde{\mathbf{G}}^F \mathcal{A})_{\chi_0} = \tilde{\mathbf{G}}_{\chi_0}^F \mathcal{A}_{\chi_0}$, and
- (ii) χ_0 extends to $\mathbf{G}^F \mathcal{A}_{\chi_0}$.

Proof. Clearly Definition 5.1 (G) implies (G') above. Conversely let $\chi \in \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ and consider a $\tilde{\mathbf{G}}^F$ -conjugate χ_1 of χ satisfying the required properties. As explained in Remark 4.5, if $\text{Out}(\mathbf{G}^F)_B$ is abelian, then χ also satisfies the required properties (see [BS22, Lemma 4.7]) and we set $\chi_0 := \chi$. On the other hand, by using the argument of Remark 4.5, if B is $\tilde{\mathbf{G}}^F$ -invariant and $(\mathbf{L}, \lambda)$ is e -Brauer–Lusztig-cuspidal, then $\chi_1 \in \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ and we set $\chi_0 := \chi_1$. This shows that there exists $\chi_0 \in \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ and $x \in \tilde{\mathbf{G}}^F$ such that $\chi_0 = \chi^x$ satisfies the required properties. In particular $\chi_0 \in \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) \cap \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)^x)$ and [Ros24a, Proposition 4.10] implies that $(\mathbf{L}, \lambda) = (\mathbf{L}, \lambda)^{xy}$ for some $y \in \mathbf{G}^F$. It follows that $\chi_0 = \chi^{xy}$ with $xy \in \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})_F^F$ as required by Definition 5.1 (G). $\square$

Moreover, we make another remark on the local condition in Definition 5.1. This should be compared with condition A(d) of [CS19, Definition 2.2] with $d = e$.

Remark 5.3. Let $\mathbf{L}$ be an e -split Levi subgroup of $\mathbf{G}$. Then condition A(d) of [CS19, Definition 2.2] states that for every $\psi \in \text{Irr}(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F)$ there exists an $\mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F$ -conjugate ψ_0 such that:

- (i) $(\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}, \psi_0} = \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})_{\psi_0}^F (\mathbf{G}^F \mathcal{A})_{\mathbf{L}, \psi_0}$, and
- (ii) ψ_0 extends to $(\mathbf{G}^F \mathcal{A})_{\mathbf{L}, \psi_0}$.

Condition (L) of Definition 5.1 gives a more precise description by saying that whenever $\lambda \in \text{Irr}(\mathbf{L})^F$ is e -cuspidal and ψ lies above λ , then ψ_0 can be chosen to lie above λ .

We point out that a detailed inspection of the argument used in all instances where condition A(d) of [CS19, Definition 2.2] has been verified would actually provide the more precise condition (L) of Definition 5.1. This is explained in more details in Lemma 5.7 below.

Before proving Theorem D, we prove a similar result for the simpler Parametrisation 2.7.

Theorem 5.4. Assume Hypothesis 3.1 with $\mathbf{G}$ simple and simply connected and suppose that $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$. Let $\mathbf{L}$ be an e -split Levi subgroup of $\mathbf{G}$ and suppose that the following conditions hold:

- (i) maximal extendibility holds with respect to $\mathbf{G}^F \trianglelefteq \tilde{\mathbf{G}}^F$ and to $\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F$;
- (ii) there exists a $((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}} \rtimes \mathcal{K})$ -equivariant extension map for $\text{Cusp}_e(\tilde{\mathbf{L}}, F)$ with respect to $\tilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F$;
- (iii) the requirement from Definition 5.1 holds for $\mathbf{L} \leq \mathbf{G}$.

Then Parametrisation 2.7 holds for every e -cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$.

Proof. Fix an e -cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$. We want to find a bijection $\Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}}$ as in Parametrisation 2.7. Let $\mathcal{T}$ be the $\tilde{\mathbf{L}}^F$ -transversal in $\text{Cusp}_e(\mathbf{L}, F)$ given by Definition 5.1. Since $\mathbf{G}^F$ -central isomorphisms of character triples are compatible with conjugation, it is no loss of generality to assume $\lambda \in \mathcal{T}$. Now Assumption 4.1 (iii) and (iv) hold by Definition 5.1 (G) and (L) respectively, while the bijection from Assumption 4.1 (ii) exists by Theorem 3.19. Since under our assumption we also have Hypothesis 2.4, we apply Theorem 4.3 and obtain Parametrisation 2.7 for $(\mathbf{L}, \lambda)$ and $\mathbf{G}$. $\square$

The same proof can be used to obtain Theorem D. Here, we prove a slightly more general result which allows us to consider a larger class of blocks. The additional block theoretic requirements are inspired by [CS15, Theorem 4.1 (v)] and [BS20, Theorem 2.4 (v)] and hold automatically for unipotent blocks, blocks with maximal defect and in general for every group of type **B**, **C** or **E**₇ (see Remark 4.5).

Theorem 5.5. *Assume Hypothesis 3.1 with $\mathbf{G}$ simple and simply connected and suppose that $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$. Let $\mathbf{L}$ be an e -split Levi subgroup of $\mathbf{G}$, $B \in \text{Bl}(\mathbf{G}^F)$ and suppose that the following conditions hold:*

- (i) *maximal extendibility holds with respect to $\mathbf{G}^F \trianglelefteq \tilde{\mathbf{G}}^F$ and to $\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F$;*
- (ii) *there exists a $((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}} \rtimes \mathcal{K})$ -equivariant extension map for $\text{Cusp}_e(\tilde{\mathbf{L}}, F)$ with respect to $\tilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F$;*
- (iii) *the requirement from Definition 5.1 holds for $\mathbf{L} \leq \mathbf{G}$;*
- (iv) *the block B satisfies either*
 - (a) *$\text{Out}(\mathbf{G}^F)_{\mathcal{B}}$ is abelian, where $\mathcal{B}$ is the $\tilde{\mathbf{G}}^F$ -orbit of B , or*
 - (b) *for every subgroup $\mathbf{G}^F \leq J \leq \tilde{\mathbf{G}}^F$, we have that every block C of J covering B is $\tilde{\mathbf{G}}^F$ -invariant.*

Then Parametrisation C holds for every e -Brauer–Lusztig-cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$ with $\text{bl}(\lambda)^{\mathbf{G}^F} = B$.

Proof. Consider an e -Brauer–Lusztig-cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$ as in the statement. Let $\mathcal{T}$ be the $\tilde{\mathbf{L}}^F$ -transversal in $\text{Cusp}_e(\mathbf{L}, F)$ given by Definition 5.1. Since $\mathbf{G}^F$ -block isomorphisms of character triples are compatible with conjugation and our block theoretic hypothesis (iv) is preserved by $\tilde{\mathbf{G}}^F$ -conjugation, it is no loss of generality to assume $\lambda \in \mathcal{T}$. Now Assumption 4.4 (iii) and (iv) hold by Definition 5.1 (G) and (L) respectively, while the bijection from Assumption 4.4 (ii) exists by Theorem 3.19. Finally notice that Assumption 4.4 (v) and (vi) hold by our hypothesis. Since Hypothesis 2.4 holds under Hypothesis 3.1 for ℓ as above, we can apply Theorem 4.8 to conclude that Parametrisation C holds for $(\mathbf{L}, \lambda)$ and $\mathbf{G}$. $\square$

The extendibility conditions in Theorem 5.4 (i)-(ii) and Theorem 5.5 (i)-(ii) should be compared with condition B(d) of [CS19, Definition 2.2] with $d = e$.

The reader should compare Theorem 3.20 with Theorem 5.4 and Theorem 5.5. We want to stress, when proving Parametrisation C and Parametrisation 2.7, that the hardest task is to show that the associated character triples are $\mathbf{G}^F$ -central isomorphic and $\mathbf{G}^F$ -block isomorphic respectively.

5.2. Results for groups of type A and C

Finally, by applying the main results of [BS20] and [Bro22], we obtain consequences of Theorem 3.20, Theorem 5.4 and Theorem 5.5 for groups of type **A** and **C**. We start by considering Theorem 3.20 which allows us to prove the following corollary.

Corollary 5.6. *Let $\mathbf{G}, F : \mathbf{G} \rightarrow \mathbf{G}$, q, ℓ and e be as in Notation 2.3 and suppose that $\mathbf{G}$ is simple, simply connected of type **A** or **C** and that $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$. For every e -cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$, there exists an $\text{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{(\mathbf{L}, \lambda)}$ -equivariant bijection*

$$\Omega_{(\mathbf{L}, \lambda)}^{\mathbf{G}} : \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) \rightarrow \text{Irr}(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \mid \lambda)$$

that preserves the ℓ -defect of characters.

Proof. We apply Theorem 3.20 with $\mathbf{K} = \mathbf{G}$. Suppose first that $\mathbf{G}$ is of type **A**. Then [BS20, Corollary 4.7 (b)] shows that there exists a $((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}} \rtimes \mathcal{K})$ -equivariant extension map with respect to $\tilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F$ and therefore the result follows. On the other hand, if $\mathbf{G}$ is of type **C**, then we obtain a $((\tilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}} \rtimes \mathcal{K})$ -equivariant extension map for $\text{Cusp}_e(\tilde{\mathbf{L}}, F)$ with respect to $\tilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F$ by applying [BS20, Theorem 4.1 (b)] whose conditions are satisfied thanks to [Bro22, Corollary 4.13, Proposition 4.18, Lemma 5.11, Proposition 5.18] (see the proof of [Bro22, Theorem 1.1] for a detailed explanation). This concludes the proof. $\square$

Our next aim is to show that the bijections obtained in Corollary 5.6 can be chosen in such a way that the corresponding character triples are $\mathbf{G}^F$ -central isomorphic and even $\mathbf{G}^F$ -block isomorphic as predicted by Parametrisation 2.7 and Parametrisation C respectively. In order to apply Theorem 5.4 and Theorem 5.5, we need to prove the requirements of Definition 5.1. In the next lemma we consider the local condition from Definition 5.1 (L).

Lemma 5.7. *If $\mathbf{G}$ is simple, simply connected of type A or C and $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$, then Definition 5.1 (L) holds with respect to every e -split Levi subgroup $\mathbf{L}$.*

Proof. First, observe that [BS20, Theorem 1.1] and [Bro22, Theorem 1.1] rely on the proof of [CS17b, Theorem 4.3], and therefore on the arguments introduced in [CS17a, Section 5], via an application of [BS20, Theorem 4.1]. In particular, we focus on the argument used in [CS17a, Proposition 5.13]. Consider $\psi \in \text{Irr}(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \mid \lambda)$ and notice that λ has an extension $\hat{\lambda} \in \text{Irr}(\mathbf{N}_{\mathbf{G}}(\mathbf{L})_{\lambda}^F)$ by [BS20, Theorem 1.2 (a)] (if $\mathbf{G}$ is of type A) and [Bro22, Theorem 1.2] (if $\mathbf{G}$ is of type C). Using Gallagher's theorem and the Clifford correspondence, we can write $\psi = (\hat{\lambda}\eta)^{\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F}$ for some $\eta \in \text{Irr}(\mathbf{N}_{\mathbf{G}}(\mathbf{L})_{\lambda}^F / \mathbf{L}^F)$. By the argument of [CS17a, Proposition 5.13], there exists $\eta_0 \in \text{Irr}(\mathbf{N}_{\mathbf{G}}(\mathbf{L})_{\lambda}^F / \mathbf{L}^F)$ such that $\psi_0 := (\hat{\lambda}\eta_0)^{\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F}$ satisfies Definition 5.1 (L.i)-(L.ii) and $\psi = \psi_0^x$ for some $x \in \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F$. By the definition of ψ_0 , we deduce that ψ_0 lies above λ and therefore ψ lies above λ and λ^x . Then Clifford's theorem implies that $\lambda = \lambda^{xy}$ for some $y \in \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F$ and we conclude that $\psi = \psi_0^{xy}$ with $xy \in \mathbf{N}_{\mathbf{G}}(\mathbf{L})_{\lambda}^F$. $\square$

Using Lemma 5.7 we can prove Parametrisation 2.7 under suitable hypotheses. We start by considering groups of type A.

Corollary 5.8. *Consider $\mathbf{G}$, $F : \mathbf{G} \rightarrow \mathbf{G}$, q , ℓ and e as in Notation 2.3 and suppose that $\mathbf{G}$ is simple, simply connected of type A and that $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$. Let $(\mathbf{L}, \lambda)$ be an e -cuspidal pair of $\mathbf{G}$ and set $B := \text{bl}(\lambda)^{\mathbf{G}^F}$. If one of the following is satisfied:*

- (i) $\text{Out}(\mathbf{G}^F)_{\mathcal{B}}$ is abelian, where $\mathcal{B}$ is the $\tilde{\mathbf{G}}^F$ -orbit of B ; or
- (ii) $(\mathbf{L}, \lambda)$ is e -Brauer–Lusztig-cuspidal and either B is unipotent or B has maximal defect,

then Parametrisation 2.7 holds for $(\mathbf{L}, \lambda)$.

Proof. We prove the statement via an application of Theorem 5.4. First, we notice that Hypothesis 3.1 and Hypothesis 2.4 are satisfied under our assumptions thanks to Remark 2.6. Noticing that $\tilde{\mathbf{G}}^F / \mathbf{G}^F \simeq \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F / \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F$ is cyclic, [Isa76, Corollary 11.22] implies that maximal extendibility holds with respect to $\mathbf{G}^F \trianglelefteq \tilde{\mathbf{G}}^F$ and to $\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F$. Moreover, as in the proof of Corollary 5.6, we obtain a $((\tilde{\mathbf{G}}^F \mathcal{A}) \rtimes \mathcal{K})$ -equivariant extension map with respect to $\tilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F$ by applying [BS20, Corollary 4.7 (b)]. It remains to check the requirements of Definition 5.1. We obtain Definition 5.1 (L) by applying Lemma 5.7. To prove Definition 5.1 (G), we observe that condition (G') in Remark 5.2 is satisfied by [CS17a, Theorem 4.1]. Then, in order to apply Remark 5.2 we only need to show that if B is unipotent or has maximal defect, then it is $\tilde{\mathbf{G}}^F$ -invariant. This follows by Remark 4.5. $\square$

Next, we consider Parametrisation 2.7 for groups of type C.

Corollary 5.9. *Consider $\mathbf{G}$, $F : \mathbf{G} \rightarrow \mathbf{G}$, q , ℓ and e as in Notation 2.3 and suppose that $\mathbf{G}$ is simple, simply connected of type C and that $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$. Then Parametrisation 2.7 holds for every e -cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$.*

Proof. We argue as in the proof of Corollary 5.8 and prove the result via an application of Theorem 5.4. By Remark 2.6, we notice that Hypothesis 3.1 and Hypothesis 2.4 are satisfied under our assumptions. Since $\tilde{\mathbf{G}}^F / \mathbf{G}^F \simeq \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F / \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F$ is cyclic, maximal extendibility holds with respect to $\mathbf{G}^F \trianglelefteq \tilde{\mathbf{G}}^F$ and to $\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F$ by [Isa76, Corollary 11.22]. Moreover, as shown in the proof of Corollary 5.6, we obtain a $((\tilde{\mathbf{G}}^F \mathcal{A}) \rtimes \mathcal{K})$ -equivariant extension map for $\text{Cusp}_e(\tilde{\mathbf{L}}, F)$ with respect to $\tilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{L})^F$ by applying [BS20, Theorem 4.1] together with [Bro22, Corollary 4.13, Proposition 4.18, Lemma 5.11, Proposition 5.18]. Finally, noticing that $\text{Out}(\mathbf{G}^F)$ is abelian, we obtain Definition 5.1 (G) by applying Remark 5.2 and noticing that condition (G') in Remark 5.2 is satisfied by [CS17b, Theorem 3.1]. $\square$

Finally, we prove Parametrisation C for certain e -Brauer–Lusztig-cuspidal pairs of groups of type A and C. First, we deal with groups of type A. Notice that the hypothesis of the following result is the same as the one of Corollary 5.8 with the additional assumption that $(\mathbf{L}, \lambda)$ is e -Brauer–Lusztig cuspidal even when considering the case where $\text{Out}(\mathbf{G}^F)_{\mathcal{B}}$ is abelian.

Corollary 5.10. *Consider $\mathbf{G}$, $F : \mathbf{G} \rightarrow \mathbf{G}$, q , ℓ and e as in Notation 2.3 and suppose that $\mathbf{G}$ is simple, simply connected of type A and that $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$. Let $(\mathbf{L}, \lambda)$ be an e -Brauer–Lusztig-cuspidal pair of $\mathbf{G}$ and set $B := \text{bl}(\lambda)^{\mathbf{G}^F}$. If one of the following is satisfied:*

- (i) $\text{Out}(\mathbf{G}^F)_{\mathcal{B}}$ is abelian, where $\mathcal{B}$ is the $\tilde{\mathbf{G}}^F$ -orbit of B ;
- (ii) B is unipotent; or
- (iii) B has maximal defect,

then Parametrisation C holds for $(\mathbf{L}, \lambda)$.

Proof. We show that Parametrisation C holds for the e -cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$ by applying Theorem 5.5. By the proof of Corollary 5.8 it only remains to verify Theorem 5.5 (iv). Therefore, it is enough to show that if B is unipotent or has maximal defect then Theorem 5.5 (iv.b) is satisfied. This fact follows by using [DM91, Proposition 13.20] and the results of [CS15, Section 5] as explained in Remark 4.5. Now the result follows by recalling that $(\mathbf{L}, \lambda)$ is e -Brauer–Lusztig-cuspidal by assumption. $\square$

To conclude, we consider groups of type C and prove Parametrisation C for all e -Brauer–Lusztig-cuspidal pairs under suitable hypotheses on ℓ .

Corollary 5.11. *Consider $\mathbf{G}$, $F : \mathbf{G} \rightarrow \mathbf{G}$, q , ℓ and e as in Notation 2.3 and suppose that $\mathbf{G}$ is simple, simply connected of type C and that $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$. Then Parametrisation C holds for every e -Brauer–Lusztig-cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$.*

Proof. As in the proof of Corollary 5.10 we apply Theorem 5.5. Using the proof of Corollary 5.9 we therefore only need to check Theorem 5.5 (iv). Since $\text{Out}(\mathbf{G}^F)$ is abelian, Theorem 5.5 (iv.a) holds and then the result follows because $(\mathbf{L}, \lambda)$ is e -Brauer–Lusztig-cuspidal by hypothesis. $\square$

Acknowledgements. The content of this paper is part of the author’s doctoral thesis. The author would like to thank Britta Späth for proposing a fascinating research topic and for providing countless suggestions, Julian Brough for multiple discussions on the extendibility of characters of e -split Levi subgroups and Gunter Malle for helpful comments on generalised Harish-Chandra theory and for a thorough reading of an earlier version of this paper. The author also wants to express his gratitude to the anonymous referee for spotting some inaccuracies and providing comments and suggestions.

Competing interest. The author has no competing interests to declare.

Funding statement. This work is partially supported by the GRK2240: *Algebra-geometric Methods in Algebra, Arithmetic and Topology* funded by the DFG, by the grant EP/T004592/1 of the EPSRC, and by the Walter Benjamin Programme of the DFG - Project number 525464727.

References

- [BDR17] C. Bonnafé, J.-F. Dat and R. Rouquier, ‘Derived categories and Deligne–Lusztig varieties II’, *Ann. of Math. (2)* **185**(2) (2017), 609–670.
- [BM11] C. Bonnafé and J. Michel, ‘Computational proof of the Mackey formula for $q > 2$ ’, *J. Algebra* **327** (2011), 506–526.
- [BMM93] M. Broué, G. Malle and J. Michel, ‘Generic blocks of finite reductive groups’, *Astérisque* **212** (1993), 7–92.
- [Bon06] C. Bonnafé, ‘Sur les caractères des groupes réductifs finis à centre non connexe: applications aux groupes spéciaux linéaires et unitaires’, *Astérisque* **306** (2006), vi+165.
- [Bro22] J. Brough, ‘Characters of normalisers of d -split Levi subgroups in $Sp_{2n}(q)$ ’, preprint (2022). [arXiv:2203.06072](https://arxiv.org/abs/2203.06072).
- [BS20] J. Brough and B. Späth, ‘On the Alperin–McKay conjecture for simple groups of type A’, *J. Algebra* **558** (2020), 221–259.
- [BS22] J. Brough and B. Späth, ‘A criterion for the inductive Alperin weight condition’, *Bull. Lond. Math. Soc.* **54**(2) (2022), 466–481.
- [CE94] M. Cabanes and M. Enguehard, ‘On unipotent blocks and their ordinary characters’, *Invent. Math.* **117**(1) (1994), 149–164.

- [CE99] M. Cabanes and M. Enguehard, ‘On blocks of finite reductive groups and twisted induction’, *Adv. Math.* **145**(2) (1999), 189–229.
- [CE04] M. Cabanes and M. Enguehard, *Representation Theory of Finite Reductive Groups*, vol. 1 of *New Mathematical Monographs* (2004), (Cambridge University Press, Cambridge).
- [CS] M. Cabanes and B. Späth, ‘The McKay conjecture on character degrees’, to appear in *Ann. of Math.* (2)
- [CS13] M. Cabanes and B. Späth, ‘Equivariance and extendibility in finite reductive groups with connected center’, *Math. Z.* **275**(3–4) (2013), 689–713.
- [CS15] M. Cabanes and B. Späth, ‘On the inductive Alperin–McKay condition for simple groups of type A' ’, *J. Algebra* **442** (2015), 104–123.
- [CS17a] M. Cabanes and B. Späth, ‘Equivariant character correspondences and inductive McKay condition for type A' ’, *J. Reine Angew. Math.* **728** (2017), 153–194.
- [CS17b] M. Cabanes and B. Späth, ‘Inductive McKay condition for finite simple groups of type C' ’, *Represent. Theory* **21** (2017), 61–81.
- [CS19] M. Cabanes and B. Späth, ‘Descent equalities and the inductive McKay condition for types B and E' ’, *Adv. Math.* **356** (2019), Article 106820, 48 pp.
- [Dad73] E. C. Dade, ‘Block extensions’, *Illinois J. Math.* **17** (1973), 198–272.
- [DL76] P. Deligne and G. Lusztig, ‘Representations of reductive groups over finite fields’, *Ann. of Math.* (2) **103**(1) (1976), 103–161.
- [DM90] F. Digne and J. Michel, ‘On Lusztig’s parametrization of characters of finite groups of Lie type’, *Astérisque* **181**–182 (1990), 113–156.
- [DM91] F. Digne and J. Michel, *Representations of Finite Groups of Lie Type*, vol. 21 of *London Mathematical Society Student Texts* (1991), (Cambridge University Press, Cambridge).
- [FS86] P. Fong and B. Srinivasan, ‘Generalized Harish-Chandra theory for unipotent characters of finite classical groups’, *J. Algebra* **104**(2) (1986), 301–309.
- [GLS98] D. Gorenstein, R. Lyons and R. Solomon, *The Classification of the Finite Simple Groups. Number 3. Part I. Chapter A*, vol. 40 of *Mathematical Surveys and Monographs* (1998), (American Mathematical Society, Providence, RI).
- [GM20] M. Geck and G. Malle, *The Character Theory of Finite Groups of Lie Type: A Guided Tour*, vol. 187 of *Cambridge Studies in Advanced Mathematics* (2020), (Cambridge University Press).
- [HC70] Harish-Chandra, ‘Eisenstein series over finite fields’, in *Functional Analysis and Related Fields* (Proc. Conf. M. Stone, Univ. Chicago, Chicago, Ill., 1968) (1970), 76–88.
- [HL80] R. B. Howlett and G. I. Lehrer, ‘Induced cuspidal representations and generalised Hecke rings’, *Invent. Math.* **58**(1) (1980), 37–64.
- [Isa76] I. M. Isaacs, *Character Theory of Finite Groups* (Academic Press [Harcourt Brace Jovanovich, Publishers], New York–London, 1976).
- [KM13] R. Kessar and G. Malle, ‘Quasi-isolated blocks and Brauer’s height zero conjecture’, *Ann. of Math.* (2) **178**(1) (2013), 321–384.
- [KS15] S. Koshitani and B. Späth, ‘Clifford theory of characters in induced blocks’, *Proc. Amer. Math. Soc.* **143**(9) (2015), 3687–3702.
- [KS16a] S. Koshitani and B. Späth, ‘The inductive Alperin–McKay and blockwise Alperin weight conditions for blocks with cyclic defect groups and odd primes’, *J. Group Theory* **19**(5) (2016), 777–813.
- [KS16b] S. Koshitani and B. Späth, ‘The inductive Alperin–McKay condition for 2-blocks with cyclic defect groups’, *Arch. Math. (Basel)* **106**(2) (2016), 107–116.
- [Lus76] G. Lusztig, ‘On the finiteness of the number of unipotent classes’, *Invent. Math.* **34**(3) (1976), 201–213.
- [Mal07] G. Malle, ‘Height 0 characters of finite groups of Lie type’, *Represent. Theory* **11** (2007), 192–220.
- [Mal14] G. Malle, ‘On the inductive Alperin–McKay and Alperin weight conjecture for groups with abelian Sylow subgroups’, *J. Algebra* **397** (2014), 190–208.
- [MS16] G. Malle and B. Späth, ‘Characters of odd degree’, *Ann. of Math.* (2) **184**(3) (2016), 869–908.
- [MT11] G. Malle and D. Testerman, *Linear Algebraic Groups and Finite Groups of Lie Type*, vol. 133 of *Cambridge Studies in Advanced Mathematics* (2011), (Cambridge University Press, Cambridge).
- [Mur13] M. Murai, ‘On blocks of normal subgroups of finite groups’, *Osaka J. Math.* **50**(4) (2013), 1007–1020.
- [NS14] G. Navarro and B. Späth, ‘On Brauer’s height zero conjecture’, *J. Eur. Math. Soc. (JEMS)* **16**(4) (2014), 695–747.
- [Ros] D. Rossi, ‘A reduction theorem for the Character Triple Conjecture’, to appear in *Proc. Lond. Math. Soc.*
- [Ros22] D. Rossi, *Character Triple Conjecture, Towards the Inductive Condition for Dade’s Conjecture for Groups of Lie Type*, PhD thesis (Bergische Universität Wuppertal, 2022).
- [Ros23] D. Rossi, ‘The Brown complex in non-defining characteristic and applications’, preprint (2023). [arXiv:2303.13973](https://arxiv.org/abs/2303.13973).
- [Ros24a] D. Rossi, ‘Counting conjectures and e -local structures in finite reductive groups’, *Adv. Math.* **436** (2024), Article 109403, 61 pp.
- [Ros24b] D. Rossi, ‘The simplicial complex of Brauer pairs of a finite reductive group’, *Math. Z.* **308**(2) (2024), Article 27.
- [Ros24c] D. Rossi, ‘A local-global principle for unipotent characters’, *Forum Math. Sigma* **12** (2024), Article e125.
- [Ruh22a] L. Ruhstorfer, ‘The Alperin–McKay and Brauer’s Height Zero Conjecture for the prime 2’, *Ann. of Math.* (2) **201**(2) (2024), 379–457.

- [Ruh22b] L. Ruhstorfer, ‘Jordan decomposition for the Alperin–McKay conjecture’, *Adv. Math.* **394** (2022), Article 108031.
- [Ruh22c] L. Ruhstorfer, ‘Quasi-isolated blocks and the Alperin–McKay conjecture’, *Forum Math. Sigma* **10** (2022), Article e48, 43 pp.
- [SF14] A. A. Schaeffer Fry, ‘ $Sp_6(2^a)$ is “good” for the McKay, Alperin weight, and related local-global conjectures’, *J. Algebra* **401** (2014), 13–47.
- [Spä12] B. Späth, ‘Inductive McKay condition in defining characteristic’, *Bull. Lond. Math. Soc.* **44**(3) (2012), 426–438.
- [Spä13] B. Späth, ‘A reduction theorem for the Alperin–McKay conjecture’, *J. Reine Angew. Math.* **680** (2013), 153–189.
- [Spä17] B. Späth, ‘A reduction theorem for Dade’s projective conjecture’, *J. Eur. Math. Soc. (JEMS)* **19**(4) (2017), 1071–1126.
- [Spä23] B. Späth, ‘Extensions of characters in type D and the inductive McKay condition, I’, *Nagoya Math. J.* **252** (2023), 906–958.