Canad. Math. Bull. Vol. **61** (3), 2018 pp. 483–494
http://dx.doi.org/10.4153/CMB-2017-084-3

Tannakian Duality for Affine Homogeneous Spaces

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Abstract. Associated with any closed quantum subgroup $G \subset U_N^+$ and any index set $I \subset \{1, \dots, N\}$ is a certain homogeneous space $X_{G,I} \subset S^{N-1}_{\mathbb{C},+}$, called an affine homogeneous space. Using Tannakian duality methods, we discuss the abstract axiomatization of the algebraic manifolds $X \subset S^{N-1}_{\mathbb{C},+}$ that can appear in this way.

Introduction

Compact quantum groups were introduced by Woronowicz in [13, 14]. These are abstract objects, having no points in general, generalizing at the same time the usual compact groups and the duals of discrete groups. Compact quantum groups have no Lie algebra in general, but analogues of the Peter–Weyl theory, Tannakian duality, and the Weingarten integration formula are available. Thus, we have here some interesting examples of noncommutative manifolds that are definitely algebraic and which are probably a bit Riemannian too, because we can integrate on them.

This is a continuation of [1], which was concerned with integration theory over the associated homogeneous spaces. The main finding there was the fact that, in order to have a good integration theory, one must restrict attention to a certain special class of homogeneous spaces, called "affine". To be more precise, associated with any closed subgroup $G \subset U_N^+$ and any index set $I \subset \{1, \dots, N\}$ is a certain homogeneous space $X_{G,I} \subset S^{N-1}_{\mathbb{C},+}$, called affine. In the classical case this space appears as $X_{G,I} = G/(G \cap C_N^I)$, where $C_N^I \subset U_N$ is the group of unitaries fixing the vector $\xi_I = \frac{1}{\sqrt{|I|}} \sum_{i \in I} e_i$. In general, however, there are many new twists and questions coming from noncommutativity. Importantly, this construction covers many interesting examples; see [1].

One question left open in [1] was that of finding an abstract axiomatization of the algebraic manifolds $X \subset S^{N-1}_{\mathbb{C},+}$ that can appear in this way. We will answer this question with a Tannakian characterization of such manifolds. We believe that some further improvements of this result can lead to an axiomatization of the "easy algebraic manifolds", which was the main question in [1] and which remains open.

The paper is organized as follows: Section 1 is a preliminary section; in Sections 2 and 3, we state and prove the Tannakian duality result, and Section 4 contains a number of further results.

Received by the editors June 11, 2017; revised December 21, 2017.
Published electronically February 28, 2018.
AMS subject classification: **46L65**, 46L89.
Keywords: quantum isometry, noncommutative manifold.

1 Homogeneous Spaces

We use Woronowicz's quantum group formalism in [13, 14], with the extra axiom $S^2 = \mathrm{id}$. To be more precise, we will need the following definition.

Definition 1.1 Assume that A is a unital C^*-algebra, and that $u \in M_N(A)$ is a unitary matrix whose coefficients generate A such that the following formulae define morphisms of C^*-algebras $\Delta\colon A \to A \otimes A$, $\varepsilon\colon A \to \mathbb{C}$, and $S\colon A \to A^{\mathrm{opp}}$:

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}, \quad \varepsilon(u_{ij}) = \delta_{ij}, \quad S(u_{ij}) = u_{ji}^*.$$

We then write $A = C(G)$ and call G a compact matrix quantum group.

The above maps Δ, ε, S are called *comultiplication*, *counit*, and *antipode*, respectively. The basic examples include compact Lie groups $G \subset U_N$, their q-deformations at $q = -1$, and the duals of the finitely generated discrete groups $\Gamma = \langle g_1, \dots, g_N \rangle$; see [13].

We recall that the free unitary quantum group U_N^+, constructed by Wang in [11], and the corresponding free complex sphere $S^{N-1}_{\mathbb{C},+}$, from [3], are constructed as follows:

$$C(U_N^+) = C^*\Big((u_{ij})_{i,j=1,\dots,N} \;\Big|\; u^* = u^{-1}, u^t = \bar{u}^{-1} \Big)$$
$$C(S^{N-1}_{\mathbb{C},+}) = C^*\Big(x_1, \dots, x_N \;\Big|\; \sum_i x_i x_i^* = \sum_i x_i^* x_i = 1 \Big).$$

Here both algebras on the right are by definition universal C^*-algebras.

Following [1], we can now formulate the following definition.

Definition 1.2 An *affine homogeneous space* over a closed subgroup $G \subset U_N^+$ is a closed subset $X \subset S^{N-1}_{\mathbb{C},+}$, such that there exists an index set $I \subset \{1, \dots, N\}$ such that

$$\alpha(x_i) = \frac{1}{\sqrt{|I|}} \sum_{j \in I} u_{ij}, \quad \Phi(x_i) = \sum_j u_{ij} \otimes x_j$$

define morphisms of C^*-algebras satisfying $(\int_G \otimes \mathrm{id})\Phi = \int_G \alpha(\cdot)1$.

Here, and in what follows, a closed subspace $Y \subset Z$ corresponds by definition to a quotient map $C(Z) \to C(Y)$. As for $\int_G$, this is Haar integration; see [13].

As a first observation, the coaction condition $(\mathrm{id} \otimes \Phi)\Phi = (\Delta \otimes \mathrm{id})\Phi$ is satisfied, and we also have $(\mathrm{id} \otimes \alpha)\Phi = \Delta\alpha$. In the case where α is injective, we have the following proposition.

Proposition 1.3 *When α is injective, we must have $X = X^{\min}_{G,I}$, where*

$$C(X^{\min}_{G,I}) = \Big\langle \frac{1}{\sqrt{|I|}} \sum_{j \in I} u_{ij} \;\Big|\; i = 1, \dots, N \Big\rangle \subset C(G).$$

Moreover, $X^{\min}_{G,I}$ is affine homogeneous, for any $G \subset U_N^+$ and any $I \subset \{1, \dots, N\}$.

https://doi.org/10.4153/CMB-2017-084-3 Published online by Cambridge University Press

Proof The first assertion is clear from the definitions. Regarding the second assertion, consider the variables $z_i = \frac{1}{\sqrt{|I|}} \sum_{j \in I} u_{ij} \in C(G)$. Then we have

$$\Delta(z_i) = \frac{1}{\sqrt{|I|}} \sum_{j \in I} \sum_k u_{ik} \otimes u_{kj} = \sum_k u_{ik} \otimes z_k.$$

Thus, we have a coaction map as in Definition 1.2, given by $\Phi = \Delta$, and the ergodicity condition, namely $(\int_G \otimes \mathrm{id})\Delta = \int_G(\cdot)1$, holds as well, by the definition of $\int_G$; see [1]. ∎

Given exponents $e_1, \dots, e_k \in \{1, *\}$, consider the following quantities:

$$P_{i_1, \cdots i_k j_1 \cdots j_k} = \int_G u_{i_1 j_1}^{e_1} \cdots u_{i_k j_k}^{e_k}.$$

Once again following [1], we have the following result.

Proposition 1.4 *We must have $X \subset X_{G,I}^{\max}$, as subsets of $S_{\mathbb{C},+}^{N-1}$, where*

$$C(X_{G,I}^{\max}) = C(S_{\mathbb{C},+}^{N-1}) \Big/ \Big\langle (Px^{\otimes k})_{i_1 \cdots i_k} = \frac{1}{\sqrt{|I|^k}} \sum_{j_1, \dots, j_k \in I} P_{i_1 \cdots i_k j_1 \cdots j_k} \Bigm| \forall k, \forall i_1, \dots, i_k \Big\rangle.$$

Moreover, $X_{G,I}^{\max}$ is affine homogeneous, for any $G \subset U_N^+$, and any $I \subset \{1, \dots, N\}$.

Proof The idea here is that the ergodicity condition $(\int_G \otimes \mathrm{id})\Phi = \int_G \alpha(\cdot)1$ produces the relations in the statement. To be more precise, observe that we have

$$\begin{aligned}
&\Big(\int_G \otimes \mathrm{id}\Big)\Phi = \int \alpha(\cdot)1 \\
&\qquad \Longleftrightarrow \Big(\int_G \otimes \mathrm{id}\Big)\Phi(x_{i_1}^{e_1} \cdots x_{i_k}^{e_k}) = \frac{1}{\sqrt{|I|^k}} \int_G \alpha(x_{i_1}^{e_1} \cdots x_{i_k}^{e_k}), \quad \forall k, \forall i_1, \dots, i_k \\
&\qquad \Longleftrightarrow \sum_{j_1, \dots, j_k} P_{i_1 \cdots i_k j_1 \cdots j_k} x_{j_1}^{e_1} \cdots x_{j_k}^{e_k} = \frac{1}{\sqrt{|I|^k}} \sum_{j_1, \dots, j_k \in I} P_{i_1 \cdots i_k j_1 \cdots j_k}, \quad \forall k, \forall i_1, \dots, i_k
\end{aligned}$$

Thus, we have $X \subset X_{G,I}^{\max}$, and the last assertion is standard as well; see [1]. ∎

We will need one more general result from [1], namely an extension of the Weingarten integration formula [2,5,6,12] to the affine homogeneous space setting.

Proposition 1.5 *Assuming that $G \to X$ is an affine homogeneous space with index set $I \subset \{1, \dots, N\}$, the Haar integration functional $\int_X = \int_G \alpha$ is given by*

$$\int_X x_{i_1}^{e_1} \cdots x_{i_k}^{e_k} = \sum_{\pi, \sigma \in D} (\xi_\pi)_{i_1 \cdots i_k} K_I(\sigma) W_{kN}(\pi, \sigma),$$

where $\{\xi_\pi | \pi \in D\}$ is a basis of $\mathrm{Fix}(u^{\otimes k})$, $W_{kN} = G_{kN}^{-1}$ with $G_{kN}(\pi, \sigma) = \langle \xi_\pi, \xi_\sigma \rangle$ is the associated Weingarten matrix, and $K_I(\sigma) = \frac{1}{\sqrt{|I|^k}} \sum_{b_1, \dots, b_k \in I} \overline{(\xi_\sigma)_{b_1 \cdots b_k}}$.

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Proof By using the Weingarten formula for the quantum group G, we have

$$\int_X x_{i_1}^{e_1}\cdots x_{i_k}^{e_k} = \frac{1}{\sqrt{|I|^k}}\sum_{b_1,\ldots,b_k\in I}\int_G u_{i_1b_1}^{e_1}\cdots u_{i_kb_k}^{e_k}$$
$$= \frac{1}{\sqrt{|I|^k}}\sum_{b_1,\ldots,b_k\in I}\sum_{\pi,\sigma\in D}(\xi_\pi)_{i_1\cdots i_k}\overline{(\xi_\sigma)_{b_1\cdots b_k}}W_{kN}(\pi,\sigma).$$

But this gives the formula in the statement, and we are done; see [1]. ∎

Finally, here is a natural example of an intermediate space $X_{G,I}^{\min}\subset X\subset X_{G,I}^{\max}$.

Proposition 1.6 *Given a closed quantum subgroup $G\subset U_N^+$, and a set $I\subset\{1,\ldots,N\}$, if we consider the quotient algebra*

$$C(X_{G,I}^{\mathrm{med}}) =$$
$$C(S_{\mathbb{C},+}^{N-1})\Big/\Big\langle \sum_{a_1,\ldots,a_k}\xi_{a_1\cdots a_k}x_{a_1}^{e_1}\cdots x_{a_k}^{e_k} = \frac{1}{\sqrt{|I|^k}}\sum_{b_1,\ldots,b_k\in I}\xi_{b_1\cdots b_k}\ \Big|\ \forall k,\forall\xi\in\mathrm{Fix}(u^{\otimes k})\Big\rangle,$$

we obtain in this way an affine homogeneous space $G\to X_{G,I}$.

Proof We know from Proposition 1.4 that $X_{G,I}^{\max}\subset S_{\mathbb{C},+}^{N-1}$ is constructed by imposing on the standard coordinates the conditions $Px^{\otimes k}=P^I$, where

$$P_{i_1\cdots i_kj_1\cdots j_k} = \int_G u_{i_1j_1}^{e_1}\cdots u_{i_kj_k}^{e_k},\quad P_{i_1\cdots i_k}^I = \frac{1}{\sqrt{|I|^k}}\sum_{j_1,\ldots,j_k\in I}P_{i_1\cdots i_kj_1\cdots j_k}.$$

According to the Weingarten integration formula for G, we have

$$(Px^{\otimes k})_{i_1\cdots i_k} = \sum_{a_1,\ldots,a_k}\sum_{\pi,\sigma\in D}(\xi_\pi)_{i_1\cdots i_k}\overline{(\xi_\sigma)_{a_1\cdots a_k}}W_{kN}(\pi,\sigma)x_{a_1}^{e_1}\cdots x_{a_k}^{e_k},$$
$$P_{i_1\cdots i_k}^I = \frac{1}{\sqrt{|I|^k}}\sum_{b_1,\ldots,b_k\in I}\sum_{\pi,\sigma\in D}(\xi_\pi)_{i_1\cdots i_k}\overline{(\xi_\sigma)_{b_1\cdots b_k}}W_{kN}(\pi,\sigma).$$

Thus, $X_{G,I}^{\mathrm{med}}\subset X_{G,I}^{\max}$, and the other assertions are standard as well; see [1]. ∎

We can now put everything together, as follows.

Theorem 1.7 *Given a closed subgroup $G\subset U_N^+$ and a subset $I\subset\{1,\ldots,N\}$, the affine homogeneous spaces over G with index set I have the following properties.*

(i) *These are exactly the intermediate subspaces $X_{G,I}^{\min}\subset X\subset X_{G,I}^{\max}$ on which G acts affinely, with the action being ergodic.*
(ii) *For the minimal and maximal spaces $X_{G,I}^{\min}$ and $X_{G,I}^{\max}$, as well as for the intermediate space $X_{G,I}^{\mathrm{med}}$ constructed above, these conditions are satisfied.*
(iii) *By performing the GNS construction with respect to the Haar integration functional $\int_X=\int_G\alpha$, we obtain the minimal space $X_{G,I}^{\min}$.*

We agree to identify all these spaces, via the GNS construction, and denote them by $X_{G,I}$.

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Proof Indeed, this follows by combining the various results and observations formulated above. Once again, for full details on all these facts, we refer the reader to [1]. ■

Observe the similarity with what happens for the C^*-algebras of discrete groups, where the various intermediate algebras $C^*(\Gamma) \to A \to C^*_{\mathrm{red}}(\Gamma)$ must be identified as well, in order to reach to a unique noncommutative space $\widehat{\Gamma}$. For details here, see [13].

Regarding the basic examples of such spaces, we have the following proposition.

Proposition 1.8 *Given $N \in \mathbb{N}$ and $I \subset \{1, \dots, N\}$, the following hold:*

(i) *In the classical case, $G \subset U_N$, we have $X_{G,I} = G/(G \cap C_N^I)$, where $C_N^I \subset U_N$ is the group of unitaries fixing the vector $\xi_I = \frac{1}{\sqrt{|I|}}(\delta_{i\in I})_i$.*

(ii) *In the group dual case, $G = \widehat{\Gamma} \subset U_N^+$ with $\Gamma = \langle g_1, \dots, g_N \rangle$, embedded via $u_{ij} = \delta_{ij} g_i$, we have $X_{G,I} = \widehat{\Gamma_I}$, with $\Gamma_I = \langle g_i | i \in I \rangle \subset \Gamma$.*

Proof In this statement, (i) follows from the fact that the action $G \curvearrowright X_{G,I}$ can be shown to be transitive, and the stabilizer of ξ_I is the group $G \cap C_N^I$ in the statement. As for (ii), this follows directly from Definition 1.2, by using $u_{ij} = \delta_{ij} g_i$; see [1]. ■

One interesting question is that of understanding how much of (i) can apply to the general case. The answer here is as follows, with (ii) providing counterexamples.

Proposition 1.9 *The quotient map*

$$G/(G \cap C_N^{I+}) \longrightarrow X_{G,I},$$

is, in general, proper, where $C_N^{I+} \subset U_N^+$ is the subgroup defined by

$$C(C_N^{I+}) = C(U_N^+)/\langle \xi_I = \xi_I \rangle$$

and the relation $u\xi_I = \xi_I$ is interpreted as an equality of column vectors over $C(U_N^+)$.

Proof Observe first that C_N^{I+} is indeed a quantum group. In fact, it is standard to exhibit an isomorphism $C_N^{+I} \simeq U_{N-1}^+$, by reasoning as in [10]. We must check that the defining relations for $C(G/(G \cap C_N^{I+}))$ hold for the standard generators $x_i \in C(X_{G,I})$. But if we denote the quotient map by $\pi\colon C(G) \to C(G \cap C_N^{I+})$, we have

$$(\mathrm{id} \otimes \pi)\Delta x_i = (\mathrm{id} \otimes \pi)\Big(\frac{1}{\sqrt{|I|}} \sum_{j\in I} \sum_k u_{ik} \otimes u_{kj} \Big) = \sum_k u_{ik} \otimes (\xi_I)_k = x_i \otimes 1,$$

as desired.

Finally, for the group duals this quotient map is given by $\widehat{\Gamma'_I} \to \widehat{\Gamma_I}$, where $\Gamma'_I \subset \Gamma$ is the normal closure of Γ_I, and so this map can be indeed proper; see [1]. ■

2 Algebraic Manifolds

In what follows, we discuss the axiomatization of affine homogeneous spaces, as algebraic submanifolds of the free sphere $S^{N-1}_{\mathbb{C},+}$. We use the following formalism.

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Definition 2.1 A closed subset $X \subset S^{N-1}_{\mathbb{C},+}$ is called *algebraic* when

$$C(X) = C(S^{N-1}_{\mathbb{C},+}) \Big/ \Big\langle P_i(x_1,\dots,x_N) = 0, \forall i \in I \Big\rangle$$

for a certain family of noncommutative $*$-polynomials $P_i \in \mathbb{C}\langle x_1,\dots,x_N\rangle$.

There are many examples of such manifolds, as for instance all the compact matrix quantum groups. Indeed, assuming that we have a closed subgroup $G \subset U_N^+$, by rescaling the standard coordinates we obtain an embedding $G \subset U_N^+ \subset S^{N^2-1}_{\mathbb{C},+}$, and the following result, coming from [14], shows that we indeed have an algebraic manifold.

Proposition 2.2 *Given a closed subgroup $G \subset U_N^+$, with the corresponding fundamental corepresentations denoted $u \to v$, we have the formula*

$$C(G) = C(U_N^+) \Big/ \Big(T \in \mathrm{Hom}(u^{\otimes k}, u^{\otimes l}), \forall k,l, \forall T \in \mathrm{Hom}(v^{\otimes k}, v^{\otimes l}) \Big)$$

with $k,l = \cdots \circ \bullet \bullet \circ \bullet \cdots$ being colored integers, with the tensor power conventions $w^\circ = w, w^\bullet = \overline{w}, w^{xy} = w^x \otimes w^y$, and with the notation $\mathrm{Hom}(r,p) = \{T | Tr = pT\}$.

Proof For any choice of two colored integers k,l and of an intertwiner $T \in \mathrm{Hom}(v^{\otimes k}, v^{\otimes l})$, the formula $T \in \mathrm{Hom}(u^{\otimes k}, u^{\otimes l})$, with $u = (u_{ij})$ being the fundamental corepresentation of $C(U_N^+)$, corresponds to a collection of N^{k+l} relations between the variables u_{ij}. By dividing $C(U_N^+)$ by the ideal generated by all these relations when k, l, and T vary, we obtain a certain algebra A, which is the algebra on the right in the statement.

It is clear that we have a surjective morphism $A \to C(G)$, and by using Woronowicz's Tannakian results in [14], this surjective morphism follows to be an isomorphism. For a short recent proof of this fact using basic Hopf algebra theory, see [9]. ■

In relation to affine homogeneous spaces, we have the following proposition.

Proposition 2.3 *Any affine homogeneous space $X_{G,I} \subset S^{N-1}_{\mathbb{C},+}$ is algebraic, with*

$$\sum_{i_1,\dots,i_k} \xi_{i_1\cdots i_k} x_{i_1}^{e_1} \cdots x_{i_k}^{e_k} = \frac{1}{\sqrt{|I|^k}} \sum_{b_1,\dots,b_k \in I} \xi_{b_1\cdots b_k} \quad \forall k, \forall \xi \in \mathrm{Fix}(u^{\otimes k})$$

as defining relations. Moreover, we can use vectors ξ belonging to a basis of $\mathrm{Fix}(u^{\otimes k})$.

Proof Indeed this follows from the various results in Section 1 and, more specifically, from Proposition 1.6, using the identifications made in Theorem 1.7. ■

In order to reach a more categorical description of $X_{G,I}$, we will use Frobenius duality. We use colored indices, and we denote by $k \to \overline{k}$ the operation on the colored indices that consists in reversing the index and switching all the colors. Also, we agree to identify the linear maps $T: (\mathbb{C}^N)^{\otimes k} \to (\mathbb{C}^N)^{\otimes l}$ with the corresponding rectangular matrices $T \in M_{N^l \times N^k}(\mathbb{C})$, written $T = (T_{i_1\cdots i_l j_1 \cdots j_k})$. With these conventions, the precise formulation of Frobenius duality that we will need is as follows.

Proposition 2.4 *We have an isomorphism of complex vector spaces*

$$T \in \mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) \longleftrightarrow \xi \in \mathrm{Fix}(u^{\otimes l} \otimes u^{\otimes \overline{k}})$$

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given by the formulae $T_{i_1\cdots i_l j_1\cdots j_k} = \xi_{i_1\cdots i_l j_k\cdots j_1}$ *and* $\xi_{i_i\cdots i_l j_1\cdots j_k} = T_{i_1\cdots i_l j_k\cdots j_1}$.

Proof This is a well-known result that follows from the general theory in [13]. To be more precise, given integers $K, L \in \mathbb{N}$, consider the following standard isomorphism, which in matrix notation makes $T = (T_{IJ}) \in M_{L\times K}(\mathbb{C})$ correspond to $\xi = (\xi_{IJ})$:

$$T \in \mathcal{L}(\mathbb{C}^{\otimes K}, \mathbb{C}^{\otimes L}) \longleftrightarrow \xi \in \mathbb{C}^{\otimes L+K}.$$

Given two arbitrary corepresentations $v \in M_K(C(G))$ and $w \in M_L(C(G))$, the abstract Frobenius duality result established in [13] states that the above isomorphism restricts into an isomorphism of vector spaces as follows:

$$T \in \mathrm{Hom}(v, w) \longleftrightarrow \xi \in \mathrm{Fix}(w \otimes \overline{v}).$$

In our case, we can apply this result with $v = u^{\otimes k}$ and $w = u^{\otimes l}$. Since, according to our conventions, we have $\overline{v} = u^{\otimes \overline{k}}$, this gives the isomorphism in the statement. ∎

With the above result in hand, we can enhance the construction of $X_{G,I}$, as follows.

Theorem 2.5 *Any affine homogeneous space* $X_{G,I}$ *is algebraic, with*

$$\sum_{\substack{i_1\cdots i_l\\ j_1\cdots j_k}} T_{i_1\cdots i_l j_1\cdots j_k} x_{i_1}^{e_1}\cdots x_{i_l}^{e_l}(x_{j_1}^{f_1}\cdots x_{j_k}^{f_k})^* = \frac{1}{\sqrt{|I|^{k+l}}}\sum_{\substack{b_1\cdots b_l\in I\\ c_1,\ldots,c_k\in I}} T_{b_1\cdots b_l c_1\cdots c_k}$$

for any k, l, *and any* $T \in \mathrm{Hom}(u^{\otimes k}, u^{\otimes l})$, *as defining relations.*

Proof We must prove that the relations in the statement are satisfied, over $X_{G,I}$. We know from Proposition 2.3, with $k \to l\overline{k}$, that the following relation holds:

$$\sum_{\substack{i_1,\ldots,i_l\\ j_1\cdots j_k}} \xi_{i_1\cdots i_l j_k\cdots j_1} x_{i_1}^{e_1}\cdots x_{i_l}^{e_l} x_{j_k}^{\overline{f}_k}\cdots x_{j_1}^{\overline{f}_1} = \frac{1}{\sqrt{|I|^{k+l}}}\sum_{\substack{b_1,\ldots,b_l\in I\\ c_1,\ldots,c_k\in I}} \xi_{b_1\cdots b_l c_k\cdots c_1}.$$

In terms of the matrix $T_{i_1\cdots i_l j_1\cdots j_k} = \xi_{i_1\cdots i_l j_k\cdots j_1}$ from Proposition 2.3, we obtain

$$\sum_{\substack{i_1,\ldots,i_l\\ j_1\cdots j_k}} T_{i_1\cdots i_l j_1\cdots j_k} x_{i_1}^{e_1}\cdots x_{i_l}^{e_l} x_{j_k}^{\overline{f}_k}\cdots x_{j_1}^{\overline{f}_1} = \frac{1}{\sqrt{|I|^{k+l}}}\sum_{\substack{b_1\cdots b_l\in I\\ c_1\cdots c_k\in I}} T_{b_1\cdots b_l c_1\cdots c_k}.$$

This gives the formula in the statement, and we are done. ∎

3 Tannakian Duality

In this section we state and prove our main result. The description of the affine homogeneous spaces found in Theorem 2.5 suggests the following notion.

Definition 3.1 Given an algebraic submanifold $X \subset S^{N-1}_{\mathbb{C},+}$ and a subset $I \subset \{1,\ldots,N\}$, we say that X is *I-affine* when $C(X)$ is presented by relations of type

$$\sum_{\substack{i_1,\ldots,i_l\\ j_1\cdots j_k}} T_{i_1\cdots i_l j_1\cdots j_k} x_{i_1}^{e_1}\cdots x_{i_l}^{e_l}(x_{j_1}^{f_1}\cdots x_{j_k}^{f_k})^* = \frac{1}{\sqrt{|I|^{k+l}}}\sum_{\substack{b_1,\ldots,b_l\in I\\ c_1,\ldots,c_k\in I}} T_{b_1\cdots b_l c_1\cdots c_k}$$

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with the operators T belonging to certain linear spaces $F(k,l) \subset M_{N^l \times N^k}(\mathbb{C})$, which altogether form a tensor category $F = (F(k,l))$.

According to Theorem 2.5, any affine homogeneous space $X_{G,I}$ is an I-affine manifold, with the corresponding tensor category being the one associated with the quantum group $G \subset U_N^+$ that produces it, formed by the linear spaces $F(k,l) = \mathrm{Hom}(u^{\otimes k}, u^{\otimes l})$.

We will need some basic facts regarding quantum affine actions. Following Definition 1.2, we say that a closed subgroup $G \subset U_N^+$ *acts affinely* on a closed subset $X \subset S^{N-1}_{\mathbb{C},+}$ when the formula $\Phi(x_i) = \sum_j u_{ij} \otimes x_j$ defines a morphism of C^*-algebras.

We have the following standard result from [4], inspired by [7, 8].

Proposition 3.2 *Given an algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$, the quantum group*

$$G^+(X) = \max\left\{ G \subset U_N^+ \,\middle|\, G \curvearrowright X \right\}$$

exists and is unique. We call it the affine quantum isometry group of X.

Proof In order to have a universal coaction, the relations defining $G^+(X) \subset U_N^+$ must be those making $x_i \to X_i = \sum_j u_{ij} \otimes x_j$ a morphism of algebras. Thus, in order to construct $G^+(X)$, we just have to clarify how the relations $P_\alpha(x_1, \dots, x_N) = 0$ defining X are interpreted inside $C(U_N^+)$. So, pick one such polynomial, $P = P_\alpha$, and write it as follows:

$$P(x_1, \dots, x_N) = \sum_r \alpha_r \cdot x_{i_1^r} \cdots x_{i_{s(r)}^r}.$$

Now, if we formally replace each coordinate $x_i \in C(X)$ by the corresponding element $X_i = \sum_j u_{ij} \otimes x_j \in C(U_N^+) \otimes C(X)$, the following formula must hold:

$$P(X_1, \dots, X_N) = \sum_r \alpha_r \sum_{j_1^r, \dots, j_{s(r)}^r} u_{i_1^r j_1^r} \cdots u_{i_{s(r)}^r j_{s(r)}^r} \otimes x_{j_1^r} \cdots x_{j_{s(r)}^r}.$$

Thus, the relations $P(X_1, \dots, X_N) = 0$ correspond to certain polynomial relations between the standard generators $u_{ij} \in C(U_N^+)$, and this gives the result; see [4]. ■

Now by getting back to our questions, let us study the quantum isometry groups of the manifolds $X \subset S^{N-1}_{\mathbb{C},+}$ that are I-affine. We have here the following result.

Proposition 3.3 *For an I-affine manifold $X \subset S^{N-1}_{\mathbb{C},+}$, we have $G \subset G^+(X)$, where $G \subset U_N^+$ is the Tannakian dual of the associated tensor category F.*

Proof We recall from the proof of Proposition 3.2 that the relations defining $G^+(X)$ are those expressing the vanishing of the following quantities:

$$P(X_1, \dots, X_N) = \sum_r \alpha_r \sum_{j_1^r, \dots, j_{s(r)}^r} u_{i_1^r j_1^r} \cdots u_{i_{s(r)}^r j_{s(r)}^r} \otimes x_{j_1^r} \cdots x_{j_{s(r)}^r}.$$

In the case of an I-affine manifold, the defining relations are those from Definition 3.1, with the corresponding polynomials P being indexed by the elements of F. But the vanishing of the associated relations $P(X_1, \dots, X_N) = 0$ corresponds precisely to the Tannakian relations defining $G \subset U_N^+$, and so we obtain $G \subset G^+(X)$, as claimed. ■

We now have all the needed ingredients, and we can prove the following theorem.

Theorem 3.4 *Assume that an algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$ is I-affine, with associated tensor category F.*

(i) *We have an inclusion $G \subset G^+(X)$, where G is the Tannakian dual of F.*
(ii) *X is an affine homogeneous space, $X = X_{G,I}$, over this quantum group G.*

Proof In the context of Definition 3.1, the tensor category F there gives rise, by the Tannakian result in Proposition 2.2, to a quantum group $G \subset U_N^+$. What is left now is to construct the affine space morphisms α, Φ, and the proof here goes as follows:

(i) Construction of α. We want to construct a morphism,

$$\alpha\colon C(X) \longrightarrow C(G)\colon x_i \longrightarrow X_i = \frac{1}{\sqrt{|I|}} \sum_{j \in I} u_{ij}.$$

In view of Definition 3.1, we must therefore prove that

$$\sum_{\substack{i_1,\dots,i_l\\ j_1,\dots,j_k}} T_{i_1\cdots i_l j_1\cdots j_k} X_{i_1}^{e_1} \cdots X_{i_l}^{e_l} (X_{j_1}^{f_1} \cdots X_{j_k}^{f_k})^* = \frac{1}{\sqrt{|I|^{k+l}}} \sum_{\substack{b_1,\dots,b_l \in I\\ c_1,\dots,c_k \in I}} T_{b_1\cdots b_l c_1\cdots c_k}.$$

By replacing the variables X_i by their above values, we want to prove that

$$\sum_{\substack{i_1,\dots,i_l\\ j_1,\dots,j_k}} \sum_{\substack{r_1,\dots,r_l \in I\\ s_1,\dots,s_k \in I}} T_{i_1\cdots i_l j_1\cdots j_k} u_{i_1 r_1}^{e_1} \cdots u_{i_l r_l}^{e_l} (u_{j_1 s_1}^{f_1} \cdots u_{j_k s_k}^{f_k})^* = \sum_{\substack{b_1,\dots,b_l \in I\\ c_1,\dots,c_k \in I}} T_{b_1\cdots b_l c_1\cdots c_k}.$$

Now observe that from the relation $T \in \mathrm{Hom}(u^{\otimes k}, u^{\otimes l})$ we obtain

$$\sum_{\substack{i_1,\dots,i_l\\ j_1,\dots,j_k}} T_{i_1\cdots i_l j_1\cdots j_k} u_{i_1 r_1}^{e_1} \cdots u_{i_l r_l}^{e_l} (u_{j_1 s_1}^{f_1} \cdots u_{j_k s_k}^{f_k})^* = T_{r_1\cdots r_l s_1\cdots s_k}.$$

Thus, by summing over indices $r_i \in I$ and $s_i \in I$, we obtain the desired formula.

(ii) Construction of Φ. We want to construct a morphism,

$$\Phi\colon C(X) \longrightarrow C(G) \otimes C(X)\colon x_i \longrightarrow X_i = \sum_j u_{ij} \otimes x_j.$$

But this is precisely the coaction map constructed in Proposition 3.3.

(iii) Proof of ergodicity. If we go back to Proposition 1.4, we see that the ergodicity condition is equivalent to a number of Tannakian conditions, which are automatic in our case. Thus, the ergodicity condition is automatic, and we are done. ■

4 Further Results

The Tannakian result obtained in Section 3, based on the notion of I-affine manifold from Definition 3.1, remains quite theoretical. The problem is that Definition 3.1 still makes reference to a tensor category, and so the abstract characterization of affine homogeneous spaces that we obtain in this way is not totally intrinsic.

We believe that some deeper results should hold as well. To be more precise, the work on noncommutative spheres in [4] suggests that the relevant category F should appear in a more direct way from X. In analogy with Definition 3.1, let us formulate the following definition.

https://doi.org/10.4153/CMB-2017-084-3 Published online by Cambridge University Press

Definition 4.1 Given a submanifold $X \subset S^{N-1}_{\mathbb{C},+}$ and a subset $I \subset \{1,\dots,N\}$, we let $F_{X,I}(k,l) \subset M_{N^l \times N^k}(\mathbb{C})$ be the linear space of linear maps T such that

$$\sum_{\substack{i_1,\dots,i_l\\ j_1,\dots,j_k}} T_{i_1\cdots i_l j_1\cdots j_k} x_{i_1}^{e_1}\cdots x_{i_l}^{e_l}(x_{j_1}^{f_1}\cdots x_{j_k}^{f_k})^* = \frac{1}{\sqrt{|I|^{k+l}}}\sum_{\substack{b_1,\dots,b_l\in I\\ c_1,\dots,c_k\in I}} T_{b_1\cdots b_l,c_1\cdots c_k}$$

holds over X. We say that X is *I-saturated* when $F_{X,I} = (F_{X,I}(k,l))$ is a tensor category and the collection of the above relations presents the algebra $C(X)$.

Observe that any I-saturated manifold is automatically I-affine. The point is that the results in [4] seem to suggest that the converse of this fact should hold, in the sense that any I-affine manifold should be automatically I-saturated. Such a result would of course substantially improve Theorem 3.4, and make it ready for applications.

We do not have a proof of this fact, but we would like to present now a few preliminary observations on this subject. First, we have the following result.

Proposition 4.2 *The linear spaces $F_{X,I}(k,l) \subset M_{N^l\times N^k}(\mathbb{C})$ constructed above have the following properties:*

(i) *They contain the units.*
(ii) *They are stable by conjugation.*
(iii) *They satisfy the Frobenius duality condition.*

Proof All of these assertions are elementary.

(i) Consider the unit map. The associated relation is

$$\sum_{i_1,\dots,i_k} x_{i_1}^{e_1}\cdots x_{i_k}^{e_k}(x_{i_1}^{e_1}\cdots x_{i_k}^{e_k})^* = 1.$$

But this relation holds indeed, due to the defining relations for $S^{N-1}_{\mathbb{C},+}$.

(ii) We have the following sequence of equivalences:

$$\begin{aligned}
T^* &\in F_{X,I}(l,k)\\
&\iff \sum_{\substack{i_1,\dots,i_l\\ j_1,\dots,j_k}} T^*_{j_1\cdots j_k i_1\cdots i_l} x_{j_1}^{f_1}\cdots x_{j_k}^{f_k}(x_{i_1}^{e_1}\cdots x_{i_l}^{e_l})^*\\
&\qquad = \frac{1}{\sqrt{|I|^{k+l}}}\sum_{b_1,\dots,i_l\in I}\ \sum_{c_1,\dots,c_k\in I} T^*_{c_1\cdots c_k b_1\cdots b_l}\\
&\iff \sum_{\substack{i_1,\dots,i_l\\ j_1,\dots,j_k}} T_{i_1\cdots i_l j_1\cdots j_k} x_{i_1}^{e_1}\cdots x_{i_l}^{e_l}(x_{j_1}^{f_1}\cdots x_{j_k}^{f_k})^*\\
&\qquad = \frac{1}{\sqrt{|I|^{k+l}}}\sum_{\substack{b_1,\dots,b_l\in I\\ c_1,\dots,c_k\in I}} T_{b_1\cdots b_l c_1\cdots c_k}\\
&\iff T \in F_{X,I}(k,l).
\end{aligned}$$

(iii) We have a correspondence $T \in F_{X,I}(k,l) \leftrightarrow \xi \in F_{X,I}(\emptyset, l\bar{k})$, given by the usual formulae for the Frobenius isomorphism, from Proposition 2.4. ■

Based on the above result, we can now formulate our observations, as follows.

https://doi.org/10.4153/CMB-2017-084-3 Published online by Cambridge University Press

Theorem 4.3 *Given a closed subgroup $G \subset U_N^+$, and an index set $I \subset \{1, \dots, N\}$, consider the corresponding affine homogeneous space $X_{G,I} \subset S^{N-1}_{\mathbb{C},+}$.*

(i) *$X_{G,I}$ is I-saturated precisely when the collection of spaces $F_{X,I} = (F_{X,I}(k,l))$ is stable under compositions and tensor products.*

(ii) *We have $F_{X,I} = F$ precisely when $\sum_{j_1,\dots,j_l \in I}(\sum_{i_1,\dots,i_l} \xi_{i_1\cdots i_l} u^{e_1}_{i_1 j_1} \cdots u^{e_l}_{i_l j_l} - \xi_{j_1\cdots j_l}) = 0$ implies that $\sum_{i_1,\dots,i_l} \xi_{i_1\cdots i_l} u^{e_1}_{i_1 j_1} \cdots u^{e_l}_{i_l j_l} - \xi_{j_1\cdots j_l} = 0$, for any $j_1, \dots, j_l$.*

Proof From Theorem 2.5, we know that with $F(k,l) = \mathrm{Hom}(u^{\otimes k}, u^{\otimes l})$, we have inclusions of vector spaces $F(k,l) \subset F_{X,I}(k,l)$. Moreover, once again by Theorem 2.5, the relations coming from the elements of the category formed by the spaces $F(k,l)$ present $X_{G,I}$. Thus, the relations coming from the elements of $F_{X,I}$ present $X_{G,I}$ as well.

With this observation in hand, our assertions follow from Proposition 4.2.

(i) According to Proposition 4.2(i) and (ii), the unit and conjugation axioms are satisfied, so the spaces $F_{X,I}(k,l)$ form a tensor category precisely when the remaining axioms, namely the composition and the tensor product one, are satisfied. Now by assuming that these two axioms are satisfied, it follows by the above observation that X is I-saturated.

(ii) Since we already have inclusions in one sense, the equality $F_{X,I} = F$ from the statement means that we must have inclusions in the other sense:

$$F_{X,I}(k,l) \subset F(k,l).$$

By using Proposition 4.2(iii), it is enough to discuss the case $k = 0$. And here, assuming that we have $\xi \in F_{X,L}(0,l)$, the following condition must be satisfied:

$$\sum_{i_1,\dots,i_l} \xi_{i_1\cdots i_l} x^{e_1}_{i_1} \cdots x^{e_l}_{i_l} = \sum_{j_1,\dots,j_l \in I} \xi_{j_1\cdots j_l}.$$

By applying the morphism $\alpha\colon C(X_{G,I}) \to C(G)$, we deduce that we have

$$\sum_{i_1,\dots,i_l} \xi_{i_1\cdots i_l} \sum_{j_1,\dots,j_l \in I} u^{e_1}_{i_1 j_1} \cdots u^{e_l}_{i_l j_l} = \sum_{j_1,\dots,j_l \in I} \xi_{j_1\cdots j_l}.$$

Now recall that $F(0,l) = \mathrm{Fix}(u^{\otimes l})$ consists of the vectors ξ satisfying

$$\sum_{i_1,\dots,i_l} \xi_{i_1\cdots i_l} u^{e_1}_{i_1 j_1} \cdots u^{e_l}_{i_l j_l} = \xi_{j_1\cdots j_l}, \forall j_1, \dots, j_l.$$

We are therefore led to the conclusion in the statement. ∎

It is quite unclear as to how to make progress on these questions, and a more advanced algebraic trick, in the spirit of those used in [4], seems to be needed. Nor is it clear how to explicitly "capture" the relevant subgroup $G \subset G^+(X)$, in terms of our given manifold $X = X_{G,I}$, in a direct, geometric way. Summarizing, further improving Theorem 3.4 is an interesting question that we would like to raise here.

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