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LOCALLY NILPOTENT GROUPS WITH THE MAXIMUM CONDITION ON NON-NILPOTENT SUBGROUPS

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Abstract. In this paper the authors consider the class of locally nilpotent groups that have the maximum condition on non-nilpotent subgroups.

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1. Introduction. The class of groups in which every proper subgroup is nilpotent has been considered in several papers. Finite groups with all proper subgroups nilpotent were considered by O. Yu. Schmidt in [**16**], where he showed that such groups are always soluble. Subsequent papers on the subject include [**14**] and [**17**]. Such groups are, of course, locally nilpotent or finitely generated. In either case the structure of groups with all proper subgroups nilpotent can be very complicated, as is seen from the Heineken-Mohamed examples. (See, for example, [**6**], [**4**], [**13**].)

In [**17**], H. Smith also considered the class of locally nilpotent groups in which each group satisfied a certain finiteness condition on its non-nilpotent subgroups. We continue this theme in the current paper. A group G is said to satisfy *max-(non-nil)*, the maximum condition on non-nilpotent subgroups, if every non-empty set of non-nilpotent subgroups has a maximal element, or equivalently, if every ascending chain of non-nilpotent subgroups terminates in finitely many steps. The class of groups with the maximum condition on non-abelian subgroups was considered in [**12**], where a complete structure theorem was given. In [**17**], Smith proved that a torsionfree locally nilpotent group with max-(non-nil) is necessarily nilpotent; (in fact Smith's result had apparently weaker hypotheses than max-(non-nil)). In this paper we consider further the class of groups with max-(non-nil). It is easy to construct locally nilpotent groups with max-(non-nil) in which not every proper subgroup is nilpotent; for example the direct product of a Heineken-Mohamed type group with a finite nilpotent group will do. (Indeed, it is not difficult to show that an infinite locally finite, non-nilpotent group with max-(non-nil) is a direct product of finitely many Sylow p-subgroups, exactly one of which is infinite and also a locally nilpotent non-nilpotent group). As further motivation for our work we note that Smith also proved in [**17**] that if p is a prime then a locally nilpotent p-group G with max-(non-nil) is nilpotent provided $G/G^{\mathfrak{F}}$ is infinite; here $G^{\mathfrak{F}}$ denotes the finite residual of G. In our work the finite residual again plays an important role.

We cannot obtain a detailed description of locally nilpotent groups with max-(non-nil), because of examples such as those of Heineken-Mohamed type. Our results divide naturally into two cases. In Section 2 we give a number of preliminary results on groups with max-(non-nil). In Section 3 we show that if G is a non-nilpotent, locally nilpotent group with max-(non-nil) and if T is the torsion subgroup of G then $G^{\mathfrak{F}} \leq T$. Our main result in Section 3 (Theorem A) partially generalizes Theorem 2.2 of [**17**]. We consider the case in which $G/G^{\mathfrak{F}}$ is not finitely generated and G is non-nilpotent. We prove, among other things, that in this case $T/G^{\mathfrak{F}}$ is finite and G/T is a nilpotent minimax group.

In Section 4 we consider the much harder case, in which $G/G^{\mathfrak{F}}$ is finitely generated. In this case $G^{\mathfrak{F}}$ need not be nilpotent. We show that $G^{\mathfrak{F}}$ is $\mathfrak{F}$-perfect if G is non-nilpotent; in the case of locally nilpotent p-groups this is an immediate corollary of Theorem 2.3 of [**17**]. In Theorem B we indicate that if G is non-nilpotent, but $G^{\mathfrak{F}}$ is nilpotent, and if $G/G^{\mathfrak{F}}$ is finitely generated then the structure of $G^{\mathfrak{F}}$ is very restricted, as consequently is that of G. In Theorem C we consider the case in which $G^{\mathfrak{F}}$ is non-nilpotent. In this case $G^{\mathfrak{F}}$ is a p-group, for some prime p, and every proper subgroup of $G^{\mathfrak{F}}$ is nilpotent.

Our notation, when not explained, is that in standard use. We should like to thank Professor A. O. Asar for sending us a preprint of his paper [**1**] and also the referee for his numerous suggestions which improved this paper.

2. Preliminaries. In this section we collect some easy preliminary results together that we use throughout the paper. The first two of these are quite straightforward and so we omit their proof. As usual we let max denote the maximum condition on subgroups.

LEMMA 2.1. *Let G be a group satisfying max-(non-nil).*
(i) *If H is a subgroup of G, then H satisfies max-(non-nil).*
(ii) *If H is a normal subgroup of G, then G/H satisfies max-(non-nil).*

The next result uses the well-known fact that a soluble group with the maximum condition on subgroups is polycyclic.

LEMMA 2.2. *Let G be a group satisfying max-(non-nil), and suppose H is a non-nilpotent subgroup of G. Then* $N_G(H)/H$ *satisfies max. In particular, if* $N_G(H)$ *is locally (soluble-by-finite), then* $N_G(H)/H$ *is polycyclic-by-finite.*

LEMMA 2.3. *Let G be a group satisfying max-(non-nil). Suppose that* H, A, B *are subgroups of G such that* $A \triangleleft B$*, A and B are H-invariant, and* $H \cap B \leq A$*. If AH is non-nilpotent, then B/A satisfies the maximal condition on H-invariant subgroups.*

Proof. Suppose that $B_1/A \leq B_2/A \dots$ is an ascending chain of H-invariant subgroups of B/A. Then there is an integer k such that $B_nH = B_kH$, for all $n \geq k$. Since $H \cap B \leq A$ we have $B_n = B_k$, by the Dedekind law. □

It is clear that any strictly ascending infinite chain of subgroups in a group with max-(non-nil) has the property that every subgroup in the chain is nilpotent. We record this in our next lemma.

LEMMA 2.4. *Let G be a group satisfying max-(non-nil) and suppose that* $G_1 \leq G_2 \leq \dots \leq G_n \leq \dots$ *is a strictly ascending chain of subgroups. Then* G_n *is nilpotent, for each* $n \in \mathbb{N}$.

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There are some natural, simple consequences of these easy results that will be very useful in what follows. First we have the following result.

Corollary 2.5. *Let G be a group satisfying max-(non-nil). Suppose that H, K are subgroups of G such that $K \triangleleft H$. If $H/K = A/K \times B/K$ and $A/K, B/K$ do not satisfy max, then H is nilpotent.*

Proof. Since $H/B \cong A/K$ we see that H/B does not satisfy max. By Lemma 2.2 the subgroup B is nilpotent. Similarly the subgroup A is nilpotent. By Fitting's Theorem $H = AB$ is also nilpotent. □

Corollary 2.6. *Let G be a group satisfying max-(non-nil). Suppose that H, K are subgroups of G such that $K \triangleleft H$. If $H/K = \mathrm{Dr}_{\lambda \in \Lambda} A_\lambda/K$ is a direct product of infinitely many non-trivial groups, then H is nilpotent.*

Proof. Since Λ is infinite it contains infinite subsets Λ_1, Λ_2 such that $\Lambda_1 \cup \Lambda_2 = \Lambda$, $\Lambda_1 \cap \Lambda_2 = \emptyset$. Now put $A/K = \mathrm{Dr}_{\lambda \in \Lambda_1} A_\lambda/K$, $B/K = \mathrm{Dr}_{\lambda \in \Lambda_2} A_\lambda/K$. Clearly, both A/K and B/K do not satisfy max and we can apply Corollary 2.5. □

Lemma 2.7. *Let G be a group satisfying max-(non-nil). Suppose that H, K, S are subgroups of G such that $K \triangleleft H$. Suppose also that*

(i) *$H/K = \mathrm{Dr}_{\lambda \in \Lambda} A_\lambda/K$, where $A_\lambda \neq K$ for every $\lambda \in \Lambda$ and the set Λ is infinite;*

(ii) *A_λ is S-invariant for every $\lambda \in \Lambda$;*

(iii) *$H \cap S \leq K$.*

Then the subgroup HS is nilpotent.

Proof. Write $H/K = U/K \times V/K$, where each of U/K and V/K is a direct product modulo K of infinitely many A_λ's. There is an infinite strictly ascending chain from US to HS and so US is nilpotent, as is VS. Thus there is an integer k such that $[U,_k S] \leq S$ and $[V,_k S] \leq S$. It is easy to see that, by induction, $[H,_k S] \leq KS$ and, since KS is nilpotent, it follows that S is subnormal in HS. Since H, S are nilpotent it follows that HS is also nilpotent, as required. □

3. The case in which $G/G^{\mathfrak{F}}$ is not finitely generated.

In this section we consider the case of a locally nilpotent group G such that $G/G^{\mathfrak{F}}$ is not finitely generated. We shall require some terminology which is well known, but which we briefly explain. Let G be an abelian group of finite (special) rank, and let H be a finitely generated subgroup of G such that G/H is periodic. Let $Sp(G)$, the Spectrum of G, denote the set of primes p such that a Sylow p-subgroup of G/H is infinite.

If L is another finitely generated subgroup of G such that G/L is periodic, then both factors $H/(H \cap L)$ and $L/(H \cap L)$ are finite. It follows that the set $Sp(G)$ is independent of the choice of finitely generated subgroup H.

Now let G be a nilpotent group of finite rank and let

$$1 = G_0 \leq G_1 \leq \ldots \leq G_n = G$$

be a central series of G. We let

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$$Sp(G) = Sp(G_1) \cup Sp(G_2/G_1) \cup \ldots \cup Sp(G_n/G_{n-1}).$$

Clearly the set $Sp(G)$ is independent of the choice of central series.

With these preliminaries out of the way we can prove a crucial lemma.

LEMMA 3.1. *Let G be a locally nilpotent group satisfying max-(non-nil) and let T be the torsion subgroup of G. If G is non-nilpotent, then G/T is either finitely generated or nilpotent minimax with $Sp(G/T) = \{p\}$, for some prime p. Moreover, in the latter case, G contains a normal nilpotent subgroup U such that G/U is a Prüfer p-group.*

Proof. We may assume that G/T is non-trivial. By Theorem 2.2 of [**17**] G/T is nilpotent. In particular, $G \neq G'$. Let $H = G/G'$. Let P be the torsion subgroup of H and let $K = H/P$. Note that if H/H^n is infinite, for some natural number n, then H/H^n is a bounded abelian group and hence is a direct sum of infinitely many non-trivial cyclic groups. Also the same is true of $G/G'G^n$. It follows from Corollary 2.6 that G is then nilpotent, a contradiction. Thus H/H^n is finite for each n.

Suppose first that K has infinite rank. Let $\mathcal{M}$ be a maximal linearly independent subset of K and let $M = \langle \mathcal{M} \rangle$. Then $M = \operatorname{Dr}_{i \in I} N_i$, where $N_i \cong \mathbb{Z}$. Clearly K/M is periodic and I is infinite. Let J be a countably infinite subset of I and let $\{p_j \mid j \in J\}$ be an infinite set of primes. Let $L = \operatorname{Dr}_{i \in I \setminus J} N_i \times \operatorname{Dr}_{j \in J} N_j^{p_j}$. Then K/L is periodic and $\pi(K/L)$ is infinite. Thus $K/L = \operatorname{Dr}_{p \in \mathbb{P}} X_p/L$, where for infinitely many primes p we have $X_p \neq L$. By Corollary 2.6 it follows that G is nilpotent in this case, which is a contradiction. Consequently H/P has finite rank.

Suppose next that P is trivial. Let $\{h_1, \ldots, h_r\}$ be a maximal independent subset of H, and let $R = \langle h_1, \ldots, h_r \rangle$. Then H/R is a periodic group. If H/R is finite, then H is finitely generated. If H/R is infinite then, by Corollary 2.5, $H/R = H_1/R \times H_2/R$ where H_1/R is a Prüfer p-subgroup, for some prime p, and H_2/R is finite. It follows that H is minimax and $Sp(H) = \{p\}$. Moreover in this case $H/H_2 \cong C_{p^\infty}$. Let U be a preimage of H_2 in G. By Lemma 2.4, U is nilpotent.

Suppose that $P \neq 1$. If $K = H/P$ is not finitely generated let $\{a_1 + P, \ldots, a_n + P\}$ be a maximal linearly independent subset of K. If $M = \langle a_1, \ldots, a_n \rangle$ then $P \cap M$ is finite so that if $\pi(P)$ is infinite then $\pi(H/M)$ is also infinite and we deduce that G is nilpotent, using Corollary 2.6. Consequently $\pi(P)$ is finite.

Let $P = \operatorname{Dr}_{q \in \pi(P)} S_q$, where S_q, is a Sylow q-subgroup of P. Since S_q is a pure subgroup of H and H/H^n is finite for each natural number n, it follows that S_q/S_q^n is also finite, by [**9**, Lemma 6], and Lemma 7 of [**9**] then shows that $S_q = X_q \times Y_q$, where X_q is finite and Y_q is a divisible subgroup. Since $H = Y_q \times W_q$ for some subgroup W_q, $H/W_q \cong Y_q$ and Corollary 2.6 yields that Y_q is either trivial or a Prüfer q-subgroup, for every $q \in \pi(P)$. If we assume that $Y_q \neq 1$ for some q then, since H/P is of finite rank but not finitely generated, H has a factor group which is the direct product of two Prüfer subgroups. Corollary 2.6 again shows that in this case G is nilpotent. This contradiction proves that $Y_q = 1$ for all $q \in \pi(P)$ and hence P is finite. Then H has finite rank and as above we can show H is minimax and that G contains a nilpotent subgroup U such that $G/U \cong C_{p^\infty}$.

If H/P is finitely generated, then $H = P \times Q$, for some finitely generated torsionfree abelian group Q and Corollary 2.6 implies that $\pi(P)$ is finite. Similar arguments to those above show that either $P = E \times F$, where E is a Prüfer p-subgroup for some prime p and F is finite, or P is finite. Hence either H is finitely generated or

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H has finite rank. As above H is minimax and we can construct a normal nilpotent subgroup U of G such that $G/U \cong C_{p^\infty}$. Hence in every case either G/TG' is finitely generated or G/TG' is minimax with $Sp(G/TG') = \{p\}$. By [**15**, Corollary to Theorem 2.26] it follows that either G/T is finitely generated or G/T is minimax with $Sp(G/T) = \{p\}$. □

The following corollary is immediate from Lemma 3.1 and the fact that a torsionfree nilpotent minimax group is residually finite [**15**, Theorem 9.38].

Corollary 3.2. *Let G be a locally nilpotent group satisfying max-(non-nil) and suppose that $G^{\widetilde{\mathfrak{F}}}$ is the finite residual of G. If G is non-nilpotent, then $G^{\widetilde{\mathfrak{F}}}$ is periodic.*

Next we prove that the torsion subgroup of our group $G/G^{\widetilde{\mathfrak{F}}}$ is quite restricted. First we prove the following result.

Lemma 3.3. *Let G be a locally nilpotent group satisfying max-(non-nil). Suppose that $G^{\widetilde{\mathfrak{F}}}$ is the finite residual of G. Let p be a prime and P a Sylow p-subgroup of G. If G is non-nilpotent, then $P/(P \cap G^{\widetilde{\mathfrak{F}}})$ is finite.*

Proof. Let $H = G^{\widetilde{\mathfrak{F}}}$, and suppose that $1 \neq x \in P \setminus H$. Then there is a normal subgroup G_x of finite index such that $x \notin G_x$. We have $G/G_x = U_x/G_x \times V_x/G_x$, where U_x/G_x is a Sylow p-subgroup of G/G_x, V_x/G_x is a Sylow p'-subgroup of G/G_x. Since x is a p-element, $x \notin V_x$ and G/V_x is a finite p-group. Thus we can assume that G/G_x is a p-group. Let

$$\mathcal{S} = \{S | S \text{ is a normal subgroup of } G \text{ such that } G/S \text{ is a finite } p\text{-group}\}$$

and

$$M = \bigcap \{S : S \in \mathcal{S}\}.$$

The argument above shows that $M \cap P = H \cap P$. Set $D = G/M$. By Corollary 2.6 $D/D'D^p$ is finite, so that Lemma 2 of [7] implies that the torsion subgroup of D is finite. In particular, $PM/M \cong P/(P \cap M)$ is finite. □

We require a technical result that will be needed in our proof that the torsion subgroup of $G/G^{\widetilde{\mathfrak{F}}}$ is finite. This in turn requires several small lemmas, which are probably well known, but we include them here.

Lemma 3.4. *Let G be a nilpotent group. Suppose that A is a normal infinite σ-subgroup of G, for some set of primes σ, such that G/A is finitely generated. Then there is a normal subgroup M of G such that G/M is an infinite σ-group, and if $A \leq Z(G)$, then we can choose M to be finitely generated.*

Proof. Suppose first that $A \leq Z(G)$. There exists $M \leq G$, with M finitely generated, such that $G = MA$. Clearly $M \triangleleft G$ and, since M is finitely generated, $A \cap M$ is finite. Hence G/M is of the desired type in this case. Suppose that G is nilpotent of class c. If A is not necessarily central then, since $G/\gamma_c(G)$ is not finitely generated, $A\gamma_c(G)/\gamma_c(G)$ is an infinite σ-subgroup with $G/A\gamma_c(G)$ finitely generated and so,

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inductively, we may assume that $G/\gamma_c(G)$ contains a normal subgroup $M/\gamma_c(G)$ such that G/M is an infinite σ-group. This completes the proof. □

LEMMA 3.5. *Let G be a nilpotent group and suppose that H is a normal abelian subgroup of G such that G/H is a π-group, for some set of primes π. If $K \leq H$, then $[K, G]$ is a π-group.*

Proof. Let $[K,_i G] = [K, \underbrace{G, \ldots, G}_{i}]$. We prove that, for each $i \geq 1$, $[K,_i G]/[K,_{i+1} G]$ is a π-group; then since G is nilpotent we have $[K,_c G] = 1$, for some c, and the claim follows.

Let $x \in [K,_{i-1} G]$, $g \in G$. Then there exists a π-number n such that $g^n \in H$ and so $[x, g^n] = 1$, since H is abelian and normal in G. Using commutator calculus we see that

$$1 = [x, g^n] = [x, g]^n \bmod [K,_{i+1} G]$$

so that the claim follows. □

LEMMA 3.6. *Let G be a nilpotent group and suppose that $M \lhd N \lhd G$. Suppose that for some set of primes π, G/N is an infinite π-group and N/M is an infinite π'-group. Then G has a normal subgroup K such that G/K is periodic and the direct product of an infinite π- and an infinite π'-group.*

Proof. Consider $M^G = M[M, G]$ and apply Lemma 3.5 to the group G/N'. We have that

$$[M, G]N'/N' = [MN'/N', G/N']$$

is a π-group. However $[M, G]MN'/MN'$ is an image of this and so is also a π-group. Thus $M^G N'/MN'$ is a π-group. However N/M is a π'-group, so that N/MN' is also π' and, since $M^G N'/MN' \leq N/MN'$, we have $M^G \leq M^G N' = MN'$. Since G is nilpotent and N/M is infinite, it follows that N/MN' is infinite and hence so is N/M^G. Thus G/M^G has the required properties. □

LEMMA 3.7. *Suppose that G is a nilpotent group with torsion subgroup T satisfying*
(i) *T is an infinite π-group, for some set of primes π;*
(ii) *G/T has finite rank;*
(iii) *G has a series $1 \leq T = Z_0 \leq X_1 \leq Z_1 \leq X_2 \leq Z_2 \leq \ldots \leq X_c \leq Z_c = G$, where $\{Z_i/T\}$ is the upper central series of G/T, X_{i+1}/Z_i is finitely generated and Z_i/X_i is either trivial or an infinite periodic group, and at least one of these factors contains an infinite π'-subgroup.*

Then G has a periodic factor group G/H such that the Sylow π and Sylow π'-subgroups of G/H are infinite.

Proof. Note that by changing the X_i accordingly, we may always suppose that the Sylow π- (respectively π'-) subgroup of Z_i/X_i is trivial or infinite. We let

$$1 \leq T = Z_0 \leq X_1 \leq Z_1 \leq X_2 \leq Z_2 \leq \ldots \leq X_c \leq Z_c = G, \tag{1}$$

be a series as in the statement of the lemma.

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We use induction on the total number of infinite periodic factors in G/T in the above series. By assumption at least one of the factors is infinite. If there is only one, say Z_k/X_k, then G/Z_k is finitely generated and the result follows if Z_k/X_k has both an infinite π- and infinite π'-subgroup, using Lemma 3.4 applied to G/X_k. Thus we may assume that Z_k/X_k is an infinite π'-group. Then, according to Lemma 3.4 applied to appropriate images of G/X_k, there is a finitely generated normal subgroup M/X_k such that G/M is an infinite π'-group. Then M/T is finitely generated and, again by Lemma 3.4, M contains a normal subgroup L such that M/L is an infinite π-group. Then, by Lemma 3.6, the result holds for G.

Consequently in the series (1) we may assume that at least two of the factors Z_i/X_i are infinite. Let k be largest such that Z_k/X_k is infinite. Then G/Z_k is finitely generated. If Z_k/X_k has both an infinite π- and an infinite π'-factor, then we apply Lemma 3.4 to G/X_k and deduce the result.

If Z_k/X_k is an infinite π-group then, by hypothesis, there exists $l < k$ such that Z_l/X_l has an infinite π'-factor. If $Z_l/X_l = A/X_l \times B/X_l$, where A/X_l is an infinite π'-group, then we apply the induction hypothesis to the group G/B, (interchanging the roles of the sets of π and π') and hence obtain the result for G. If Z_k/X_k is an infinite π'-group and for some l we have Z_l/X_l has an infinite π-factor, then the same argument proves the result again. Thus we may assume that all infinite factors Z_i/X_i are π'-groups. But then applying Lemma 3.4 to G/X_k gives us a finitely generated normal subgroup M/X_k such that G/M is an infinite π'-group. Then applying the induction hypothesis to M yields that M has a factor group M/L which is the direct product of an infinite π-group R/L with an infinite π'-group S/L. We have $S \triangleleft M \triangleleft G$ with M/S an infinite π-group and G/M an infinite π'-group. Now apply Lemma 3.6. The result follows. □

With this digression out of the way we now have the following result.

Proposition 3.8. *Let G be a locally nilpotent group satisfying max-(non-nil). Let T be the torsion subgroup of G and $G^{\mathfrak{F}}$ the finite residual of G. If G is non-nilpotent and G/T is not finitely generated, then $T/G^{\mathfrak{F}}$ is finite. In particular, $G/G^{\mathfrak{F}}$ is a nilpotent minimax group with finite torsion subgroup.*

Proof. By Corollary 3.2, $G^{\mathfrak{F}} \leq T$. Suppose, to the contrary, that $T/G^{\mathfrak{F}}$ is infinite. Let q be a prime. Then if S_q is a Sylow q-subgroup of G it follows that $S_qG^{\mathfrak{F}}/G^{\mathfrak{F}}$ is finite, by Lemma 3.3, and hence $\pi = \pi(T/G^{\mathfrak{F}})$ is infinite. By Lemma 3.1, $Sp(G/T) = \{p\}$, for some prime p. For every prime $q \in \pi$ with $q \neq p$ choose a maximal G-invariant subgroup $M_q \leq S_q$ such that $S_q \cap G^{\mathfrak{F}} \leq M_q$. Set

$$M = (\operatorname*{Dr}_{p\neq q\in\pi} M_q) \times (\operatorname*{Dr}_{q\notin\pi} S_q) \times S_p.$$

The factors S_q/M_q for $q \neq p$ are G-chief factors and are therefore central; (see, for example, [**15**, Corollary 1 to Theorem 5.27]). It follows that $T/M \leq Z(G/M)$ and so G/M is nilpotent. Now we can apply Lemma 3.7 to the group G/M and deduce that G contains a normal subgroup H such that G/H is periodic and the Sylow p-subgroup and Sylow p'-subgroup of G/H are infinite. Corollary 2.5 then shows that G is nilpotent, a contradiction which proves that $T/G^{\mathfrak{F}}$ is finite. □

Next, we obtain more information concerning $G^{\mathfrak{F}}$.

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PROPOSITION 3.9. *Let G be a locally nilpotent group satisfying max-(non-nil). Let T be the torsion subgroup of G and $G^{\mathfrak{F}}$ the finite residual of G. Suppose that G is non-nilpotent and G/T is not finitely generated. Then $G^{\mathfrak{F}}$ is a p-group, where $p = Sp(G/T)$.*

Proof. By Lemma 3.1, G contains a nilpotent, normal subgroup $U \geq T$ such that G/U is a Prüfer p-group, for some prime p. By Proposition 3.8, $G/G^{\mathfrak{F}}$ is nilpotent and minimax. If H is a G-invariant subgroup of $G^{\mathfrak{F}}$ such that $G^{\mathfrak{F}}/H$ is finite, then G/H is residually finite; (see, for example, [**15**, Theorem 9.38]). This contradiction shows that $G^{\mathfrak{F}}$ contains no proper G-invariant subgroups of finite index. Suppose that $G^{\mathfrak{F}}$ is not a p-group. Then the observation above implies that the Sylow p'-subgroup of $G^{\mathfrak{F}}$ is infinite. It follows that the Sylow p'-subgroup of T is infinite. Let $T = P \times Q$, where P (respectively Q) is a Sylow p-subgroup (respectively Sylow p'-subgroup) of T. We first suppose that $P = 1$. Since U is nilpotent and normal in G, T has a finite U-central series

$$1 = T_0 \leq T_1 \leq \ldots \leq T_n = T,$$

the terms of which are normal in G.

Let $x \in G$, and let $V = \langle U, x\rangle$. Since G/V is a Prüfer p-group, V is nilpotent, by Lemma 2.4. Let

$$1 = Y_0 \leq Y_1 \leq \ldots \leq Y_m = T_1$$

be the upper V-central series of T_1. Suppose that $Y_2 \neq Y_1$. If $y \in Y_2 \setminus Y_1$, then $[T_1, U] = 1$ implies $1 \neq [y, x] \in Y_1$, so $y^x = yy_1$, for some element $y_1 \in Y_1$. Let $k = |xU|$, $q = |y_1|$, so that $(k, q) = 1$. We have $y = x^{-k}yx^k = yy_1^k$; thus $y_1^k = 1$. Since $y_1^q = 1$ it follows that $y_1 = 1$. This means that $Y_1 = T_1$ and hence, by induction, $\{T_i | 0 \leq i \leq n\}$ is also a V-central series. Since x is an arbitrary element of G, $\{T_i | 0 \leq i \leq n\}$ is therefore a G-central series. Since G/T is nilpotent it follows that G is also nilpotent. In the general case, when $P \neq 1$, the argument above shows that G/P is nilpotent. By Lemma 3.7 applied to G/P, we see that there exists a normal subgroup $S \geq P$ such that G/S is periodic and the Sylow p-subgroup and Sylow p'-subgroup of G/S are infinite. This is a contradiction, since Corollary 2.5 then implies that G is nilpotent. Hence $G^{\mathfrak{F}}$ is a p-group, as required. □

We have now obtained rather a lot of information concerning a locally nilpotent group G with max-(non-nil) in the case in which $G/G^{\mathfrak{F}}$ is not finitely generated. We summarize our results as Theorem A.

THEOREM A. *Let G be a locally nilpotent group satisfying max-(non-nil). Let T be the torsion subgroup of G, and let $G^{\mathfrak{F}}$ be the finite residual of G. If G is non-nilpotent and $G/G^{\mathfrak{F}}$ is not finitely generated then the following hold.*

(1) $G^{\mathfrak{F}} \leq T$;
(2) *$T/G^{\mathfrak{F}}$ is finite;*
(3) *G/T is a nilpotent minimax group;*
(4) *$Sp(G/T) = \{p\}$, for some prime p;*
(5) *$G^{\mathfrak{F}}$ is a p-subgroup;*
(6) *G contains a normal nilpotent subgroup U such that G/U is a Prüfer p-group;*
(7) *if S is a non-nilpotent subgroup of G, then $G = SU$.*

Proof. First note that (1) follows from Corollary 3.2. Next we note that if $G/G^{\mathfrak{F}}$ is not finitely generated then G/T is not finitely generated either. For, suppose that G/T is finitely generated. Then $T/G^{\mathfrak{F}} = \mathrm{Dr}_{\lambda \in \Lambda}(P_\lambda/G^{\mathfrak{F}})$ is the direct product of its non-trivial Sylow p_λ-subgroups $P_\lambda/G^{\mathfrak{F}}$. The groups $P_\lambda/G^{\mathfrak{F}}$ are finite, by Lemma 3.3, and, by assumption, Λ is infinite. Each of the groups P_λ is characteristic in G. By assumption, G contains a finitely generated subgroup H such that $G = HT$. Since $H \cap T$ is finite it follows that if $p_\lambda > |H \cap T|$ then for such $\lambda \in \Lambda$, $P_\lambda H$ is nilpotent; otherwise the chain $P_1H \leq P_1P_2H \leq \dots$ would be an infinite ascending chain of subgroups all but finitely many of which are non-nilpotent, a contradiction. We write $T/G^{\mathfrak{F}} = U/G^{\mathfrak{F}} \times V/G^{\mathfrak{F}}$, where each of $U/G^{\mathfrak{F}}$ and $V/G^{\mathfrak{F}}$ is a direct product modulo $G^{\mathfrak{F}}$ of infinitely many of the P_λ and, since $H \cap T$ is finite, we see that $G = HT$ is nilpotent, as in the proof of Lemma 2.7. This contradiction shows that G/T is not finitely generated.

Now Lemma 3.1 implies (3), (4) and (6). Proposition 3.8 implies (2) and Proposition 3.9 implies (5). To prove (7) let S be a non-nilpotent subgroup of G. Then SU is a normal non-nilpotent subgroup of G so that G/SU is finitely generated, by Lemma 2.2, and divisible. Hence $G = SU$. □

4. The case in which $G/G^{\mathfrak{F}}$ is finitely generated. In this section we require a knowledge of modules over principal ideal domains. First we gather together some of the relevant notions.

Let J be a principal ideal domain, and let Spec(J) denote the set of all its maximal ideals. For every ideal $P \in \mathrm{Spec}\, J$ we choose an element x_P such that $P = x_P J$ and we let $\sigma(J)$ be the set of such elements. Then every element $y \neq 0$ has a presentation $y = u(x_1)^{l_1} \dots (x_n)^{l_n}$, where $u \in U(J)$, the set of units of J, $x_i \in \sigma(J)$, $l_i \in \mathbb{N}$, $1 \leq i \leq n$. This presentation is unique.

Now let A be a J-module. Then the set $t_J(A) = \{a \in A | \mathrm{Ann}_J(a) \neq 0\}$ is a submodule of A. We call $t_J(A)$ the *J-torsion submodule of A*. If $a \in t_J(A)$, then $\mathrm{Ann}_A(a) = yJ$ and $y = u(x_1)^{l_1} \dots (x_n)^{l_n}$, as above. Put $\pi(a) = \{x_1, \dots, x_n\}$ and $\pi(A) = \bigcup_{a \in t_J(A)} \pi(a)$. Let $x \in \sigma(J)$. An element $a \in A$ is called an *x-element* if $ax^n = 0$ for some $n \in \mathbb{N}$. The set A_x of all x-elements of A is a submodule. Moreover,

$$t_J(A) = \bigoplus_{x \in \pi(A)} A_x.$$

The submodule A_x is called the *x-component of A*. The module A is called *J-periodic* if $A = t_J(A)$; the module A is called *J-torsionfree* if $t_J(A) = 0$.

In our first result we consider a natural example of a locally nilpotent group that does not satisfy max-(non-nil).

LEMMA 4.1. *Let $G = A \rtimes \langle g \rangle$, where $A = \mathrm{Dr}_{n \in \mathbb{N}} \langle a_n \rangle$ is an elementary abelian p-subgroup, for some prime p, and g is an element of infinite order such that*

$$a_1^g = a_1,\ a_{n+1}^g = a_{n+1}a_n, \ \textit{for each } n \in \mathbb{N}.$$

Then G does not satisfy max-(non-nil).

Proof. We think of A as an $F_p\langle g \rangle$-module. Then A is an $F_p\langle g \rangle$-periodic module; indeed, A coincides with its $(g-1)$-component. Clearly, $A = [A, g]$, and so A is a

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$(g-1)$-divisible module. Let $y = g^p$. Then A is $(y-1)$-divisible also. Hence there are elements $b_1 = a_1, b_2, \dots, b_n, \dots$ such that $b_1(y-1) = 0, b_{n+1}(y-1) = b_n, (n \in \mathbb{N})$. Let $B = \langle b_n | n \in \mathbb{N} \rangle$. Then B is an $F_p\langle y \rangle$-submodule of A and the subgroup $\langle B, y \rangle$ is hypercentral, but non-nilpotent. However,

$$A = B \oplus Bg \oplus \dots \oplus Bg^{p-1},$$

and, in particular, A/B does not satisfy max-$\langle y \rangle$. The result follows from Lemma 2.3. □

Clearly if G is locally nilpotent and satisfies max-(non-nil), then G involves no group of the type indicated in Lemma 4.1. Now suppose that A is a normal elementary abelian p-subgroup of G and suppose that $g \in G$ has infinite order. The submodule B of the $F_p\langle g \rangle$-module A is called a *basic submodule* (more precisely $(g-1)$-*basic*) if B satisfies the following conditions:

(i) B is a direct sum of cyclic submodules;

(ii) $B(g-1)^n = B \cap A(g-1)^n$, for all $n \in \mathbb{N}$;

(iii) A/B is $(g-1)$-divisible.

As in Abelian Group Theory (see, for example, [**3**, Chapter 6]) one can prove that A contains a basic submodule B.

Lemma 4.2. *Let G be a group satisfying max-(non-nil) and suppose that A is an elementary abelian normal p-subgroup of G, for some prime p. Suppose that g is an element of infinite order and that the $F_p\langle g \rangle$-module A coincides with its $(g-1)$-component. Then there is an integer n such that* $[A, \underbrace{g, \dots, g}_{n}] = 1$.

Proof. We suppose the contrary. Suppose first that there exists a natural number n such that $g^n \in C_G(A)$. We claim that $\langle A, g \rangle$ is nilpotent in this case. Factoring by g^n we only need to show that $\langle A, g \rangle / \langle g^n \rangle$ is nilpotent. However if $a \in A$, then a has only finitely many conjugates in $\langle A, g \rangle$ and so $\langle a \rangle^{\langle g \rangle}$ is finite. We may then construct an infinite strictly ascending chain of finite subgroups $N_1\langle g \rangle / \langle g^n \rangle \lneqq N_2\langle g \rangle / \langle g^n \rangle \lneqq \dots$ with $a \in N_i \leq A$. By Lemma 2.4, each $N_i\langle g \rangle / \langle g^n \rangle$ is nilpotent; if n is relatively prime to p this implies $[N_i, \langle g \rangle] \leq \langle g^n \rangle$ for each i and hence $[A, g, g] = 1$, since a is an arbitrary element of A. We may therefore assume n is a power of p but, in this case, $\langle A, g \rangle$ is nilpotent, by [**15**, Lemma 6.34]. Thus, if $\langle g \rangle \cap C_G(A) \neq 1$, the result follows.

Hence we may suppose that $\langle g \rangle \cap C_G(A) = 1$. Let B be a $(g-1)$-basic submodule of A. Clearly A is infinite. Suppose that $B \neq A$. Then A/B contains a non-trivial submodule $\bar{C} = \langle \bar{c}_n | n \in \mathbb{N} \rangle$ such that $\bar{c}_1(g-1) = 0, \bar{c}_{n+1}(g-1) = \bar{c}_n, n \in \mathbb{N}$. However Lemma 4.1 shows that in this case A does not satisfy max-(non-nil). It follows that $A = B$ and so

$$A = \bigoplus_{\lambda \in \Lambda} A_\lambda,$$

where $A_\lambda = a_\lambda F_p\langle g \rangle$ is a cyclic submodule, for each $\lambda \in \Lambda$. Let $s_\lambda \in \mathbb{N}$ be such that $\mathrm{Ann}_{F_p\langle g \rangle}(a_\lambda) = (g-1)^{s_\lambda} F_p\langle g \rangle$. Our assumption shows that the set $\{s_\lambda | \lambda \in \Lambda\}$ is infinite. Hence there are subsets Λ_1, Λ_2 such that $\Lambda_1 \cap \Lambda_2 = \emptyset$, $\Lambda_1 \cup \Lambda_2 = \Lambda$ and both sets $\{s_\lambda | \lambda \in \Lambda_1\}$ and $\{s_\lambda | \lambda \in \Lambda_2\}$ are infinite. Put

$$E_i = \bigoplus_{\lambda \in \Lambda_i} A_\lambda, \text{ for } i = 1, 2.$$

Then for $i = 1, 2$, $\langle E_i, g \rangle$ is non-nilpotent. In particular, Lemma 2.3 implies that $A/E_1 \cong E_2$ satisfies max-$\langle g \rangle$, which gives a contradiction since $|\Lambda_2|$ is infinite. Hence there is an integer n such that $A_\lambda(g-1)^n = 0$, for every $\lambda \in \Lambda$. Hence $A(g-1)^n = 0$ and the result follows. □

Lemma 4.3. *Let G be a locally nilpotent group satifying max-(non-nil) and suppose that A is a normal elementary abelian p-subgroup, for some prime p. If $G/C_G(A)$ is finitely generated, then there is an integer n such that $[A,_n G] = 1$.*

Proof. If A is finite the result is clear. We suppose that A is infinite. Let $T/C_G(A)$ be the (finite) torsion subgroup of $G/C_G(A)$ and let G/T have the ascending central series

$$T = Z_0 \leq Z_1 \leq \ldots \leq Z_m = G$$

such that $Z_{i+1}/Z_i = \langle g_{i+1} Z_i \rangle$ is an infinite cyclic group, for $0 \leq i \leq m-1$. Since $T/C_G(A)$ is finite, A has a finite upper T-central series

$$1 = A_0 \leq A_1 \leq \ldots \leq A_{n_1} = A$$

(see, for example, [**15**, Lemma 6.34]) and, in particular, $[A,_{n_1} T] = 1$. Thus we may suppose that $G \neq T$. Since T is normal in G the subgroups A_i are G-invariant. Let $aA_i \in A_{i+1}/A_i$ and $B/A_i = \langle aA_i \rangle^{\langle g_1 \rangle}$. Since $\langle a, g_1 \rangle$ is nilpotent, B/A_i is finite. Thus it has a finite $\langle g_1 \rangle$-central series and it follows that A_{i+1}/A_i coincides with its $(g_1 - 1)$-component. By Lemma 4.2 there is an integer $n_{2,i}$ such that $[A_{i+1},_{n_{2,i}} g_1] \leq A_i$, for $1 \leq i \leq n_1$. Let n_2 be the sum of the $n_{2,i}$. Then $[A,_{n_2} Z_1] = 1$. Similar arguments imply that there is an integer n such that $[A,_n G] = 1$. □

As an immediate corollary we have the following result.

Corollary 4.4. *Let G be a locally nilpotent group satisfying max-(non-nil) and suppose that A is a normal elementary abelian p-subgroup, for some prime p. Suppose that G/A is finitely generated. Then G is nilpotent.*

We can put this information to work to obtain some further information concerning the finite residual.

Lemma 4.5. *Let G be a locally nilpotent group satisfying max-(non-nil). Let $G^{\mathfrak{F}}$ be the finite residual of G and suppose that $G/G^{\mathfrak{F}}$ is finitely generated. If G is non-nilpotent, then $G^{\mathfrak{F}}$ is $\mathfrak{F}$-perfect.*

Proof. Let $R = G^{\mathfrak{F}}$ and suppose that R contains a proper subgroup of finite index. Then there is a prime p such that R contains a subgroup H of index p. Put $U = \text{core}_G H$; then R/U is an elementary abelian p-group. By Corollary 4.4, G/U is nilpotent. By Proposition 2 of [**5**], G contains a normal subgroup $V \geq U$ such that V/U is torsionfree and G/V is a bounded p-group. Thus $V \cap R = U$. If G/V is finite,

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then $R/U = R/(V \cap R) \cong RV/V$ is also finite, so that G/U is finitely generated and hence residually finite. This contradicts the fact that $R = G^{\mathfrak{F}}$. Hence G/V is infinite. Since G/V is nilpotent, $(G/V)/(G/V)'$ is infinite by [**15**, Corollary to Theorem 2.26]. It follows from Corollary 2.6 that G is nilpotent. This contradiction shows that R is $\mathfrak{F}$-perfect. □

Our next few results show that periodic normal abelian subgroups of our group G are quite restricted, at least when an element of infinite order does not act nilpotently.

LEMMA 4.6. *Let G be a locally nilpotent group satisfying max-(non-nil). Let A be a normal abelian p-subgroup, for some prime p, and suppose that $g \in G$ is an element of infinite order. Suppose that $A = [A, g]$. Then $C_A(g)$ is not divisible.*

Proof. For every element $a \in A$ the subgroup $\langle a, g\rangle$ is nilpotent. It follows that $C_A(g) \neq 1$ and it is easy to see that $\langle A, g\rangle$ is hypercentral. Thus A has upper $\langle g\rangle$-central series

$$1 = C_0 \leq C_1 \leq \dots C_\alpha \leq C_{\alpha+1} \leq \dots C_\gamma = A,$$

where $C_1 = C_A(g)$ and $C_{\alpha+1}/C_\alpha = C_{A/C_\alpha}(g)$ for $\alpha < \gamma$. Since $\langle a, g\rangle$ is nilpotent there is an integer n such that $a \in C_n$. Hence $A = \bigcup_{n\in\mathbb{N}} C_n$, and $A = C_\omega$. Assume that C_1 is divisible. Since $A = [A, g]$, $[C_{n+1}, g] = C_n$, for each $n \in \mathbb{N}$, and the map

$$\phi : a \longmapsto [a, g], a \in C_{n+1},$$

from C_{n+1} onto C_n is an epimorphism with kernel C_1. Thus if C_n is divisible, then so is C_{n+1} and it follows by induction on n that C_n is divisible, for all $n \in \mathbb{N}$. Now $C_n \neq A$, for any $n \in \mathbb{N}$, since $A = [A, g]$. We have $C_2 = C_1 \times D_1$, for some divisible subgroup D_1. Since $[C_2, g] = C_1$ we have $[D_1, g] = C_1$. Let $c_1 \in \Omega_1(C_1)$. Then there is an element $c_2 \in D_1$ such that $c_1 = [c_2, g]$. Suppose that $|c_2| > p$. Then $c_2^p \neq 1$ and $1 = c_1^p = [c_2, g]^p = [c_2^p, g]$ so that $1 \neq c_2^p \in C_A(g) = C_1$. But $c_2^p \in D_1$. This contradiction shows that $|c_2| = p$. Inductively we can construct a set of elements $\{c_n | n \in \mathbb{N}\}$ such that $\langle c_n | n \in \mathbb{N}\rangle = \bigoplus_{n\in\mathbb{N}}\langle c_n\rangle$, $|c_n| = p$, $[c_1, g] = 1$ and $[c_{n+1}, g] = c_n$, for all $n \in \mathbb{N}$. Lemma 4.1 now gives a contradiction. Hence $C_1 = C_A(g)$ is not divisible. □

COROLLARY 4.7. *Let G be a locally nilpotent group satisfying max-(non-nil). Let A be a normal abelian p-subgroup, for some prime p, and suppose that $g \in G$ is an element of infinite order. Assume that $A = [A, g]$. Then $C_A(g)$ contains no non-trivial divisible subgroups.*

Proof. We may as well suppose that $G = \langle A, g\rangle$. Suppose, by way of contradiction, that D is a non-trivial divisible subgroup of $C_A(g)$. Then there exists a subgroup E such that $C_A(g) = D \times E$. Clearly, E is $\langle g\rangle$-invariant. Let U be a maximal $\langle g\rangle$-invariant subgroup of A such that $E \leq U$ and $D \cap U = 1$. Let $C/U = C_{A/U}(g)$. Since DU/U is divisible there is a subgroup F of C such that $C/U = DU/U \times F/U$. Then $C = DF$ and $[F, g] \leq U \leq F$ so that F is $\langle g\rangle$-invariant. However $D \cap F \leq D \cap U = 1$ and so the maximal choice of U yields $F = U$. Hence $C/U = DU/U$ is a divisible group. We thus obtain a contradiction on applying Lemma 4.6 to $\langle A/U, gU\rangle$. □

Corollary 4.8. *Let G be a locally nilpotent group satisfying max-(non-nil). Let A be a normal abelian p-subgroup, for some prime p, and suppose that $g \in G$ is an element of infinite order. Assume that $A = [A, g]$. Then $C_A(g)$ is a bounded subgroup.*

Proof. Suppose, on the contrary, that $C = C_A(g)$ is not bounded. Let B be a basic subgroup of C. If $B \neq C$, then C/B is a divisible group. Thus $C_{A/B}(g)$ contains a divisible subgroup, which is impossible by Corollary 4.7. Thus $B = C$. Since C is not bounded, C contains a subgroup $E = \mathrm{Dr}_{n \in \mathbb{N}} \langle e_n \rangle$ such that $|e_n| = p^n$, for $n \in \mathbb{N}$. Let $U = \langle (e_n)(e_{n+1})^{-p} | n \in \mathbb{N} \rangle$. Now E/U is a Prüfer p-group. Hence $C_{A/U}(g)$ contains a divisible subgroup. This contradiction shows that $C_A(g)$ is bounded. □

Corollary 4.9. *Let G be a locally nilpotent group satisfying max-(non-nil). Let A be a normal abelian p-subgroup, for some prime p, and suppose that $g \in G$ is an element of infinite order. Assume that $A = [A, g]$. If B is a $\langle g \rangle$-invariant subgroup of A such that $\langle B, g \rangle$ is nilpotent, then B is bounded.*

Proof. Clearly $\langle A, g \rangle$ is hypercentral. Let

$$1 = C_0 \leq C_1 \leq \ldots \leq C_n \leq \ldots C_\omega = A$$

be the upper $\langle g \rangle$-central series of A. Since $[C_{n+1}, g] = C_n$ and $C_{n+1}/C_1 \cong C_n$, it follows from Corollary 4.8 and induction on n that C_n is bounded, for every $n \in \mathbb{N}$. Let

$$1 = B_0 \leq B_1 \leq \ldots \leq B_m = B$$

be the upper $\langle g \rangle$-central series of B. Clearly $B_i \leq C_i$ for all $i \in \mathbb{N}$ and in particular, $B = B_m \leq C_m$. Hence B is bounded. □

Lemma 4.10. *Let G be a locally nilpotent group satisfying max-(non-nil). Let A be a normal abelian p-subgroup, for some prime p, and suppose that $g \in G$ is an element of infinite order. If $A = [A, g]$, then A is a Černikov subgroup.*

Proof. Let $C_1 = C_A(g)$. By Corollary 4.8, C_1 is bounded and so $C_1 = \mathrm{Dr}_{\lambda \in \Lambda} \langle c_\lambda \rangle$, for some index set Λ. Choose an index λ_0 and let $a = c_{\lambda_0}$. Then $C_1 = \langle a \rangle \times B_1$ where $B_1 = \mathrm{Dr}_{\lambda \neq \lambda_0} \langle c_\lambda \rangle$, and the subgroup B_1 is $\langle g \rangle$-invariant. Let M be a $\langle g \rangle$-invariant subgroup of A, maximal subject to $B_1 \leq M$ and $M \cap \langle a \rangle = 1$. Put $C_2/M = C_{A/M}(g)$. If X/M is a non-trivial subgroup of C_2/M, then $[C_2, g] \leq M$ implies $[X, g] \leq M \leq X$ and so X is $\langle g \rangle$-invariant. It follows from the choice of M that every non-trivial subgroup of C_2/M has non-trivial intersection with $\langle aM \rangle$. Hence C_2/M is finite, since C_2/M is bounded, by Corollary 4.8. In particular, C_2/M is an Artinian $\mathbb{Z}\langle g \rangle$-module. Since A/M has its upper $\langle g \rangle$-central series of length ω, Lemma 1 of **[18]** implies that A/M is an Artinian $\mathbb{Z}\langle g \rangle$-module. If $g^n \in C_{G/M}(A/M)$ for some natural number n, then the subgroup $\langle A/M, gM \rangle$ is a locally nilpotent group satisfying min-n and hence is Černikov. Suppose then that $\langle gM \rangle \cap C_{G/M}(A/M) = 1$ and that A/M is not Černikov. Then, by [**11**, Theorem 1], A contains $\langle g \rangle$-invariant subgroups U, V satisfying the following conditions: $M \leq V \leq U$, V/M is a Černikov group, $U/V = \mathrm{Dr}_{n \in \mathbb{N}} \langle u_n V \rangle$ is an elementary abelian p-group and $[u_1 V, g] = 1$, $[u_{n+1} V, g] = u_n V$, for all $n \in \mathbb{N}$. This contradicts Lemma 4.1.

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Hence A/M is a Černikov group, in any case. Since $A/M = [A/M, g]$, A/M is infinite and therefore A/M does not satisfy the maximal condition for $\langle g \rangle$-invariant subgroups. By Lemma 2.3, it follows that $\langle M, g \rangle$ is a nilpotent subgroup. By Corollary 4.9, M is bounded. Using a corollary to Lemma 1 of **[8]**, we can write $A = D \times E$, where D is a divisible Černikov subgroup and E is a bounded subgroup. Clearly, D is $\langle g \rangle$-invariant. Since A/D is bounded it has a finite series of $\langle g \rangle$-invariant subgroups with elementary abelian p-factors. Lemma 4.2 then implies that $\langle A/D, gD \rangle$ is nilpotent. Hence $[A/D, g] \neq A/D$, which is a contradiction unless $A = D$. Thus A is a Černikov group. □

LEMMA 4.11. *Let G be a locally nilpotent group satisfying max-(non-nil). Let A be a normal abelian p-subgroup of G, for some prime p, and suppose that $g \in G$ is an element of infinite order. Let $A_0 = A$ and $A_{n+1} = [A_n, g]$, for all $n \in \mathbb{N}$. Then there is an integer m such that $A_m = A_{m+1}$.*

Proof. Suppose, on the contrary, that $A_{n+1} \neq A_n$ for all $n \in \mathbb{N}$. Suppose first that $A_\omega = \bigcap_{n \in \mathbb{N}} A_n = 1$. For each $a \in A$, $\langle a \rangle^{\langle g \rangle}$ is finite, since $\langle a, g \rangle$ is nilpotent.

Let $1 \neq d_1 \in A$, $D_1 = \langle d_1 \rangle^{\langle g \rangle}$. Since D_1 is finite, from $\bigcap_{n \in \mathbb{N}} A_n = 1$, we see that there is an integer k_1 such that $D_1 \cap A_{k_1} = 1$. Since $\langle D_1, g \rangle = \langle d_1, g \rangle$ is nilpotent there is an integer m_1 such that $[D_1, \underbrace{g, \ldots, g}_{m_1}] = [D_{1},{}_{m_1} g] = 1$. If there is an integer r such that $[A_{k_1},{}_r g] = 1$, then $A_{k_1+r} = 1$. Hence there is an element $d_2 \in A_{k_1}$ and an integer $m_2 > m_1$ such that $[d_2,{}_{m_2} g] = 1$ and $[d_2,{}_{m_2-1} g] \neq 1$.

Set $D_2 = \langle d_2 \rangle^{\langle g \rangle}$ so that D_2 is again finite. Clearly $D_2 \leq A_{k_1}$ and, in particular, $D_1 \cap D_2 = 1$. The subgroup $D_1 D_2$ is finite and so there exists an integer k_2 such that $D_1 D_2 \cap A_{k_2} = 1$. In this way we constuct an infinite family of finite $\langle g \rangle$-invariant subgroups $\{D_n | n \in \mathbb{N}\}$ and integers $\{m_n | n \in \mathbb{N}\}$ such that

$$\langle D_n | n \in \mathbb{N} \rangle = \operatorname*{Dr}_{n \in N} D_n, \text{ and } [D_i,{}_{m_i} g] = 1 \text{ but } [D_i,{}_{m_i - 1} g] \neq 1,$$

and

$$m_1 < m_2 < \ldots < m_n < \ldots .$$

The subgroup $\langle g, D_n | n \in \mathbb{N} \rangle$ is clearly non-nilpotent, a contradiction to Lemma 2.7.

Finally, if $A_\omega \neq 1$ we apply the argument above to A/A_ω and obtain the result in general. □

COROLLARY 4.12. *Let G be a locally nilpotent group satisfying max-(non-nil). Let A be a normal abelian p-subgroup of G, for some prime p, and suppose that $g \in G$ is an element of infinite order. Then either $\langle A, g \rangle$ is nilpotent or A contains a divisible Černikov subgroup D such that $D = [D, g]$ and A/D is finite.*

Proof. Let $A_0 = A$, and $A_{n+1} = [A_n, g] = [A,{}_{n+1} g]$, for all $n \in \mathbb{N}$. By Lemma 4.11 there is an integer m such that $A_{m+1} = A_m$. If $A_m = 1$, then $\langle A, g \rangle$ is nilpotent. Suppose that $D = A_m \neq 1$. Then $D = [D, g]$, and by Lemma 4.10, D is a Černikov group. Clearly D is divisible and $\langle D, g \rangle$ is non-nilpotent. By Lemma 2.3, A/D satisfies max-$\langle g \rangle$ and, since $\langle A, g \rangle$ is hypercentral, it follows that A/D is finite. □

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COROLLARY 4.13. *Let G be a locally nilpotent group satisfying max-(non-nil). Let A be a normal abelian periodic subgroup of G and suppose that $g \in G$ is an element of infinite order. Then either $\langle A, g\rangle$ is nilpotent or A contains a divisible Černikov subgroup D such that $D = [D, g]$ and A/D is finite.*

Proof. If the set $\pi(A)$ is infinite, then $\langle A, g\rangle$ is nilpotent, by Lemma 2.7. Suppose that $\pi(A)$ is finite, let $p \in \pi(A)$ and let A_p be a Sylow p-subgroup of A. If $\langle A_p, g\rangle$ is nilpotent, for every $p \in \pi(A)$, then $\langle A, g\rangle$ is also nilpotent. Suppose that $\langle A_p, g\rangle$ is non-nilpotent for some prime p. Then Corollary 4.12 yields that A_p contains a divisible Černikov subgroup D such that $D = [D, g]$. In particular $\langle D, g\rangle$ is non-nilpotent. By Lemma 2.3, A/D satisfies max-$\langle g\rangle$. Hence A/D is finite since $\langle A, g\rangle$ is hypercentral. □

Let G be a group and A a divisible normal subgroup of G. We say that A is *divisibly irreducible* in G if A contains no proper G-invariant divisible subgroups.

LEMMA 4.14. *Let G be a group and A a normal abelian subgroup of G satisfying the following conditions:*

(i) *$G/C_G(A)$ is nilpotent;*
(ii) *$A \cap Z(G)$ contains a subgroup B such that A/B is a divisible Černikov group;*
(iii) *A/B is divisibly irreducible in G;*
(iv) *$A/B = [A/B, G]$.*

Then A contains a G-invariant divisible Černikov subgroup D such that $D = [D, G]$, D is divisibly irreducible in G, $A = BD$ and $B \cap D$ is finite.

Proof. Let $C_G(A) \neq zC_G(A) \in Z(G/C_G(A))$. Then the mapping $\phi : a \longmapsto [a, z]$, $(a \in A)$, is a $\mathbb{Z}G$-endomorphism of A. Thus $K = \ker\phi$ and $D = \operatorname{Im}\phi$ are G-invariant subgroups of A and, since $B \leq Z(G)$, we have $B \leq K$. It follows that either K/B is finite or $K = A$, since A/B is divisibly irreducible in G. However, by choice of z, $K \neq A$. Hence K/B is finite. It follows that D is a divisible Černikov subgroup, D is divisibly irreducible and either $D = [D, G]$ or $[D, G] = 1$. In the latter case $D = [A, z]$ is central in G and so $[A, z, G] = 1$. By definition of z we also have $[z, G, A] = 1$ and hence $[G, A, z] = 1$. However (ii) and (iv) imply $1 \neq D = [A, z] = \big[[G, A]B, z\big] = [G, A, z]$. This contradiction implies $D = [D, G]$. Thus $D \cap B$ is finite and $DB/B \cong D/(D \cap B)$ is an infinite G-invariant divisible subgroup of A/B, so that $BD = A$. □

LEMMA 4.15. *Let G be a locally nilpotent group satisfying max-(non-nil). Let $G^{\mathfrak{F}}$ be the finite residual of G. Suppose that $G/G^{\mathfrak{F}}$ is finitely generated and $G^{\mathfrak{F}}$ is not Černikov. If $G^{\mathfrak{F}}$ is nilpotent, then G is nilpotent.*

Proof. Suppose, on the contrary, that G is non-nilpotent. By Lemma 4.5, $R = G^{\mathfrak{F}}$ is $\mathfrak{F}$-perfect. Since R is periodic, by Corollary 3.2, and nilpotent, then R is a divisible abelian subgroup. (See, for example, [**15**, Corollary 2 of Theorem 9.23].) Let T/R be the torsion subgroup of G/R. Then G/T is a finitely generated torsion-free nilpotent group and so G/T has a central series

$$T = C_0 \leq C_1 \leq \ldots \leq C_m = G$$

such that $C_{i+1}/C_i = \langle g_{i+1}C_i \rangle$, for $0 \leq i \leq m-1$. Let P be a Prüfer p-subgroup of R, and let $U = P^T$. Since T/R is finite, U is a divisible Černikov group. Since R is not Černikov, $R = U \times V$ for some non-trivial subgroup V. By Theorem 2.7 of [**2**], there exists a T-invariant subgroup W such that $R = UW$ and $U \cap W$ is bounded. Since T/R is finite there is a finite subgroup F such that $T = FR$. Since the factor groups R/U and R/W are both divisible, they do not satisfy max-F. By Lemma 2.3, UF and WF are both nilpotent and so there is an integer k such that $[U,_k T] = [W,_k T] = 1$. Since $R = UW$, we see that $[R,_k T] = 1$ and hence T is nilpotent. Thus we may assume that $T \neq G$. Since R is divisible, $R \leq Z(T)$ by [**15**, Lemma 3.29.1]) and $\langle R, g_1 \rangle$ is nilpotent.

Let

$$1 = R_0 \leq R_1 \leq \ldots \leq R_{l_1} = R$$

be the upper $\langle g_1 \rangle$-central series of R. By Corollary 4.13, there is an integer l_2 such that $[R,_{l_2} g_2] = 1$. It follows that $[R,_{l_1 l_2} C_2] = 1$. Using similar arguments and a simple induction we see that the group G is nilpotent, a contradiction, which proves the result. □

Theorem B. *Let G be a locally nilpotent group satisfying max-(non-nil). Let $G^{\mathfrak{F}}$ be the finite residual of G and suppose that $G/G^{\mathfrak{F}}$ is finitely generated. If G is non-nilpotent and $G^{\mathfrak{F}}$ is nilpotent, then $G^{\mathfrak{F}}$ is a divisible Černikov group. Moreover, $G^{\mathfrak{F}}$ is divisibly irreducible and $[G^{\mathfrak{F}}, G] = G^{\mathfrak{F}}$.*

Proof. Let $R = G^{\mathfrak{F}}$. By Lemma 4.15, R is Černikov and hence, by Lemma 4.5, divisible Černikov. Let A be a minimal G-invariant divisible subgroup of R and set $A_1 = [A, G]$. Since $[A, G]$ is a divisible G-invariant subgroup of A we have $A_1 = A$ or A_1 is trivial. Since R is a Černikov group it has a finite series of G-invariant subgroups

$$1 = R_0 \leq R_1 \leq \ldots \leq R_t = R,$$

every factor of which is divisibly irreducible. In particular, either $[R_{i+1}, G] \leq R_i$ or $[R_{i+1}/R_i, G] = R_{i+1}/R_i$, for each i, with $0 \leq i \leq t-1$. Since G is non-nilpotent there exists an integer l such that $[R_{l+1}/R_l, G] = R_{l+1}/R_l$. Let l be the least integer with this property. Suppose that $l > 0$. By Lemma 4.14, R_{l+1}/R_{l-1} contains a G-invariant divisible Černikov subgroup Q_l/R_{l-1} such that $R_{l+1}/R_{l-1} = (R_l/R_{l-1})(Q_l/R_{l-1})$, Q_l/R_{l-1} is divisibly irreducible and $[Q_l/R_{l-1}, G] = Q_l/R_{l-1}$. Using similar arguments we see that R contains a G-invariant divisibly irreducible subgroup D such that $[D, G] = D$. Since G/R is finitely generated, $G = RF$ for some finitely generated subgroup F. Then $\langle D, F \rangle$ is non-nilpotent, and Lemma 2.3 implies that R/D satisfies max-F. Since R/D is divisible and periodic, R/D is finite. Hence $R = D$ and the result follows. □

Theorem C. *Let G be a locally nilpotent group satisfying max-(non-nil). Let $G^{\mathfrak{F}}$ be the finite residual of G and suppose that $G/G^{\mathfrak{F}}$ is finitely generated. Let G be non-nilpotent and non-minimax.*

(i) *$G^{\mathfrak{F}}$ is periodic.*

(ii) *$G^{\mathfrak{F}}$ is $\mathfrak{F}$-perfect.*

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(iii) $G^{\mathfrak{F}}$ *is non-nilpotent and every proper subgroup of* $G^{\mathfrak{F}}$ *is nilpotent. Furthermore* $G^{\mathfrak{F}}$ *is nilpotent-by-Černikov and G is soluble.*

In particular, $G^{\mathfrak{F}}$ *is a p-group, for some prime p, having an ascending series of G-invariant subgroups*

$$1 = A_0 \leq A_1 \leq \ldots \leq A_n \leq \ldots \text{ such that } \bigcup_{n \in \mathbb{N}} A_n = G^{\mathfrak{F}}$$

and such that every subgroup A_n *is nilpotent.*

Proof. Corollary 3.2 implies (i) and Lemma 4.5 implies (ii). By Theorem B, $G^{\mathfrak{F}}$ is not nilpotent. If H is a non-nilpotent normal subgroup of G, then G/H satisfies max by Lemma 2.2. In particular G/H is residually finite and so $G^{\mathfrak{F}} \leq H$. Thus the proper G-invariant subgroups of $G^{\mathfrak{F}}$ are nilpotent and hence $G^{\mathfrak{F}}$ is a p-group, for some prime p, by Corollary 2.6 and Lemma 3.3. Thus $G^{\mathfrak{F}}$ is a locally nilpotent $\mathfrak{F}$-perfect p-group and if H_i is a non-nilpotent subgroup of $G^{\mathfrak{F}}$, for some natural number i, then $|G^{\mathfrak{F}} : H_i|$ is infinite. Since a maximal subgroup of a locally nilpotent group is normal and of finite index, H_i is not maximal in $G^{\mathfrak{F}}$ and so there is a proper subgroup H_{i+1} of $G^{\mathfrak{F}}$ properly containing H_i. Since H_{i+1} is not nilpotent either, this argument can be repeated and, in this way, we construct an infinite ascending chain of non-nilpotent subgroups of $G^{\mathfrak{F}}$, contrary to the condition max-(non-nil). Hence every proper subgroup of $G^{\mathfrak{F}}$ is nilpotent. A recent result of Asar [**1**] now shows that $G^{\mathfrak{F}}$ is nilpotent-by-Černikov and the result follows. □

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