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RESEARCH ARTICLE

Worst-case reinsurance strategy with likelihood ratio uncertainty

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Abstract

In this paper, we explore a non-cooperative optimal reinsurance problem incorporating likelihood ratio uncertainty, aiming to minimize the worst-case risk of the total retained loss for the insurer. We establish a general relation between the optimal reinsurance strategy under the reference probability measure and the strategy in the worst-case scenario. This relation can further be generalized to insurance design problems quantified by tail risk measures. We also characterize distortion risk measures for which the insurer's optimal strategy remains the same in the worst-case scenario. As an application, we determine the optimal policies for the worst-case scenario using an expectile risk measure. Additionally, we propose and explore a cooperative problem, which can be viewed as a general risk sharing problem between two agents in a comonotonic market. We determine the risk measure value and the optimal reinsurance strategy in the worst-case scenario for the insurer and compare the results from the non-cooperative and cooperative models.

1. Introduction

The study on reinsurance design was pioneered by Borch (1960). Since then, optimal reinsurance problems have received significant attention in insurance and actuarial science. Typically, an optimal reinsurance model is formulated as an optimization problem, with the aim of identifying indemnity functions that satisfy specific optimization targets. In such optimization problems, the ways of quantifying risk exposure level is one of the most crucial concerns, which is typically conducted under the assumption that the distribution of the underlying loss variable is known or given. However, this assumption is often unrealistic, as the actual loss distribution is generally unavailable in most practical scenarios. Estimating distributions from available data is particularly challenging for catastrophic risks due to limited data, making such estimates unreliable. Moreover, there is no universal consensus on which loss distribution or model should be used in real-world applications. This uncertainty motivates us to explore distributional model uncertainty in determining optimal reinsurance policies. The selection of the uncertainty set of distributions is a central question in problems incorporating model uncertainty. Two common approaches to constructing uncertainty sets are moment-based and distance-based methods. Moment-based sets include distributions whose moments, such as mean and variance, meet specific conditions. For instance, Ghaoui *et al*. (2003) study the worst-case Value-at-Risk (VaR) in a portfolio optimization problem with only bounds on the mean and covariance matrix. Chen *et al*. (2011) demonstrate that the worst-case Tail Value-at-Risk (TVaR) shares the same closed-form solution as the worst-case VaR within a set of distributions with the same mean and covariance. These results are extended by Li (2018) to law-invariant coherent risk measures. Liu *et al*. (2020) determine the worst-case law-invariant convex risk functionals for a risk with a given mean and a higher moment. A distance-based uncertainty set incorporates measures that are close to the reference measure according to a chosen distance. Embrechts *et al*. (2020) model the heterogeneous beliefs by different probability measures. Liu *et al*. (2022) introduce uncertainty into risk sharing models, characterizing either by the likelihood ratio between probability measures (i.e., the Radon–Nikodym derivative) or by the Wasserstein metric between distributions. The study of uncertainty sets that combine moment and Wasserstein distance constraints can be found in Bernard *et al*. (2024), which apply isotonic projections to derive expressions for the distortion risk measure value and the distribution of random losses in worst-case and best-case scenarios. The study of isotonic projections was introduced to the field of quantitative finance by Rüschendorf and Vanduffel (2020). Following the research findings of Bernard *et al*. (2024), Cai *et al*. (2024) use the same uncertainty set and further impose stop-loss and limited-loss functions on the random variable, characterizing the worst-case distribution.

In the research area of insurance and actuarial science, optimal insurance and reinsurance design problems under distributional uncertainty become more important nowadays. Hu *et al*. (2015) introduce model uncertainty to the optimal reinsurance framework by considering the set of all distributions with the same first two moments. Asimit *et al*. (2017) consider an uncertainty set including finite alternative distributions and determine the optimal reinsurance strategies. Birghila and Pflug (2019) use Wasserstein metric to define the uncertainty set and impose a premium constraints into the uncertainty model. In Liu and Mao (2022), moments constraints are adopted, and the authors aim to minimize the worst-case VaR of the insurer's total risk exposure. Recently, Boonen and Jiang (2024) determine the optimal insurance strategy when the insurer faces uncertainty about the underlying distribution and considers all the distributions within a Wasserstein ball of a reference distribution.

In aforementioned models, the main focus is to determine the optimal reinsurance strategies in the worst-case scenario, which is a common approach to address model uncertainty in robust risk management introduced by Scarf *et al*. (1957). However, it remains unclear how the optimal strategy in a worst-case scenario differs from that in a regular scenario without the concern of model uncertainty. In practice, the insurer needs to conduct multiple scenario tests including regular-case and worst-case scenarios. Establishing a relation between the optimal solutions in different scenarios can effectively support the insurer's risk management in the following key aspects: (1) directly derive the optimal reinsurance strategy in the worst-case scenario from the solution in the regular-case scenario, if such a solution is already known; (2) assess the adequacy of the insurer's reinsurance coverage in the worst-case scenario; and (3) quantify the robustness of the insurer's optimal strategies in different scenarios. To address those concerns, we explore in this work an optimal reinsurance problem in the worst-case scenario incorporating likelihood ratio uncertainty set, as introduced in Liu *et al*. (2022). The insurer chooses a reference probability measure which can reflect the regular-case scenario in the best way and determines her optimal reinsurance strategy. Any alternative probability measure in the uncertainty set should agree with the reference one on events with zero probability. For non-zero probability random events, the likelihood ratio between the reference and alternative measures should be bounded by a predetermined tolerance level. For a law-invariant risk measure, which assigns the same risk value to random variables with the same distribution, each probability measure in the uncertainty set leads to a specific distribution of the underlying random loss, thereby affecting the risk exposure quantified by the risk measure. The insurer's objective is to minimize the total loss exposure in the worst-case scenario by choosing the optimal reinsurance policy which is modified from the solution in the regular-case scenario. To the best of our knowledge, our model is the first one solving the optimal reinsurance strategy in the worst-case scenario using the likelihood ratio uncertainty set. Furthermore, contributions of our results are four-folds. First, we establish the relation between solutions in the regular-case and worst-case scenarios for any law-invariant monetary risk measure, which is of special interests to the insurer when the robust risk measure in the worst-case scenario, such as robust expectile, has no closed-form formulation. It is essential to highlight that this relation is not limited to problems involving likelihood ratio uncertainty; it is applicable to any optimal reinsurance problem characterized by tail risk measures.

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For the general framework of tail risk measures, we refer to Liu and Wang (2021). Second, we demonstrate that for certain distortion risk measures, such as TVaR, the optimal reinsurance strategy remains the same in both the worst-case and regular-case scenarios. Such a characterization of risk measures is desirable for insurers concerned with distributional model uncertainty. Third, we propose two concepts of robustness in term of the solvency gaps and worst-case strategies, respectively, for the insurer, and characterize the robustness for Range Value-at-Risk (RVaR) which covers regulatory risk measures VaR and TVaR as special cases. Finally, we propose an alternative risk sharing model in which the reinsurer is cooperative with the insurer in term of the model uncertainty and re-calculate premium in different scenarios.

This paper is organized as follows. Section 2 presents the preliminaries. In Section 3, we examine the relation between the optimal solution to a reinsurance design problem and its worst-case counterpart under likelihood ratio uncertainty. We present the necessary and sufficient conditions under which the two problems yield the same solution. The robustness of the solution is analyzed in this section. Building on this relation, we present the robust solution to the optimal reinsurance problem quantified by the expectile risk measure in Section 4. In Section 5, we explore the cooperative reinsurance problem with likelihood ratio uncertainty and compare its results with those of its non-cooperative counterpart. Finally, Section 6 provides the conclusion of the paper.

2. Preliminaries

2.1 Risk measure and tail risks

We work with an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the reference probability measure. Without specifying otherwise, all risks are quantified under the reference probability measure $\mathbb{P}$. Let L^q be the set of all random variables in $(\Omega, \mathcal{F}, \mathbb{P})$ with finite q-th moment, $q \in (0, \infty)$. Throughout, for any random loss X, a positive value of X represents a financial loss, F_X denotes the distribution function of X, $S_X = 1 - F_X$ is the survival function of X, and the (left-continuous) quantile function is given by $F_X^{-1}(p) = \inf\{x \in \mathbb{R} : F_X(x) \geqslant p\}$ for $p \in (0, 1)$. By convention, we define $F_X^{-1}(0) = \operatorname{ess\,inf}(X)$ and $F_X^{-1}(1) = \operatorname{ess\,sup}(X)$ as the essential infimum and essential supremum of X, respectively. Let U_X be a uniform random variable on [0,1] such that $F_X^{-1}(U_X) = X$ almost surely (a.s.). The existence of such uniform random variable U_X for any X is established, for example, in Lemma A.32 of Föllmer and Schied (2016). Let $X \overset{\mathrm{d}}{=} Y$ if the random variables X and Y have the same distribution. For $x, y \in \mathbb{R}$, $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, and $(x - y)_+ = \max\{0, x - y\}$.

Let $\mathcal{X}$ be a convex cone of random variables containing L^∞, and a risk measure ρ is a functional that maps $\mathcal{X}$ to $\mathbb{R}$. In this paper, we focus on *law-invariant* risk measures, meaning that $\rho(X)$ depends solely on the distribution of X. Mathematically, ρ has the law-invariant property if $\rho(X) = \rho(Y)$ for all $X, Y \in \mathcal{X}$ such that $X \overset{\mathrm{d}}{=} Y$. A large class of risk measures commonly used in insurance and finance literature is the *monetary risk measure*, which is often employed to quantify the capital reserve.

Definition 1. (Monetary Risk Measure). *A law-invariant risk measure ρ is defined as a monetary risk measure if it satisfies the following two properties:*

(A1) Monotonicity: $\rho(X) \leqslant \rho(Y)$ if $X \leqslant Y$ a.s., $X, Y \in \mathcal{X}$.
(A2) Translation-invariance: $\rho(X - m) = \rho(X) - m$ for any $m \in \mathbb{R}$ and $X \in \mathcal{X}$.

Comonotonic additivity is of particular importance for insurance pricing:

(A3) Comonotonic additivity: $\rho(X + Y) = \rho(X) + \rho(Y)$ if $X, Y \in \mathcal{X}$ are comonotonic.[1]

[1]Two random variables X and Y are comonotonic if there exists $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ and, for all $\omega, \omega' \in \Omega_0$, it holds true $(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geqslant 0$.

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The class of *distortion risk measures*, which satisfy comonotonic additivity, is commonly used in the insurance literature. A *distortion risk measure*, denoted by ρ^g, is defined as:

$$\rho^g(X) = \int_0^{+\infty} g(S_X(x))\,\mathrm{d}x - \int_{-\infty}^0 (1 - g(S_X(x)))\,\mathrm{d}x, \qquad X \in \mathcal{X}, \tag{2.1}$$

where $g : [0, 1] \mapsto [0, 1]$ is a non-decreasing function with $g(0) = 0$ and $g(1) = 1$, and g is called the *distortion function* of ρ^g. We always assume that $\mathcal{X}$ is a set of random variables such that $\rho^g(X)$ is finite. Two well-known risk measures used in insurance regulation are the *(left) Value-at-Risk (VaR)* at a given confidence level $\alpha \in (0, 1)$, defined as $\mathrm{VaR}_\alpha(X) = F_X^{-1}(1 - \alpha)$, and the *Tail Value-at-Risk (TVaR)* at a given confidence level $\alpha \in (0, 1)$, which is defined as $\mathrm{TVaR}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u(X)\,\mathrm{d}u$ for $X \in L^1$. Both VaR_α and TVaR_α are distortion risk measures induced by distortion functions $\mathbb{1}_{(\alpha,1]}(x)$ and $\min\{x/\alpha, 1\}$, $0 \leqslant x \leqslant 1$, respectively. Here, $\mathbb{1}_{\{\cdot\}}$ is the indicator function.

Both VaR_α and TVaR_α are α-tail risk measures. In the framework of *tail risk measures* proposed by Liu and Wang (2021), for a random variable $X \in \mathcal{X}$ and $p \in (0, 1]$,

$$X_p \triangleq F_X^{-1}(1 - p + pU_X)$$

represents the *tail risk of* X beyond its $(1 - p)$-quantile. One can easily check that $\mathbb{P}(X_p \leqslant x) = \mathbb{P}(X \leqslant x | U_X \geqslant 1 - p) = (\mathbb{P}(X \leqslant x) - (1 - p))_+/p$ for $x \in \mathbb{R}$. Furthermore, for $p \in (0, 1)$, a risk measure ρ is a *p-tail risk measure* if $\rho(X) = \rho(Y)$ for all $X, Y \in \mathcal{X}$ satisfying $X_p \overset{\mathrm{d}}{=} Y_p$. Proposition 1 of Liu and Wang (2021) uses the following equation to characterize a p-tail risk measure ρ through its unique *generator*, denoted by ρ^*,

$$\rho(X) = \rho^*(X_p), \qquad X \in \mathcal{X}.$$

Conversely, for each generator ρ^*, the p-tail risk measure induced by ρ^* is also unique.

2.2 Reinsurance design and likelihood ratio uncertainty set

In a reinsurance contract, the underlying risk X faced by the insurer is shared between the insurer and the reinsurer. Given an indemnity function denoted by I, the risk $I(X)$ is ceded to the reinsurer, while $R_I(X) = X - I(X)$ is retained by the insurer. To avoid the issue of moral hazard, it is commonly take $\mathcal{I} = \{I : [0, \infty) \to [0, \infty), I$ is non-decreasing and $0 \leqslant I(x) - I(y) \leqslant x - y$ for $0 \leqslant y \leqslant x\}$ as the set of all feasible indemnity functions. In exchange for transferring risk, the reinsurer charges a reinsurance premium, denoted by $\pi(I(X))$. Thus, the insurer's total loss becomes $\ell(X; I) = R_I(X) + \pi(I(X))$ after entering the reinsurance contract I. In a regular case, when the distribution of X is given by $\mathbb{P}$, the optimal reinsurance design problem for the insurer using the risk measure ρ can be expressed as:

$$\min_{I \in \mathcal{I}} \rho\left(R_I(X) + \pi(I(X))\right). \tag{2.2}$$

Optimization problems developed from (2.2) have been intensively discussed in the literature. Obviously, the choice of the risk measure ρ, the underlying risk X, and the reinsurance premium principle π play crucial roles in (2.2). In this current work, we define the triplet (ρ, π, X) as a *reinsurance setting* and subsequently consider the optimal reinsurance strategy for the insurer.

Assumption 1. *The risk X is a non-negative random variable with a continuous support.*

Assumption 2. *Assume that ρ is a law-invariant monetary risk measure, and the premium principle is given by $\pi = (1 + \theta)\rho^g$, where $\theta \geqslant 0$ is a safety loading, and ρ^g is a distortion risk measure induced by a continuously differentiable distortion function g.*

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The above two assumptions are commonly used in the literature on insurance/reinsurance design. Note that $\pi = (1+\theta)\rho^g$ encompasses popular premium principles, such as the expected-value premium by setting $\rho^g = \mathbb{E}$, and Wang's premium by setting $\theta = 0$.

Definition 2. (Regular-case strategy). *Given a reinsurance setting (ρ, π, X), a regular-case strategy for the insurer, denoted by $I^*_{[\rho,\pi,X]}$, is the optimal reinsurance solution to the problem (2.2) under the reference probability measure $\mathbb{P}$, that is,*

$$I^*_{[\rho,\pi,X]} \in \arg\min_{I\in\mathcal{I}} \left\{\rho\left(R_I(X) + \pi(I(X))\right)\right\}. \tag{2.3}$$

For a law-invariant risk measure ρ, the "true" distribution of X is required for quantification. The distribution of X under $\mathbb{P}$ represents the regular-case scenario for the insurer. In robust risk management, the insurer needs to quantify the distribution of X in the worst-case scenario as well. Mathematically, we use the likelihood ratio to construct the uncertainty set of probability measures: given $\lambda \in (0, 1]$, define

$$\mathcal{P}_\lambda = \{\mathbb{Q} : \text{probability measure } \mathbb{Q} \text{ is absolutely continuous with respect to } \mathbb{P} \text{ and } \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \leqslant 1/\lambda\}. \tag{2.4}$$

Here, λ represents the size of the uncertainty set. Because ρ is law-invariant, evaluating the insurer's risk measure under a probability measure $\mathbb{Q} \in \mathcal{P}_\lambda$ is equivalent to determining the distribution of X under $\mathbb{Q}$. For each $\mathbb{Q} \in \mathcal{P}_\lambda$, let $X_\mathbb{Q}$ be a random variable whose distribution under $\mathbb{P}$ is the same as the distribution of X under $\mathbb{Q}$. The existence of such $X_\mathbb{Q}$ is guaranteed by taking $(F^\mathbb{Q}_X)^{-1}(U_X)$ where $F^\mathbb{Q}_X$ is the distribution of X under $\mathbb{Q}$ and U_X is a uniform random variable under $\mathbb{P}$. Thus, the insurer's risk exposure under $\mathbb{Q}$ can be evaluated by using the distribution of $X_\mathbb{Q}$ under $\mathbb{P}$. For the insurer adopting ρ and $\mathcal{P}_\lambda$ to model her uncertainty concerns, the *robust risk measure* induced by ρ and $\mathcal{P}_\lambda$ is defined as:

$$\bar{\rho}^{\mathcal{P}_\lambda}(Y) = \sup_{\mathbb{Q}\in\mathcal{P}_\lambda} \rho(Y_\mathbb{Q}), \qquad Y \in \mathcal{X}. \tag{2.5}$$

By Proposition 2 of Liu *et al*. (2022), $\bar{\rho}^{\mathcal{P}_\lambda}$ is a λ-tail risk measure generated by ρ.

3. Non-cooperative reinsurance design in the worst-case scenario

In this section, we assume a given reinsurance setting (ρ, π, X) and uncertainty set $\mathcal{P}_\lambda$ and propose a non-cooperative reinsurance model for the insurer in the worst-case scenario. Specifically, we are going to define the worst-case strategy for the insurer and investigate its relation with the regular-case strategy. In the literature, it is a commonly accepted assumption for the problem (2.2), and therefore for the regular-case strategy, that the insurer and the reinsurer use the same probability measure for risk quantification, meaning a basic agreement on the probabilities of random events from both parties. We follow the same assumption to define the optimization problem in the worst-case scenario to be comparable with problem (2.2). Meanwhile, as the counter-party, the reinsurer does not collaborate with the insurer regarding model uncertainty and determines the reinsurance premium under the reference probability measure $\mathbb{P}$, that is, $\pi(Y) = \pi(Y_\mathbb{P})$ for all $Y \in \mathcal{X}$. On the other hand, the insurer adopts the uncertainty set $\mathcal{P}_\lambda$. For any alternative probability measure $\mathbb{Q} \in \mathcal{P}_\lambda$, the insurer uses $X_\mathbb{Q}$ to evaluate her total risk after implementing a reinsurance policy $I \in \mathcal{I}$, and her total risk exposure under $\mathbb{Q}$ becomes $\ell(X_\mathbb{Q}; I) = R_I(X_\mathbb{Q}) + \pi(I(X))$ for $I \in \mathcal{I}$. Thus, the insurer's optimization problem in the worst-case scenario can be written as:

$$\min_{I\in\mathcal{I}} \sup_{\mathbb{Q}\in\mathcal{P}_\lambda} \rho(R_I(X_\mathbb{Q}) + \pi(I(X))). \tag{3.1}$$

3.1 Optimal strategy in the worst-case scenario

To solve problem (3.1), we first establish a connection between its objective function and the robust risk measure.

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Lemma 3.1. *If $f(x)$ is a deterministic function such that $\{x : f(x) \leqslant y\}$ is a Borel set for any $y \in \mathbb{R}$, then the distributions of $f(X_{\mathbb{Q}})$ and $f(X)_{\mathbb{Q}}$ are the same under the reference measure $\mathbb{P}$.*

Proof. Arbitrarily take $y \in \mathbb{R}$. Let $A = \{x : f(x) \leqslant y\}$, which is a Borel set. The distribution of $f(X_{\mathbb{Q}})$ under $\mathbb{P}$ is given by $\mathbb{P}(f(X_{\mathbb{Q}}) \leqslant y) = \mathbb{P}(X_{\mathbb{Q}} \in A) = \mathbb{Q}(X \in A)$, where the last equality follows from the construction. In addition, the distribution of $f(X)_{\mathbb{Q}}$ under $\mathbb{P}$ is $\mathbb{P}(f(X)_{\mathbb{Q}} \leqslant y) = \mathbb{Q}(f(X) \leqslant y) = \mathbb{Q}(X \in A) = \mathbb{P}(f(X_{\mathbb{Q}}) \leqslant y)$. Since y is taken arbitrarily, we conclude that $f(X_{\mathbb{Q}})$ follows the same distribution as $f(X)_{\mathbb{Q}}$ under $\mathbb{P}$. □

It is easy to see that all indemnity functions and the associated retained loss functions satisfy the conditions in Lemma 3.1. Therefore, two random variables $R_I(X)_{\mathbb{Q}}$ and $R_I(X_{\mathbb{Q}})$ share the same distribution under $\mathbb{P}$. Since $\pi(I(X))$ is a constant for a given I and X, and ρ is law-invariant, Lemma 3.1 implies, for $\mathbb{Q} \in \mathcal{P}_\lambda$, $\rho\left(R_I(X_{\mathbb{Q}}) + \pi(I(X))\right) = \rho\left(R_I(X)_{\mathbb{Q}} + \pi(I(X))\right)$. It follows that

$$\sup_{\mathbb{Q}\in\mathcal{P}_\lambda} \rho(R_I(X_{\mathbb{Q}}) + \pi(I(X))) = \sup_{\mathbb{Q}\in\mathcal{P}_\lambda} \rho(R_I(X)_{\mathbb{Q}} + \pi(I(X))) = \overline{\rho}^{\mathcal{P}_\lambda}(R_I(X) + \pi(I(X))),$$

and the problem (3.1) can be further written as:

$$\min_{I\in\mathcal{I}} \overline{\rho}^{\mathcal{P}_\lambda}(R_I(X) + \pi(I(X))), \tag{3.2}$$

where the robust risk measure $\overline{\rho}^{\mathcal{P}_\lambda}$ is adopted by the insurer. The optimal solution to problem (3.2) represents the insurer's optimal strategy in the worst-case scenario.

Definition 3. (Worst-case strategy). *Given a reinsurance setting (ρ, π, X), a worst-case strategy with respect to the uncertainty set $\mathcal{P}_\lambda$, denoted by $I^*_{[\overline{\rho}^{\mathcal{P}_\lambda},\pi,X]}$, is an optimal solution to problem (3.2), that is,*

$$I^*_{[\overline{\rho}^{\mathcal{P}_\lambda},\pi,X]} \in \arg\min\nolimits_{I\in\mathcal{I}} \overline{\rho}^{\mathcal{P}_\lambda}(R_I(X) + \pi(I(X))). \tag{3.3}$$

Theorem 3.1 below presents the first main result of this paper, where we characterize the worst-case strategy $I^*_{[\overline{\rho}^{\mathcal{P}_\lambda},\pi,X]}$ by identifying how it differs from the regular-case strategy. To proceed, we first introduce a modified premium principle:

$$\bar{\pi}_\lambda \triangleq \begin{cases} (1+\theta)g(\lambda)\rho^{\bar{g}_\lambda}, & \text{if } g(\lambda) > 0, \\ 0, & \text{if } g(\lambda) = 0, \end{cases}$$

where g satisfies Assumption 2, and $\bar{g}_\lambda(t) = \frac{g(t\lambda)}{g(\lambda)}$, $0 \leqslant t \leqslant 1$, is a well-defined distortion function when $g(\lambda) > 0$.

Theorem 3.1. *Let Assumptions 1 and 2 hold. Define $X_{\lambda,0} = X_\lambda - v_\lambda$, where $v_\lambda \triangleq \mathrm{VaR}_\lambda(X)$. Then, the insurer's worst-case strategy, which is the optimal solution to problem (3.2), is given by:*

$$I^*_{[\overline{\rho}^{\mathcal{P}_\lambda},\pi,X]}(x) = \begin{cases} (x - v_\lambda + a^*)_+\,, & x < v_\lambda, \\ I^*_{[\rho,\bar{\pi}_\lambda,X_{\lambda,0}]}(x - v_\lambda) + a^*, & x \geqslant v_\lambda, \end{cases} \tag{3.4}$$

where $a^* = \inf\{0 \leqslant a \leqslant v_\lambda : g(S_X(v_\lambda - a)) \geqslant \frac{1}{1+\theta}\}$, and $I^*_{[\rho,\bar{\pi}_\lambda,X_{\lambda,0}]}$ is the insurer's regular-case strategy to the following problem:

$$\min_{I\in\mathcal{I}} \rho\left(R_I(X_{\lambda,0}) + \bar{\pi}_\lambda(I(X_{\lambda,0}))\right). \tag{3.5}$$

Furthermore, if $\pi = (1+\theta)\mathbb{E}$, then $a^* = \left(v_\lambda - \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\right)_+$ and $\bar{\pi}_\lambda = (1+\theta)\lambda\mathbb{E}$.

Proof. The proof of Theorem 3.1 is presented in Appendix A. □

Since the robust risk measure $\overline{\rho}^{\mathcal{P}_\lambda}$ is a λ-tail risk measure generated by ρ, from Liu *et al*. (2022) we know that the insurer's worst-case scenario occurs under the worst-case probability measure $\mathbb{Q}^*$ such that $X_{\mathbb{Q}^*} \stackrel{d}{=} X_\lambda$ under $\mathbb{P}$. Mathematically, $\mathbb{Q}^*$ can be defined via its Radon–Nikodym derivative as follows: $\frac{d\mathbb{Q}^*}{d\mathbb{P}} = \frac{\mathbb{1}_{\{U_X > 1-\lambda\}}}{\lambda}$. It is easy to see that $\mathbb{Q}^* \in \mathcal{P}_\lambda$, and $X_{\mathbb{Q}^*} \stackrel{d}{=} X_\lambda$ under $\mathbb{P}$ because

$$\mathbb{P}(X_{\mathbb{Q}^*} > z) = \mathbb{Q}^*(X > z) = \frac{\mathbb{P}(X > z, U_X > 1-\lambda)}{\lambda} = \frac{\min\{\mathbb{P}(X > z), \lambda\}}{\lambda} = \mathbb{P}(X_\lambda > z).$$

Under Assumption 2, $\pi = (1+\theta)\rho^g$ is given by a distortion risk measure, and thus the premium principle $\bar{\pi}_\lambda$ is also given by a distortion risk measure, induced by the modified distortion function $\bar{g}_\lambda$. For example, the expected-value premium principle $\pi(Y) = (1+\theta)\mathbb{E}[Y]$ is commonly used in insurance literature. In this case, $\bar{\pi}_\lambda(X) = (1+\theta)\lambda\mathbb{E}[X]$ is an expected-value premium principle with an adjusted risk loading of $(1+\theta)\lambda - 1$. Following Theorem 3.1, if the insurer has a closed-form solution for the regular-case strategy $I^*_{[\rho,\pi,X]}$ for a given reinsurance setting (ρ, π, X), then $I^*_{[\rho,\bar{\pi}_\lambda,X_{\lambda,0}]}$ can be easily determined using the distribution of the tail risk $X_{\lambda,0}$ and the modified premium principle $\bar{\pi}_\lambda$. The worst-case strategy follows directly from (3.4).

For small loss amounts $x \leqslant v_\lambda$, the worst-case strategy in (3.4) follows a stop-loss policy with a retention level $v_\lambda - a^*$. Indeed, among all indemnity functions satisfying $I(v_\lambda) = a^*$, the worst-case strategy gives the smallest ceded loss on the loss range $[0, v_\lambda]$, which saves reinsurance premium for the insurer. For large loss amounts $x > v_\lambda$, the type of the worst-case strategy depends on ρ and $\bar{\pi}_\lambda$. As we will see later, under different settings, the worst-case strategy can be full coverage policy, limited-loss policy, or more complex structures.

Example 3.1. (RVaR-based worst-case strategy) *In this example, we assume that the insurer adopts a RVaR. Given a pair of confidence levels (p,q),* $0 \leqslant q < p \leqslant 1$,

$$RVaR_{(p,q)}(Y) = \frac{1}{p-q}\int_q^p \mathrm{VaR}_u(Y)\,\mathrm{d}u, \qquad Y \in \mathcal{X}, \tag{3.6}$$

is a distortion risk measure induced by the distortion function $g(t; p, q) = \left(\mathbb{1}_{\{t \geqslant q\}}\frac{t-q}{p-q}\right) \wedge 1$, $t \in [0, 1]$. *In particular,* $\mathrm{TVaR}_p = \mathrm{RVaR}_{(p,0)}$ *and* VaR_p *can be considered a special case when* $p = q$.

When the insurer adopts $\rho = RVaR_{(p,q)}$ *and* $\pi = (1+\theta)\mathbb{E}$, *the insurer's RVaR-based optimization problem in the regular-case scenario is given by:*

$$\min_{I \in \mathcal{I}} \mathrm{RVaR}_{(p,q)}(R_I(X) + (1+\theta)\mathbb{E}[I(X)]). \tag{3.7}$$

It is easy to verify that the regular-case strategy, that is, the solution to problem (3.7), is given by:

$$I^*_{[\mathrm{RVaR}_{(p,q)},(1+\theta)\mathbb{E},X]}(x) = \begin{cases} 0, & \text{if } p \geqslant \frac{1}{1+\theta}, \\ \left(x - \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\right)_+ - \left(x - \mathrm{VaR}_{\frac{q}{1-(p-q)(1+\theta)}}(X)\right)_+, & \text{if } p < \frac{1}{1+\theta}. \end{cases} \tag{3.8}$$

Next, we consider the worst-case scenario, where the insurer's problem is given by:

$$\min_{I \in \mathcal{I}} \sup_{\mathbb{Q} \in \mathcal{P}_\lambda} \mathrm{RVaR}_{(p,q)}\left(R_I(X_{\mathbb{Q}}) + (1+\theta)\mathbb{E}[I(X)]\right) = \min_{I \in \mathcal{I}} \overline{\mathrm{RVaR}}^{\mathcal{P}_\lambda}_{(p,q)}\left(R_I(X) + (1+\theta)\mathbb{E}[I(X)]\right). \tag{3.9}$$

We now verify that the optimal solution to problem (3.9) is

$$I^*_{[\overline{\mathrm{RVaR}}^{\mathcal{P}_\lambda}_{(p,q)},(1+\theta)\mathbb{E},X]}(x) = \begin{cases} 0, & \text{if } \lambda p \geqslant \frac{1}{1+\theta}, \\ \left(x - \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\right)_+ - \left(x - \mathrm{VaR}_{\frac{\lambda q}{1-(p-q)(1+\theta)\lambda}}(X)\right)_+, & \text{if } \lambda p < \frac{1}{1+\theta}. \end{cases} \tag{3.10}$$

To this end, we first note that problem (3.5) in Theorem 3.1 becomes

$$\min_{I \in \mathcal{I}} \mathrm{RVaR}_{(p,q)}\left(R_I(X_{\lambda,0}) + (1+\theta)\lambda\mathbb{E}[I(X_{\lambda,0})]\right). \tag{3.11}$$

Applying result (3.8) to solve problem (3.11), we obtain

$$I^*(x) = \begin{cases} 0, & \text{if } \lambda p \geqslant \frac{1}{1+\theta}, \\ \left(x - \mathrm{VaR}_{1 \wedge \frac{1}{(1+\theta)\lambda}}(X_{\lambda,0})\right)_+ - \left(x - \mathrm{VaR}_{\frac{q}{1-(p-q)(1+\theta)\lambda}}(X_{\lambda,0})\right)_+, & \text{if } \lambda p < \frac{1}{1+\theta}. \end{cases} \tag{3.12}$$

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Then, by Theorem 3.1, we have

$$I^*_{[\overline{\mathrm{RVaR}}^{\mathcal{P}_\lambda}_{(p,q)},(1+\theta)\mathbb{E},X]}(x)=\begin{cases}(x-v_\lambda+a^*)_+\,, & x<v_\lambda,\\ I^*(x-v_\lambda)+a^*, & x\geqslant v_\lambda,\end{cases}$$

where $a^*=\left(v_\lambda-\mathrm{VaR}_{\frac{1}{1+\theta}}(X)\right)_+$. *If* $(1+\theta)\lambda\geqslant 1/p\geqslant 1$, *then* $a^*=0$. *Combining* $I^*(x)=0$, *no-insurance is again the optimal solution to the worst-case problem (3.9). If* $1/p>(1+\theta)\lambda\geqslant 1$, *then* $a^*=0$, *and* $I^*(x)=(x-\mathrm{VaR}_{\frac{1}{1+\theta}}(X)+v_\lambda)_+-(x-\mathrm{VaR}_{\frac{\lambda q}{1-(p-q)(1+\theta)\lambda}}(X)+v_\lambda)_+$. *Thus,* $I^*_{[\overline{\mathrm{RVaR}}^{\mathcal{P}_\lambda}_{(p,q)},(1+\theta)\mathbb{E},X]}(x)$ *is given by the second case in (3.10). If* $(1+\theta)\lambda<1$, *then* $a^*=v_\lambda-\mathrm{VaR}_{\frac{1}{1+\theta}}(X)$, *and* $I^*(x)=x-(x-\mathrm{VaR}_{\frac{\lambda q}{1-(p-q)(1+\theta)\lambda}}(X)+v_\lambda)_+$. *The indemnity function in the second case of (3.10) is the optimal solution.*

Remark 3.1 (Optimal strategy using a tail risk measure). The results of Theorem 3.1 can also be applied to characterize the optimal solutions to problem (2.2) with a tail risk measure under $\mathbb{P}$. Quantifying tail risk has been a crucial consideration in modern risk management and insurance regulation. Tail risk measures such as VaR and TVaR are often used in (2.2) to capture the tail behavior of total risk.

For a general λ-tail risk measure ρ, we denote its generator by ρ^*, that is, $\rho(Y)=\rho^*(Y_\lambda)$ for all $Y\in\mathcal{X}$. Mathematically, the structure of problem (2.2) using ρ is equivalent to that of (3.2), as both employ a λ-tail risk measure. Therefore, the optimal solution to problem (2.2) under $\mathbb{P}$ can also be expressed by (3.4):

$$I^*_{[\rho,\pi,X]}(x)=\begin{cases}(x-v_\lambda+a^*)_+\,, & \text{for } x<v_\lambda,\\ I^*_{[\rho^*,\bar{\pi}_\lambda,X_{\lambda,0}]}(x-v_\lambda)+a^*, & \text{for } x\geqslant v_\lambda,\end{cases}\tag{3.13}$$

where $a^*=\inf\{0\leqslant a\leqslant v_\lambda : g(S_X(v_\lambda-a))\geqslant\frac{1}{1+\theta}\}$ and

$$I^*_{[\rho^*,\bar{\pi}_\lambda,X_{\lambda,0}]}\in\arg\min_{I\in\mathcal{I}}\rho^*\left(R_I(X_{\lambda,0})+\bar{\pi}_\lambda(I(X_{\lambda,0}))\right).$$

The expression (3.13) breaks down the optimal solution $I^*_{[\rho,\pi,X]}$ into the "left tail," which focuses on small losses, and the "right tail," which addresses large losses. This decomposition helps the insurer capture the best strategy regarding different loss amounts. To illustrate this, consider a simple case where $\rho=\mathrm{VaR}_\lambda$. Under Assumption 1, $\rho=\mathrm{VaR}_\lambda=\mathrm{VaR}^+_\lambda\triangleq\inf\{x\in\mathbb{R}:F_X(x)>p\}$. By Liu and Wang (2021), VaR^+_λ is a λ-tail risk measure generated by the essential infimum, that is, $\mathrm{VaR}^+_\lambda(Y)=\operatorname{ess\,inf}(Y_\lambda)$ for $Y\in\mathcal{X}$. Therefore, $I^*_{[\operatorname{ess\,inf},\bar{\pi}_\lambda,X_{\lambda,0}]}$ is a solution to $\min_{I\in\mathcal{I}}\operatorname{ess\,inf}\left(R_I(X_{\lambda,0})+\bar{\pi}_\lambda(I(X_{\lambda,0}))\right)=\min_{I\in\mathcal{I}}\bar{\pi}_\lambda(I(X_{\lambda,0}))$. Intuitively, VaR_λ does not account for losses that exceed the quantile level $1-\lambda$. From this "risk tail" optimization problem, it becomes evident that the optimal strategy for "right tail" losses is no-insurance, that is, $I^*_{[\operatorname{ess\,inf},\bar{\pi}_\lambda,X_{\lambda,0}]}=0$. Then (3.13) gives that $I^*_{[\mathrm{VaR}_\lambda,\pi,X]}(x)=(x-v_\lambda+a^*)_+-(x-v_\lambda)_+$.

3.2 Comparison of regular-case and worst-case strategies

Following the general relation between the regular-case and worst-case strategies characterized in Theorem 3.1, a natural question is whether two strategies being the same. Intuitively, since the distributions of the underlying loss X differ between the regular-case and worst-case scenarios, the insurer may need to adopt different strategies. However, as we will see in the following VaR-based and TVaR-based problems, the two optimal strategies can either be identical or differ. Take an uncertainty set $\mathcal{P}_\lambda$ and a confidence level $p\in(0,1)$, and assume $\pi=(1+\theta)\mathbb{E}$ with $p(1+\theta)\lambda<1$.

- (VaR-based worst-case problem.) The VaR-based problem $\min_{I\in\mathcal{I}}\mathrm{VaR}_p\left(R_I(X)+(1+\theta)\mathbb{E}[I(X)]\right)$ has been well studied in the literature, see for example Cheung *et al*. (2014), and it has solution

$$I^*_{[\mathrm{VaR}_p,(1+\theta)\mathbb{E},X]}(x)=\left(x-\mathrm{VaR}_{\frac{1}{1+\theta}}(X)\right)_+-\left(x-\mathrm{VaR}_p(X)\right)_+.\tag{3.14}$$

The VaR-based worst-case problem is expressed as $\min_{I\in\mathcal{I}} \overline{\mathrm{VaR}}_p^{\mathcal{P}_\lambda}\,(R_I(X)+(1+\theta)\mathbb{E}[I(X)])$. Note that VaR can be derived from RVaR by taking $p=q\in(0,1)$. Substituting $p=q$ into (3.10), we find that the worst-case strategy is

$$I^*_{[\overline{\mathrm{VaR}}_\alpha^{\mathcal{P}_\lambda},(1+\theta)\mathbb{E},X]}(x)=\left(x-\mathrm{VaR}_{\frac{1}{1+\theta}}(X)\right)_+-\left(x-\mathrm{VaR}_{p\lambda}(X)\right)_+. \tag{3.15}$$

Obviously, the regular-case strategy (3.14) is smaller than the worst-case strategy (3.15). In other words, the regular-case strategy does not provide sufficient protection in the worst-case scenario for the insurer.

- (TVaR-based worst-case problem.) The TVaR-based optimization problem is formulated as $\min_{I\in\mathcal{I}} \mathrm{TVaR}_p\,(R_I(X)+(1+\theta)\mathbb{E}[I(X)])$ with solution $I^*_{[\mathrm{TVaR}_p,(1+\theta)\mathbb{E},X]}(x)=\left(x-\mathrm{VaR}_{\frac{1}{1+\theta}}(X)\right)_+$. The TVaR-based worst-case problem is $\min_{I\in\mathcal{I}} \overline{\mathrm{TVaR}}_p^{\mathcal{P}_\lambda}\,(R_I(X)+(1+\theta)\mathbb{E}[I(X)])$. Note that, for $p\in(0,1)$, $\mathrm{RVaR}_{(p,0)}=\mathrm{TVaR}_p$. We can apply (3.10) with $q=0$ to verify that

$$I^*_{[\overline{\mathrm{TVaR}}_p^{\mathcal{P}_\lambda},(1+\theta)\mathbb{E},X]}(x)=\left(x-\mathrm{VaR}_{\frac{1}{1+\theta}}(X)\right)_+=I^*_{[\mathrm{TVaR}_p,(1+\theta)\mathbb{E},X]}(x), \tag{3.16}$$

that is, if the insurer adopts TVaR, the regular-case strategy remains optimal in the worst-case scenario, and the insurer does not need to change her strategy in difference scenarios.

We can see from the above examples, comparing the regular-case and worst-case strategies under a given risk measure ρ reveals any potential shortcomings of the regular-case strategy in the worst-case scenario. Conversely, a conservative insurer may prefer a risk measure that results in the same strategies in both regular- and worst-case scenarios.

For a general risk measure ρ, neither the regular-case strategy nor the worst-case strategy has a mathematically tractable closed-form expression. To facilitate comparison between the two strategies in this section, we further assume that $\rho=\rho^h$ is a distortion risk measure induced by h, and that the expected-value premium principle is adopted. The problem (2.2) can be rewritten as:

$$\min_{I\in\mathcal{I}}\left\{\rho^h(R_I(X))+(1+\theta)\mathbb{E}[I(X)]\right\},\qquad \text{for some } \theta\geqslant 0, \tag{3.17}$$

and its corresponding worst-case problem becomes

$$\min_{I\in\mathcal{I}}\sup_{\mathbb{Q}\in\mathcal{P}_\lambda}\rho^h\left(R_I(X_{\mathbb{Q}})+(1+\theta)\mathbb{E}[I(X)]\right)=\min_{I\in\mathcal{I}}\left\{\overline{\rho^h}^{\mathcal{P}_\lambda}(R_I(X))+(1+\theta)\mathbb{E}[I(X)]\right\}. \tag{3.18}$$

The following proposition characterizes the distortion function h such that the regular-case and worst-case strategies are identical.

Proposition 3.2. *Let Assumption 1 hold. Assume $\pi=(1+\theta)\mathbb{E}$ for some $\theta\geqslant 0$, and $\rho=\rho^h$ is a distortion risk measure induced by a continuously differentiable distortion function h. Define*

$$\tilde{h}(p)=\begin{cases} h(p/\lambda), & 0\leqslant p<\lambda,\\ 1, & \lambda\leqslant p\leqslant 1.\end{cases} \tag{3.19}$$

The following statements hold.

(a) *Problems (3.17) and (3.18) have at least one common optimal solution for a given $\lambda\in(0,1)$ if and only if there exists $p_0\in[0,\frac{1}{1+\theta}]$ such that $h(p)\geqslant(1+\theta)p$ for $p\in[0,p_0]$ and $\tilde{h}(p)\leqslant(1+\theta)p$ for $p\in[p_0,1]$.*

(b) *Problems (3.17) and (3.18) have at least one common optimal solution for any $\lambda\in(0,1)$ if and only if $h(p)\geqslant(1+\theta)p$ for $p\in[0,\frac{1}{1+\theta}]$.*

Proof. The proof of Proposition 3.2 is presented in Appendix A. □

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In Proposition 3.2, the insurer's risk measure ρ^h is not necessarily coherent. If the insurer selects a concave h, which makes ρ^h a coherent distortion risk measure, the results of Proposition 3.2 can be stated more precisely.

Corollary 3.3. *Let Assumption 1 hold. Assume $\pi = (1+\theta)\mathbb{E}$ for some $\theta \geqslant 0$, and $\rho = \rho^h$ is a coherent distortion risk measure induced by a continuously differentiable concave distortion function h. The following statements hold.*

- *(a) Problems (3.17) and (3.18) have at least one common optimal solution for a given $\lambda \in (0, 1)$ if and only if one of the two conditions holds: (i) ρ^h is a $\frac{1}{1+\theta}$-tail risk measure; (ii) $h'(0) \leqslant \lambda(1+\theta)$.*
- *(b) Problems (3.17) and (3.18) have at least one common optimal solution for all $\lambda \in (0, 1)$ if and only if ρ^h is a $\frac{1}{1+\theta}$-tail risk measure.*

Remark 3.2 Following assumptions required in Corollary 3.3, we can identify the type of reinsurance policy in cases (a) and (b). Suppose that ρ^h is a $\frac{1}{1+\theta}$-tail risk measure. By Theorem F.1 in Liu and Wang (2021), $h(t) = 1$ for $t \in [p, 1]$. Since h is concave, the solution to both problems (3.17) and (3.18) is the stop-loss policy $(x - \mathrm{VaR}_{\frac{1}{1+\theta}}(X))_+$. In this case, the insurer focuses on the tail risk and already chooses full coverage on it in the regular-case strategy. Therefore, the regular-case strategy remains to be the optimal in the worst-case scenario. On the other hand, if $h'(0) \leqslant \lambda(1+\theta)$, together with the concavity of h, we can verify that $\tilde{h}(t) \leqslant (1+\theta)t$ for $t \in [0, 1]$. Therefore, the optimal solution to (3.17) and (3.18) is no-insurance, meaning that all reinsurance policies are overpriced for the insurer even in the worst-case scenario.

3.3 Robustness of the reinsurance setting using RVaR

For an uncertainty set $\mathcal{P}_\lambda$, the value of λ determines its size in the sense that a smaller λ results in a more severe worst-case scenario, as it is selected from a larger $\mathcal{P}_\lambda$. Since the worst-case strategy does not necessarily align with the regular-case strategy, the change in optimal strategies is critical for the insurer from a risk management perspective. If the change in the worst-case strategy is continuous in some sense with respect to λ, the insurer would only need to make small adjustments to the current optimal strategy as the worst-case scenario worsens. In such cases, the reinsurance setting is robust with respect to the worst-case strategy.

It is widely accepted that a monetary risk measure ρ can be used for capital calculation, and $\rho(R_I(X) + \pi(I(X)))$ represents the total capital required by the insurer after entering the reinsurance contract I. In the regular-case scenario, let $R^*_{[\rho,\pi,X]}(x) = x - I^*_{[\rho,\pi,X]}(x)$. Then, the risk measure value $\rho(R^*_{[\rho,\pi,X]}(X) + \pi(I^*_{[\rho,\pi,X]}(X)))$, using the regular-case strategy $I^*_{[\rho,\pi,X]}$ defined in (2.3), is the *perceived risk value* for the insurer. However, if the worst-case scenario occurs, the insurer adopting the regular-case strategy will experience the *actual risk value* $\overline{\rho}^{\mathcal{P}_\lambda}(R^*_{[\rho,\pi,X]}(X) + \pi(I^*_{[\rho,\pi,X]}(X)))$. The difference

$$
\begin{aligned}
\Delta_{sg}^{\mathcal{P}_\lambda} &\triangleq \underbrace{\overline{\rho}^{\mathcal{P}_\lambda}(R^*_{[\rho,\pi,X]}(X) + \pi(I^*_{[\rho,\pi,X]}(X)))}_{\text{actual risk}} - \underbrace{\rho(R^*_{[\rho,\pi,X]}(X) + \pi(I^*_{[\rho,\pi,X]}(X)))}_{\text{perceived risk}} \\
&= \overline{\rho}^{\mathcal{P}_\lambda}(R^*_{[\rho,\pi,X]}(X)) - \rho(R^*_{[\rho,\pi,X]}(X))
\end{aligned}
\tag{3.20}
$$

is the *solvency gap* for the insurer when the worst-case scenario occurs. Although the premium cancels out in this expression, it affects the optimal strategies and, consequently, the solvency gap. When λ approaches 1, the insurer may expect the solvency gap to be small enough, implying that the reinsurance setting is robust concerning the solvency gap. For a general discussions on solvency gap and risk measure robustness, we refer to Embrechts *et al*. (2022).

Definition 4. *Consider a reinsurance setting (ρ, π, X) that satisfies Assumption 1.*

1. *(ρ, π, X) is robust with respect to the solvency gap, as defined in (3.20), if $\lim_{\lambda\to 1} \Delta_{sg}^{\mathcal{P}_\lambda} = 0$;*

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2. (ρ, π, X) *is robust with respect to the worst-case strategy for* $\mathcal{P}_{\lambda_0}$ *if, for any* $\delta > 0$*, there exists a neighborhood* Λ_0 *of* λ_0 *such that whenever* $\lambda, \gamma \in \Lambda_0 \cap (0, 1]$*, there exist* $I^*_{[\bar{\rho}^{\mathcal{P}_\lambda}, \pi, X]}$ *and* $I^*_{[\bar{\rho}^{\mathcal{P}_\gamma}, \pi, X]}$ *satisfying*

$$\left| I^*_{[\bar{\rho}^{\mathcal{P}_\lambda}, \pi, X]}(x) - I^*_{[\bar{\rho}^{\mathcal{P}_\gamma}, \pi, X]}(x) \right| \leqslant \delta, \qquad \text{for all } x \in [0, \operatorname{ess\,sup} X).$$

In insurance regulation, VaR and TVaR are two of the most important risk measures. RVaR can be seen as a family of risk measures that bridge VaR and TVaR. Therefore, we focus on the optimization problem using RVaR and present the following proposition to characterize the robustness of the reinsurance setting $(\rho = \mathrm{RVaR}, \pi = (1+\theta)\mathbb{E}, X)$.

Proposition 3.4 *Let Assumption 1 hold and* $\theta \geqslant 0$.

(a) The reinsurance setting $(\mathrm{RVaR}_{(p,q)}, (1+\theta)\mathbb{E}, X)$ *is robust with respect to the solvency gap for all* $0 < p < 1$ *and* $0 \leqslant q \leqslant p$.
(b) If $q \neq 0$*, the reinsurance setting* $(\mathrm{RVaR}_{(p,q)}, (1+\theta)\mathbb{E}, X)$ *is robust with respect to worst-case strategy of* $\mathcal{P}_{\lambda_0}$ *for all* $\lambda_0 \in (0, 1]$*. If* $q = 0$*, then* $\mathrm{RVaR}_{(p,0)} = \mathrm{TVaR}_p$*, and the reinsurance setting* $(\mathrm{TVaR}_p, (1+\theta)\mathbb{E}, X)$ *is robust with respect to the worst-case strategy of* $\mathcal{P}_{\lambda_0}$ *at* $\lambda_0 = 1$ *and for all* $\lambda_0 \in (0, 1)$ *with* $\lambda_0 \neq \frac{1}{p(1+\theta)}$.

Proof. We first consider robustness with respect to worst-case strategies. Assume $q \neq 0$ and $\lambda_0 \in (0, 1]$. We rely on (3.10) to determine the worst-case strategy $I^*_\lambda = I^*_{[\overline{\mathrm{RVaR}}^{\mathcal{P}_\lambda}_{(p,q)}, (1+\theta)\mathbb{E}, X]}$ for each $\lambda \in (0, 1]$.

(i) If $(1+\theta)\lambda_0 p > 1$, there exists a neighborhood Λ_0 of λ_0 such that $(1+\theta)\lambda p > 1$ for all $\lambda \in \Lambda_0 \cap (0, 1]$. Then $I^*_\lambda(x) = 0$, $x \geqslant 0$, for all $\lambda \in \Lambda_0 \cap (0, 1]$. Clearly, the reinsurance setting $(\mathrm{RVaR}_{(p,q)}, (1+\theta)\mathbb{E}, X)$ is robust with respect to the worst-case strategy of $\mathcal{P}_{\lambda_0}$.
(ii) If $(1+\theta)\lambda_0 p = 1$, then for $x \geqslant 0$,

$$I^*_\lambda(x) = \begin{cases} 0, & \text{if } \lambda \geqslant \lambda_0, \\ (x - \mathrm{VaR}_{\frac{1}{1+\theta}}(X))_+ - (x - \mathrm{VaR}_{\frac{\lambda q}{1-\lambda(p-q)(1+\theta)}}(X))_+, & \text{if } \lambda < \lambda_0. \end{cases}$$

Under Assumption 1, $\lim_{\lambda \to \lambda_0} \mathrm{VaR}_{\frac{\lambda q}{1-\lambda(p-q)(1+\theta)}}(X) = \mathrm{VaR}_{\frac{\lambda_0 q}{1-\lambda_0(p-q)(1+\theta)}}(X) = \mathrm{VaR}_{p\lambda_0}(X)$ and $\lim_{\lambda \to \lambda_0} \mathrm{VaR}_{p\lambda}(X) = \mathrm{VaR}_{p\lambda_0}(X)$. For any neighborhood $\Lambda_0 \subseteq (0, 1)$ of λ_0, and $\lambda, \gamma \in \Lambda_0$, we have

$$\begin{aligned} \sup_{x \geqslant 0} \left| I^*_\lambda(x) - I^*_\gamma(x) \right| &\leqslant \max\left\{ \mathrm{VaR}_{\frac{\lambda q}{1-\lambda(p-q)(1+\theta)}}(X) - \mathrm{VaR}_{\frac{1}{1+\theta}}(X), \mathrm{VaR}_{\frac{\gamma q}{1-\gamma(p-q)(1+\theta)}}(X) - \mathrm{VaR}_{\frac{1}{1+\theta}}(X) \right\} \\ &= \max\left\{ \mathrm{VaR}_{\frac{\lambda q}{1-\lambda(p-q)(1+\theta)}}(X), \mathrm{VaR}_{\frac{\gamma q}{1-\gamma(p-q)(1+\theta)}}(X) \right\} - \mathrm{VaR}_{p\lambda_0}(X) \\ &\to 0, \qquad |\Lambda_0| \to 0. \end{aligned}$$

Therefore, for any $\delta > 0$, we can choose Λ_0 small enough such that $\sup_{x \geqslant 0} \left| I^*_\lambda(x) - I^*_\gamma(x) \right| \leqslant \delta$.
(iii) If $(1+\theta)\lambda_0 p < 1$, there exists a neighborhood Λ_0 of λ_0 such that $(1+\theta)\lambda p < 1$ for all $\lambda \in \Lambda_0 \cap (0, 1]$. In this case, the same argument as in (ii) applies.

Combining these three cases, for any $\lambda_0 \in (0, 1)$, the reinsurance setting $(\mathrm{RVaR}_{(p,q)}, (1+\theta)\mathbb{E}, X)$ is robust with respect to the worst-case strategy of $\mathcal{P}_{\lambda_0}$.

Next, assume $q = 0$. In this case, $\mathrm{RVaR}_{(p,q)} = \mathrm{TVaR}_p$ for some $0 < p < 1$. Take $\lambda_0 \in (0, 1]$.

(iv) If $\lambda_0 \neq \frac{1}{(1+\theta)p}$, then the same arguments in cases (i) and (iii) apply. Therefore, the reinsurance setting $(\mathrm{TVaR}_p, (1+\theta)\mathbb{E}, X)$ is robust with respect to the worst-case strategy of $\mathcal{P}_{\lambda_0}$.

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(v) If $\lambda_0 = 1 = \frac{1}{(1+\theta)p}$, then $\mathcal{P}_{\lambda_0} = \{\mathbb{P}\}$, and $I^*_{\lambda_0}$ can be any feasible contract. In particular, we can choose $I^*_{\lambda_0}(x) = (x - \mathrm{VaR}_{\frac{1}{1+\theta}}(X))_+$. For any $\lambda < \lambda_0$, we have $(1+\theta)p\lambda < 1$ and $I^*_\lambda(x) = (x - \mathrm{VaR}_{\frac{1}{1+\theta}}(X))_+ = I^*_{\lambda_0}(x)$ for $x \geqslant 0$.

(vi) If $\lambda_0 = \frac{1}{(1+\theta)p} \in (0, 1)$, take $\lambda < \lambda_0$. Then $(1+\theta)p\lambda < 1$, and $I^*_\lambda(x) = (x - \mathrm{VaR}_{\frac{1}{1+\theta}}(X))_+$ for $x \geqslant 0$ is the unique worst-case strategy. Meanwhile, take $\gamma > \lambda_0$. Then $(1+\theta)p\gamma > 1$, and $I^*_\gamma = 0$ is the unique worst-case strategy. For any neighborhood Λ_0 of λ_0, there exists $\lambda, \gamma \in \Lambda_0$ such that $\lambda < \lambda_0 < \gamma$. Thus, for any $x > \mathrm{VaR}_{\frac{1}{1+\theta}}(X)$, we have $|I^*_\lambda(x) - I^*_\gamma(x)| = x - \mathrm{VaR}_{\frac{1}{1+\theta}}(X) > 0$. Therefore, $(\mathrm{TVaR}_p, (1+\theta)\mathbb{E}, X)$ is not robust with respect to the worst-case strategy of $\mathcal{P}_{\lambda_0}$.

The robustness of $(\mathrm{RVaR}_{(p,q)}, (1+\theta)\mathbb{E}, X)$ with respect to the solvency gap can be directly obtained by substituting the regular-case strategy into the expression in (3.20) and is therefore omitted here. □

4. Optimal reinsurance design with the worst-case expectile

In this section, we apply the results from Section 3 to solve an optimal reinsurance design problem using an expectile risk measure, which has recently gained attention in statistics and actuarial science. Among the two most widely used regulatory risk measures, VaR is often criticized for not satisfying sub-additivity, which makes it non-coherent. In contrast, while TVaR is coherent, it is not elicitable, a key property for model comparison and validation. Expectiles, however, can be both coherent and elicitable. For further discussion on the properties of risk measures, including coherence and elicitability, we refer to Bellini *et al*. (2014), Ziegel (2016), Liu *et al*. (2022), Fissler *et al*. (2024), and references therein.

Definition 5. (Expectile). *The expectile risk measure at a confidence level* $\alpha \in (0, 1)$ *is defined as:*

$$\mathcal{E}_\alpha(Y) \triangleq \arg\min_{m\in\mathbb{R}} \left\{\alpha\mathbb{E}\left[(Y-m)_+^2\right] + (1-\alpha)\mathbb{E}\left[(m-Y)_+^2\right]\right\}, \qquad Y \in L^2.$$

In Cai and Weng (2016), it has been shown that

$$\mathcal{E}_\alpha(Y) = \mathbb{E}[Y] + \beta\mathbb{E}\left[(Y - \mathcal{E}_\alpha(Y))_+\right] \quad \text{with} \quad \beta = \frac{2\alpha - 1}{1-\alpha}, \tag{4.1}$$

and that $\mathcal{E}_\alpha(Y)$ is the unique constant satisfying this equation. As demonstrated in Ziegel (2016), $\mathcal{E}_\alpha$ is a coherent risk measure for $\alpha \geqslant 1/2$. Therefore, we assume $\alpha \geqslant 1/2$, which is equivalent to $\beta \geqslant 0$.

With the expectile risk measure, the insurer's problem (2.2) becomes $\min_{I\in\mathcal{I}} \mathcal{E}_\alpha\left(R_I(X) + \pi(I(X))\right)$, which was studied in Cai and Weng (2016) with a budget constraint and mild assumptions on the premium principle π. In the present work, we focus on the expected-value premium principle, that is, $\pi = (1+\theta)\mathbb{E}$, and write the *expectile-based optimization problem (2.2)* as:

$$\min_{I\in\mathcal{I}} \mathcal{E}_\alpha\left(R_I(X) + (1+\theta)\mathbb{E}\left[I(X)\right]\right), \qquad \text{where } \alpha \geqslant \frac{1}{2}. \tag{4.2}$$

Furthermore, we propose the worst-case version of this problem as:

$$\min_{I\in\mathcal{I}} \sup_{\mathbb{Q}\in\mathcal{P}_\lambda} \mathcal{E}_\alpha\left(R_I(X_\mathbb{Q}) + (1+\theta)\mathbb{E}\left[I(X)\right]\right) = \min_{I\in\mathcal{I}} \bar{\mathcal{E}}_\alpha^{\mathcal{P}_\lambda}\left(R_I(X) + (1+\theta)\mathbb{E}\left[I(X)\right]\right), \tag{4.3}$$

where $\bar{\mathcal{E}}_\alpha^{\mathcal{P}_\lambda}$ represents the robust risk measure induced by $\mathcal{E}_\alpha$. To solve (4.3), we first generalize the result of the optimal reinsurance policies of (4.2) in Cai and Weng (2016), in which θ is assumed to be strictly positive, to the more general case of $\theta \geqslant -1$.

Lemma 4.1. *Under the reinsurance setting* $(\mathcal{E}_\alpha, (1+\theta)\mathbb{E}, X)$ *with* $\alpha \geqslant 1/2$ *and* $-1 \leqslant \theta \leqslant 0$, *the regular-case strategy is the full reinsurance, that is,* $I^*_{[\mathcal{E}_\alpha,(1+\theta)\mathbb{E},X]}(x) = x$, $x \geqslant 0$.

Proof. The proof of this lemma is presented in Appendix A. □

Using Lemma 4.1 and Theorem 3.2 of Cai and Weng (2016), the insurer can apply Theorem 3.1 to solve the problem (4.3) and determine the worst-case strategy. Let $v_\alpha = \mathrm{VaR}_\alpha(X)$ for $\alpha \in [0, 1]$, $d_\lambda^0 = v_{\frac{1+\beta}{\beta(1+\theta)} - \frac{\lambda}{\beta}} - v_\lambda$, and $\tilde{d}_\lambda$ be the root of $G(d)$ for a given $\lambda \in (0, 1]$, where

$$G(d) = d - \frac{1}{\lambda}\mathbb{E}[(X - v_\lambda)_+] - \frac{\beta}{\lambda}\mathbb{E}[(X - d - v_\lambda)_+], \quad d \in (0, \infty). \tag{4.4}$$

Furthermore, let $d_\lambda^* = d_\lambda^0 \wedge \tilde{d}_\lambda$, and let m_λ^* be the solution to $K(d_\lambda^*, m) = d_\lambda^*$, where

$$K(d, m) = \frac{1}{\lambda}\mathbb{E}[(X - v_\lambda)_+] - \frac{1}{\lambda}\mathbb{E}[(X - d - v_\lambda)_+] + \frac{1+\beta}{\lambda}\mathbb{E}[(X - m - v_\lambda)_+], \quad m \geqslant d. \tag{4.5}$$

Theorem 4.1 *Let Assumption 1 hold. Under the reinsurance setting $(\mathcal{E}_\alpha, (1+\theta)\mathbb{E}, X)$ with $\alpha \geqslant 1/2$ and $\theta > 0$, the solution to the expectile-based worst-case problem (4.3) can be formulated as:*

$$I^*_{[\bar{\mathcal{E}}_\alpha^{\mathcal{P}_\lambda}, (1+\theta)\mathbb{E}, X]}(x) = \begin{cases} 0, & \text{if } \frac{1+\beta}{1+\theta} \leqslant \lambda \leqslant 1, \\ (x - d_\lambda^* - v_\lambda)_+ - (x - m_\lambda^* - v_\lambda)_+, & \text{if } \frac{1}{1+\theta} < \lambda < \frac{1+\beta}{1+\theta}, \\ (x - v_{\frac{1}{1+\theta}})_+, & \text{if } 0 < \lambda \leqslant \frac{1}{1+\theta}. \end{cases} \tag{4.6}$$

Proof. By Theorem 3.1, we begin by solving the following problem:

$$\min_{I \in \mathcal{I}} \mathcal{E}_\alpha(R_I(X_{\lambda,0}) + (1+\theta')\mathbb{E}[I(X_{\lambda,0})]), \tag{4.7}$$

where $X_{\lambda,0} = X_\lambda - v_\lambda$ and $\theta' = (1+\theta)\lambda - 1 \in (-1, \theta]$. When $\theta' \leqslant 0$, by Lemma 4.1, the optimal indemnity function is given by $I^*_{[\mathcal{E}_\alpha, (1+\theta')\mathbb{E}, X_{\lambda,0})]}(x) = x$. When $\theta' > 0$, according to Theorem 3.2 of Cai and Weng (2016), the problem (4.7) can be solved by a one-layer indemnity function $I_{d,m}(x) = (x-d)_+ - (x-m)_+$, where $0 \leqslant d = \mathcal{E}^R_{I_{d,m}}(X_{\lambda,0}) \leqslant m \leqslant v_0$, and $\mathcal{E}^R_{I_{d,m}}(X_{\lambda,0})$ is the α-expectile of $R_{I_{d,m}}(X_{\lambda,0}) = X_{\lambda,0} - I_{d,m}(X_{\lambda,0})$.

Using (4.1), we have

$$\begin{aligned} \mathcal{E}^R_{I_{d,m}}(X_{\lambda,0}) &= \mathbb{E}[R_{I_{d,m}}(X_{\lambda,0})] + \beta\mathbb{E}[(R_{I_{d,m}}(X_{\lambda,0}) - \mathcal{E}^R_{I_{d,m}}(X_{\lambda,0}))_+] \\ &= \mathbb{E}[R_{I_{d,m}}(X_{\lambda,0})] + \beta\mathbb{E}[(X_{\lambda,0} - m)_+] \\ &= \frac{1}{\lambda}\mathbb{E}[(X - v_\lambda)_+] - \frac{1}{\lambda}\mathbb{E}[(X - d - v_\lambda)_+] + \frac{1+\beta}{\lambda}\mathbb{E}[(X - m - v_\lambda)_+] = K(d, m). \end{aligned}$$

The last equality holds because $S_{X_{\lambda,0}}(x) = \frac{S_X(x+v_\lambda)}{\lambda}$. For a given d, the value of m can be determined by the equation $d = \mathcal{E}^R_{I_{d,m}}(X_{\lambda,0})$, or equivalently, by $d = K(d, m)$. Thus, $m = m(d)$ can be considered as a function of d. Define $\mathcal{D} = \{d{:}0 \leqslant d \leqslant m(d) \leqslant \infty$ such that $K(d, m(d)) = d\}$. Cai and Weng (2016) demonstrates that $\mathcal{D} \neq \emptyset$ and $m = m(d)$ is strictly decreasing with respect to d over the domain $\mathcal{D} = [0, \tilde{d}_\lambda]$, where $\tilde{d}_\lambda$ is the unique root of the function $G(d) \triangleq d - K(d, d)$ over $(0, \infty)$, which can be reformulated as in (4.4). For a given λ, it is easy to gain from $d = K(d, m)$ that $m(0) = v_0 - v_\lambda$ and $m(\tilde{d}_\lambda) = \tilde{d}_\lambda$.

Define $d_\lambda^0 = v_{\frac{1+\beta}{\beta(1+\theta)} - \frac{\lambda}{\beta}} - v_\lambda$. When $\theta' > 0$, from Example 4.1 of Cai and Weng (2016), we have the following: (i) If $\theta' \geqslant \beta$, that is, $\lambda \geqslant \frac{1+\beta}{1+\theta}$, then $d_\lambda^* = \tilde{d}_\lambda$, $m_\lambda^* = m(\tilde{d}_\lambda) = \tilde{d}_\lambda$ and thus, $I^*_{[\mathcal{E}_\alpha, (1+\theta')\mathbb{E}, X_{\lambda,0}]}(x) = 0$. (ii) If $\theta' < \beta$, that is, $\frac{1}{1+\theta} < \lambda < \frac{1+\beta}{1+\theta}$, then $d_\lambda^* = d_\lambda^0 \wedge \tilde{d}_\lambda$ and m_λ^* obtained by solving $K(d_\lambda^*, m) = d_\lambda^*$, we have $I^*_{[\mathcal{E}_\alpha, (1+\theta')\mathbb{E}, X_{\lambda,0}]}(x) = (x - d_\lambda^*)_+ - (x - m_\lambda^*)_+$.

By Theorem 3.1, we have

$$I^*_{[\bar{\mathcal{E}}_\alpha^{\mathcal{P}_\lambda}, (1+\theta)\mathbb{E}, X]}(x) = \begin{cases} (x - v_\lambda + a^*)_+, & x < v_\lambda, \\ I^*_{[\mathcal{E}_\alpha, (1+\theta')\mathbb{E}, X_{\lambda,0}]}(x - v_\lambda) + a^*, & x \geqslant v_\lambda, \end{cases} \tag{4.8}$$

where $a^* = \left(v_\lambda - v_{\frac{1}{1+\theta}}\right)_+$. Then, we have

(i) If $\lambda \geqslant \frac{1+\beta}{1+\theta}$, then $I^*_{[\mathcal{E}_\alpha, (1+\theta')\mathbb{E}, X_{\lambda,0}]}(x) = 0$, $a^* = 0$, and thus, $I^*_{[\bar{\mathcal{E}}_\alpha^{\mathcal{P}_\lambda}, (1+\theta)\mathbb{E}, X]}(x) = 0$.

(ii) If $\frac{1}{1+\theta} < \lambda < \frac{1+\beta}{1+\theta}$, it follows that $I^*_{[\mathcal{E}_\alpha,(1+\theta')\mathbb{E},X_{\lambda,0}]}(x) = (x - d^*_\lambda)_+ - (x - m^*_\lambda)_+$, $a^* = 0$, and $I^*_{[\bar{\mathcal{E}}_\alpha^{\mathcal{P}_\lambda},(1+\theta)\mathbb{E},X]}(x) = (x - d^*_\lambda - v_\lambda)_+ - (x - m^*_\lambda - v_\lambda)_+$.

(iii) If $\lambda \leqslant \frac{1}{1+\theta}$, then $I^*_{[\mathcal{E}_\alpha,(1+\theta')\mathbb{E},X_{\lambda,0}]}(x) = x$, $a^* = v_\lambda - v_{\frac{1}{1+\theta}}$, and $I^*_{[\bar{\mathcal{E}}_\alpha^{\mathcal{P}_\lambda},(1+\theta)\mathbb{E},X]}(x) = (x - v_{\frac{1}{1+\theta}})_+$.

Then, we obtain the optimal indemnity function $I^*_{[\bar{\mathcal{E}}_\alpha^{\mathcal{P}_\lambda},(1+\theta)\mathbb{E},X]}$, as formulated in the theorem. □

In the following proposition, we discuss the robustness of the reinsurance setting $(\mathcal{E}_\alpha, (1+\theta)\mathbb{E}, X)$ with respect to the worst-case strategy for $\mathcal{P}_{\lambda_0}$.

Proposition 4.2. *Let Assumption 1 hold. For a given* $\lambda_0 \in (0, 1]$ *and* $\lambda_0 \neq \frac{1}{1+\theta}$, *the reinsurance setting* $(\mathcal{E}_\alpha, (1+\theta)\mathbb{E}, X)$ *with* $\alpha \geqslant \frac{1}{2}$ *and* $\theta > 0$ *remains robust with respect to the worst-case strategy for* $\mathcal{P}_{\lambda_0}$.

Proof. Let $I^*_\lambda \triangleq I^*_{[\bar{\mathcal{E}}_\alpha^{\mathcal{P}_\lambda},(1+\theta)\mathbb{E},X]}$ for simplicity. By Implicit Function Theorem, we have that $\tilde{d}_\lambda$ is a continuous function of λ. Assumption 1 ensures that d^0_λ is continuous with respect to λ. Thus, $d^*_\lambda = \tilde{d}_\lambda \wedge d^0_\lambda$ and $m(d^*_\lambda)$ exhibit continuity concerning $\lambda \in (0, 1]$.

(i) When $\lambda_0 \in (\frac{1+\beta}{1+\theta}, 1]$, by (4.6), there exists a small neighbor Λ_0 of λ_0 such that $I^*_\lambda = 0$ for $\lambda \in \Lambda_0$.

(ii) When $\lambda_0 = \frac{1+\beta}{1+\theta}$, we have $d^0_{\lambda_0} = v_0 - v_{\lambda_0}$ and $G(d^0_{\lambda_0}) = v_0 - \mathrm{TVaR}_{\lambda_0}(X) \geqslant 0$. Since $G(\tilde{d}_{\lambda_0}) = 0$ and $G(d)$ is an increasing function of d, we have $\tilde{d}_{\lambda_0} \leqslant d^0_{\lambda_0}$. Thus, $d^*_{\lambda_0} = \tilde{d}_{\lambda_0} \wedge d^0_{\lambda_0} = \tilde{d}_{\lambda_0}$ and $m^*_{\lambda_0} = m(d^*_{\lambda_0}) = \tilde{d}_{\lambda_0}$. By (4.6), for any neighborhood $\Lambda_0 \subseteq (\frac{1}{1+\theta}, 1)$ of λ_0, and $\lambda, \gamma \in \Lambda_0$, we have $\sup_{x\geqslant 0} \left| I^*_\lambda(x) - I^*_\gamma(x) \right| \leqslant \max\left\{ m(d^*_\lambda) - d^*_\lambda, m(d^*_\gamma) - d^*_\gamma \right\} \to 0$ as $|\Lambda_0| \to 0$.

(iii) When $\lambda_0 \in (\frac{1}{1+\theta}, \frac{1+\beta}{1+\theta})$, for any neighborhood $\Lambda_0 \subseteq (\frac{1}{1+\theta}, \frac{1+\beta}{1+\theta})$ of λ_0 and $\lambda, \gamma \in \Lambda_0$, by (4.6) we have $\sup_{x\geqslant 0} \left| I^*_\lambda(x) - I^*_\gamma(x) \right| \leqslant |(m^*_\lambda - d^*_\lambda) - (m^*_\gamma - d^*_\gamma)| \to 0$ as $|\Lambda_0| \to 0$.

(iv) When $\lambda_0 = \frac{1}{1+\theta}$, let $\lambda \to \lambda_0$, we have $d^0_\lambda \to 0$, $d^*_\lambda = d^0_\lambda \wedge \tilde{d}_\lambda \to 0$, and thus, $m^*_\lambda \to v_0 - v_{\lambda_0}$. If $v_0 < \infty$, for any neighborhood Λ_0 of λ, take $\lambda, \gamma \in \Lambda_0$. By (4.6), we have $\sup_{x\geqslant 0} \left| I^*_\lambda(x) - I^*_\gamma(x) \right| \leqslant \max\{v_0 - v_{\lambda_0} - m^*_\lambda + d^*_\lambda, v_0 - v_{\lambda_0} - m^*_\gamma + d^*_\gamma\} \to 0$ as $|\Lambda_0| \to 0$. Conversely, if $v_0 = \infty$, then for any $\delta > 0$ and any neighborhood Λ_0 of λ_0, there exists $\lambda, \gamma \in \Lambda_0$ and $\lambda < \lambda_0 < \gamma$ such that for $x > m^*_\gamma + \delta$, we have $\left| I^*_\lambda(x) - I^*_\gamma(x) \right| > x - v_{\lambda_0} - m^*_\gamma + d^*_\gamma > \delta$. This implies that $(\mathcal{E}_\alpha, (1+\theta)\mathbb{E}, X)$ is not robust with respect to the worst-case strategy of $\mathcal{P}_{\lambda_0}$ by Definition 4.

(v) When $\lambda_0 \in (0, \frac{1}{1+\theta})$, there exists a neighborhood Λ_0 of λ_0 such that $I^*_\lambda = (x - v_{\frac{1}{1+\theta}})_+$ for $\lambda \in \Lambda_0$.

Therefore, when $\lambda_0 \in (0, 1]$ and $\lambda_0 \neq \frac{1}{1+\theta}$, for any $\delta > 0$, there exists a sufficiently small Λ_0 such that $\sup_{x\geqslant 0} |I^*_\lambda(x) - I^*_\gamma(x)| \leqslant \delta$. This implies that $(\mathcal{E}_\alpha, (1+\theta)\mathbb{E}, X)$ is robust with respect to the worst-case strategy of $\mathcal{P}_{\lambda_0}$ by Definition 4. □

Example 4.1 (Expectile-based reinsurance design with Pareto distribution). *Assume that* X *follows a type II Pareto distribution with shape parameter* ζ *and scale parameter* η, *that is,* $X \sim Pa(\zeta, \eta)$, *with the survival function* $S_X(x) = \left(\frac{\eta}{x+\eta}\right)^\zeta$, *where* $x \geqslant 0$. *We adopt the parameter settings:* $\zeta = 3$, $\eta = 2$, *and* $\theta = 3$. *Then* $\mathrm{VaR}_p(X) = \eta p^{-1/\zeta} - \eta$, $\mathbb{E}[X] = \frac{\eta}{\zeta-1} < \infty$, *and* $\mathbb{E}[X \wedge x] = \frac{\eta}{\zeta-1} - \frac{\eta^\zeta}{(\zeta-1)(x+\eta)^{\zeta-1}}$ *for* $x \geqslant 0$.

Figure 1 illustrates the optimal retention levels $d^*_\lambda + v_\lambda$ and $m^*_\lambda + v_\lambda$ for the worst-case strategy $I^*_{[\bar{\mathcal{E}}_\alpha^{\mathcal{P}_\lambda},(1+\theta)\mathbb{E},X]}$ as outlined in Theorem 4.1. The shaded blue area indicates the portion of the loss ceded to the reinsurer. When $\lambda \leqslant \frac{1}{1+\theta} = 0.25$, the blue line representing $d^*_\lambda + v_\lambda$ aligns with $v_{\frac{1}{1+\theta}}$, and the red line representing $m^*_\lambda + v_\lambda$ extends to infinity. In this scenario, the worst-case strategy simplifies to the stop-loss function described in Theorem 4.1. From Figure 1, it is evident that as the the uncertainty set expands, the insurer cedes more of the medium-sized losses to the reinsurer. Both retention levels change smoothly with respect to λ, supporting the result in Proposition 4.2, which asserts that the reinsurance setting is robust concerning the worst-case strategy of $\mathcal{P}_\lambda$.

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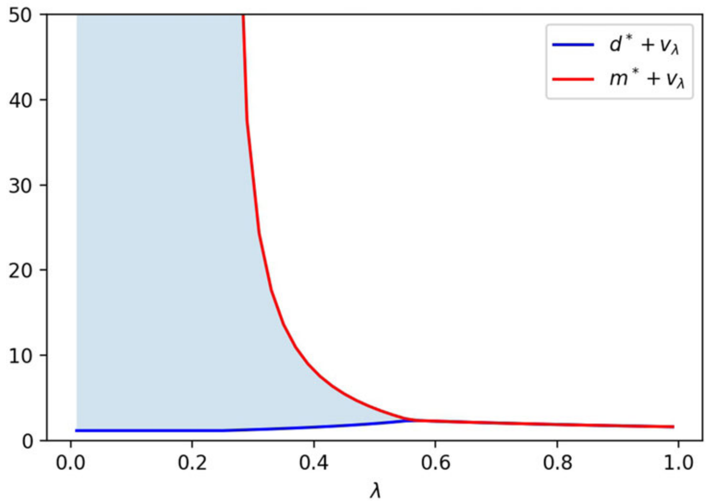

Figure 1. *Optimal ceded loss with respect to λ when $X \sim Pa(\zeta, \eta)$. The parameters are $\zeta = 3, \eta = 2, \beta = 2, \theta = 3$.*

5. Cooperative reinsurance design with likelihood ratio uncertainty

In the non-cooperative problem (3.1), the reinsurance premium is calculated under the reference probability measure and can be considered constant in the insurer's objective function, regardless of the probability measure used by the insurer. In this section, we discuss a cooperative model in which both the insurer and the reinsurer consider the distributional model uncertainty in the reinsurance design procedure while they share a common belief regarding the probability measure in different scenarios. In other words, although the insurer and the reinsurer have concerns of uncertainty, they remain to behave cooperatively whenever an alternative measure $\mathbb{Q} \in \mathcal{P}_\lambda$ is given. Such cooperative formulation is adopted for reinsurance design problem in Liu and Mao (2022). We again assume that the uncertainty set of probability measures is given by (2.4) for a given $\lambda \in (0, 1]$. Thus, the insurer's optimization problem in the worst-case scenario can be written as:

$$\min_{I \in \mathcal{I}} \sup_{\mathbb{Q} \in \mathcal{P}_\lambda} \rho\left(R_I(X_{\mathbb{Q}}) + \pi\left(I(X_{\mathbb{Q}})\right)\right). \tag{5.1}$$

It is worth noting that problem (5.1) can be viewed as a comonotonic risk sharing problem between market participants in the worst-case scenario. To see this, since ρ is translation-invariant, we have $\rho\left(R_I(X_{\mathbb{Q}}) + \pi\left(I(X_{\mathbb{Q}})\right)\right) = \rho\left(R_I(X_{\mathbb{Q}})\right) + \pi\left(I(X_{\mathbb{Q}})\right)$, where the insurer and reinsurer share the total loss X under the probability measure $\mathbb{Q}$. They use the risk measures ρ and π, respectively, to quantify their own risk exposure. Thus, problem (5.1) provides the optimal risk sharing strategy for the insurer and reinsurer in the worst-case scenario. Furthermore, we can easily derive a lower bound for problem (5.1):

$$\min_{I \in \mathcal{I}} \sup_{\mathbb{Q} \in \mathcal{P}_\lambda} \rho\left(R_I(X_{\mathbb{Q}}) + \pi\left(I(X_{\mathbb{Q}})\right)\right) \geqslant \sup_{\mathbb{Q} \in \mathcal{P}_\lambda} \min_{I \in \mathcal{I}} \left\{\rho\left(R_I(X_{\mathbb{Q}})\right) + \pi\left(I(X_{\mathbb{Q}})\right)\right\}. \tag{5.2}$$

For an arbitrary monetary risk measure ρ, neither problem on the left nor the right side of (5.2) may be mathematically tractable. To clarify this further, in the remainder of this section, we assume that the insurer uses a distortion risk measure $\rho = \rho^h$ to quantify her risk exposure. We then demonstrate that the inequality in (5.2) holds as an equality and then characterize the optimal reinsurance strategy in the worst-case scenario.

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Theorem 5.1. *Let Assumptions 1 and 2 hold, and assume that $\rho = \rho^h$ is induced by a continuously differentiable distortion function h. Then, the minimum of the cooperative problem (5.1) is*

$$\rho^{h\wedge[(1+\theta)g]}(X_\lambda), \tag{5.3}$$

which is achieved by the optimal reinsurance contract $I^*(x) = \int_0^x (I^*)'(y)\,\mathrm{d}y$, $y \geqslant 0$, with

$$(I^*)'(x) = \begin{cases} 0, & \text{if } h(\mathbb{P}(X_\lambda > x)) < (1+\theta)g(\mathbb{P}(X_\lambda > x)), \\ 1, & \text{if } h(\mathbb{P}(X_\lambda > x)) > (1+\theta)g(\mathbb{P}(X_\lambda > x)), \\ \eta(x), & o/w, \end{cases} \tag{5.4}$$

with any function $\eta : [0, \infty) \to [0, 1]$.

Furthermore, the minimum of the cooperative problem (5.1) is always greater than that of the non-cooperative problem problem (3.1), that is,

$$\min_{I\in\mathcal{I}} \sup_{\mathbb{Q}\in\mathcal{P}_\lambda} \left\{\rho\left(R_I(X_\mathbb{Q})\right) + \pi\left(I(X_\mathbb{Q})\right)\right\} \geqslant \min_{I\in\mathcal{I}} \sup_{\mathbb{Q}\in\mathcal{P}_\lambda} \left\{\rho\left(R_I(X_\mathbb{Q})\right) + \pi\left(I(X)\right)\right\}. \tag{5.5}$$

Proof. Note that $h \wedge [(1+\theta)g] = \min\{h, (1+\theta)g\}$ is a non-decreasing and continuous function with $(h \wedge [(1+\theta)g])(0) = 0$ and $(h \wedge [(1+\theta)g])(1) = 1$. Therefore, $h \wedge [(1+\theta)g]$ is a valid distortion function. For any $\mathbb{Q} \in \mathcal{P}_\lambda$, if $\rho = \rho^h$ and $\pi = (1+\theta)\rho^g$, it is well known that $\min_{I\in\mathcal{I}} \left\{\rho\left(R_I(X_\mathbb{Q})\right) + \pi\left(I(X_\mathbb{Q})\right)\right\} = \rho^{h\wedge[(1+\theta)g]}(X_\mathbb{Q})$. Therefore, (5.1) can be rewritten as:

$$\begin{aligned} \min_{I\in\mathcal{I}} \sup_{\mathbb{Q}\in\mathcal{P}_\lambda} \left\{\rho\left(R_I(X_\mathbb{Q})\right) + \pi\left(I(X_\mathbb{Q})\right)\right\} &\geqslant \sup_{\mathbb{Q}\in\mathcal{P}_\lambda} \min_{I\in\mathcal{I}} \left\{\rho\left(R_I(X_\mathbb{Q})\right) + \pi\left(I(X_\mathbb{Q})\right)\right\} \\ &= \sup_{\mathbb{Q}\in\mathcal{P}_\lambda} \rho^{h\wedge[(1+\theta)g]}(X_\mathbb{Q}) = \rho^{h\wedge[(1+\theta)g]}(X_\lambda). \end{aligned} \tag{5.6}$$

On the other hand, from the proof of Proposition 2 in Liu *et al*. (2022), for any $\mathbb{Q} \in \mathcal{P}_\lambda$ and any random variable Y, we have $\mathbb{Q}(Y \leqslant y) \geqslant \mathbb{P}(Y_\lambda \leqslant y)$ for all $y \in \mathbb{R}$. This implies that

$$\mathbb{P}(I(X_\mathbb{Q}) \leqslant x) = \mathbb{P}(I(X)_\mathbb{Q} \leqslant x) = \mathbb{Q}(I(X) \leqslant x) \geqslant \mathbb{P}(I(X)_\lambda \leqslant x) = \mathbb{P}(I(X_\lambda) \leqslant x), \qquad x \geqslant 0.$$

By the law-invariance and monotonicity of ρ^g, we have $\rho^g(I(X_\mathbb{Q})) \leqslant \rho^g(I(X_\lambda))$. Similarly, we can verify that $\rho^h(R_I(X_\mathbb{Q})) \leqslant \rho^h(R_I(X_\lambda))$. Then it follows that, for any $I \in \mathcal{I}$,

$$\sup_{\mathbb{Q}\in\mathcal{P}_\lambda} \left\{\rho^h\left(R_I(X_\mathbb{Q})\right) + (1+\theta)\rho^g\left(I(X_\mathbb{Q})\right)\right\} \leqslant \rho^h\left(R_I(X_\lambda)\right) + (1+\theta)\rho^g\left(I(X_\lambda)\right).$$

Consequently,

$$\min_{I\in\mathcal{I}} \sup_{\mathbb{Q}\in\mathcal{P}_\lambda} \left\{\rho\left(R_I(X_\mathbb{Q})\right) + \pi\left(I(X_\mathbb{Q})\right)\right\} \leqslant \min_{I\in\mathcal{I}} \left\{\rho^h\left(R_I(X_\lambda)\right) + (1+\theta)\rho^g\left(I(X_\lambda)\right)\right\} = \rho^{h\wedge[(1+\theta)g]}(X_\lambda), \tag{5.7}$$

where the optimal indemnity such that the last equality holds is defined by (5.4). Inequalities (5.6) and (5.7) together imply (5.3), and I^* is the optimal reinsurance contract.

Next, we show (5.5). If h and g are distortion functions, it is straightforward to verify that $h \wedge [(1+\theta)g]$ is also a distortion function. For function $f \in \{h, g, h \wedge [(1+\theta)g]\}$, define

$$\bar{f}(t) = \begin{cases} f(t/\lambda), & 0 \leqslant t < \lambda, \\ 1, & \lambda \leqslant t \leqslant 1. \end{cases}$$

Under Assumption 2, the problem (3.1) can be transformed into

$$\min_{I\in\mathcal{I}} \left\{\sup_{\mathbb{Q}\in\mathcal{P}_\lambda} \rho(R_I(X_\mathbb{Q}) + \pi(I(X)))\right\} = \min_{I\in\mathcal{I}} \left\{\rho^{\bar{h}}(R_I(X)) + (1+\theta)\rho^g(I(X))\right\} = \rho^{\bar{h}\wedge[(1+\theta)g]}(X).$$

Thus,

$$\rho^{h\wedge[(1+\theta)g]}(X_\lambda) = \overline{\rho^{h\wedge[(1+\theta)g]}}^{\mathcal{P}_\lambda}(X) = \rho^{\overline{h\wedge[(1+\theta)g]}}(X) = \rho^{\bar{h}\wedge[(1+\theta)\bar{g}]}(X) \geqslant \rho^{\bar{h}\wedge[(1+\theta)g]}(X). \tag{5.8}$$

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Table 1. $\rho = RVaR_{(p,q)}$.

		$\rho^{h\wedge(1+\theta)g}(X_\lambda)$	$\bar{\rho}^{\mathcal{P}_\lambda}(\ell(X;I^*_{[\bar{\rho}^{\mathcal{P}_\lambda},\pi,X]}))$
$(1+\theta)p>1$	$\lambda<\frac{1}{(1+\theta)p}$	$2*RVaR_{(\lambda p,\lambda q)}(X)$	$\frac{1}{\lambda(p-q)}\int_{\lambda q}^{c_2} v_u\,\mathrm{d}u+(1+\theta)\int_{c_2}^{\frac{1}{1+\theta}} v_u\,\mathrm{d}u$
	$\lambda\geqslant\frac{1}{(1+\theta)p}$		$RVaR_{(\lambda p,\lambda q)}(X)$
$(1+\theta)p\leqslant 1$		$\frac{1}{\lambda(p-q)}\int_{\lambda q}^{c_1} v_u\,\mathrm{d}u+\frac{1+\theta}{\lambda}\int_{c_1}^{\frac{\lambda}{1+\theta}} v_u\,\mathrm{d}u$	$\frac{1}{\lambda(p-q)}\int_{\lambda q}^{c_2} v_u\,\mathrm{d}u+(1+\theta)\int_{c_2}^{\frac{1}{1+\theta}} v_u\,\mathrm{d}u$

The parameter $c_1=\frac{\lambda q}{1-(1+\theta)(p-q)}$ and $c_2=\frac{\lambda q}{1-\lambda(1+\theta)(p-q)}$.

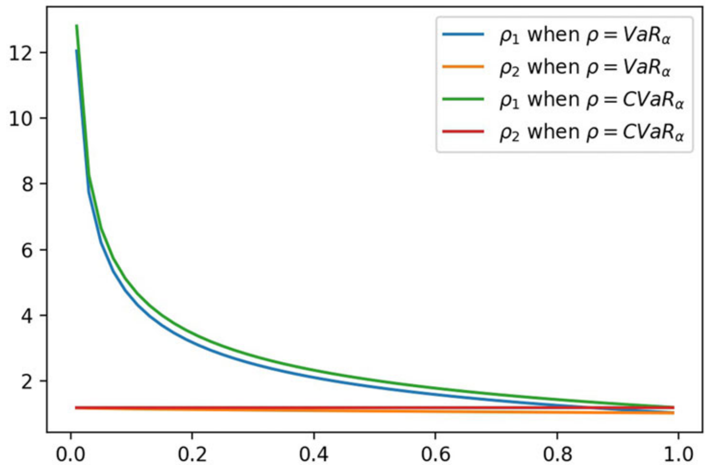

Figure 2. *For $\lambda\in(0,1]$ $\rho_1=\rho^{h\wedge(1+\theta)g}(X_\lambda)$ and $\rho_2=\bar{\rho}^{\mathcal{P}_\lambda}(\ell(X;I^*_{[\bar{\rho}^{\mathcal{P}_\lambda},\pi,X]}))$, where $\zeta=3,\eta=2$, $\theta=0.2$, and $\alpha=0.05$.*

The first equality holds because $\bar{\rho}^{\mathcal{P}_\lambda}$ is a λ-tail risk measure generated by ρ, the second equality holds because $\overline{\rho^f}^{\mathcal{P}_\lambda}$ is a distortion risk measure with distortion function $\bar{f}$, and the last inequality holds because $\bar{g}\geqslant g$. The inequality (5.5) follows from (5.8). $\square$

Example 5.1 *(Comparison of Problem (3.1) and Problem (5.1) with Pareto distribution). Assume that X follows the type II Pareto distribution, that is, $X\sim Pa(\zeta,\eta)$ with the survival function $S_X(x)=\left(\frac{\eta}{x+\eta}\right)^{\zeta}$, $x>0$. Let $\pi=(1+\theta)\mathbb{E}$ with $\theta\geqslant 0$. Table 1 provide a theoretical comparison of the objective functional values $\rho^{h\wedge(1+\theta)g}(X_\lambda)$ and $\bar{\rho}^{\mathcal{P}_\lambda}(\ell(X;I^*_{[\bar{\rho}^{\mathcal{P}_\lambda},\pi,X]}))$ for $\rho^h = RVaR_{(p,q)}$. For two special cases when $\rho^h=\mathrm{VaR}_\alpha,\mathrm{TVaR}_\alpha$, the comparison of the value functions of Problems (3.1) and (5.1) for different sizes of the uncertainty set are displayed in Figures 2 and 3.*

6. Conclusion

In this paper, we investigate insurance design problems that seek to minimize the worst-case risk of the insurer's total loss, accounting for model uncertainty. The worst-case problem involves maximization over probability measures derived from the likelihood ratio uncertainty set. We establish a relation between the optimal strategy in the regular-case strategy, which disregards model uncertainty, and the optimal strategy in the worst-case scenario, which incorporates model uncertainty. Specifically, when the insurer is aware of the regular-case strategy, our results provide a framework for determining the corresponding worst-case strategy in a non-cooperative setting. In a more general context, this relation is applicable to optimal reinsurance models quantified by tail risk measures. Our findings are applicable to a wide range of problems. We focus particularly on the model quantified by the expectile risk measure, where we derive optimal policies for the worst-case problem. We also provide a sufficient and necessary

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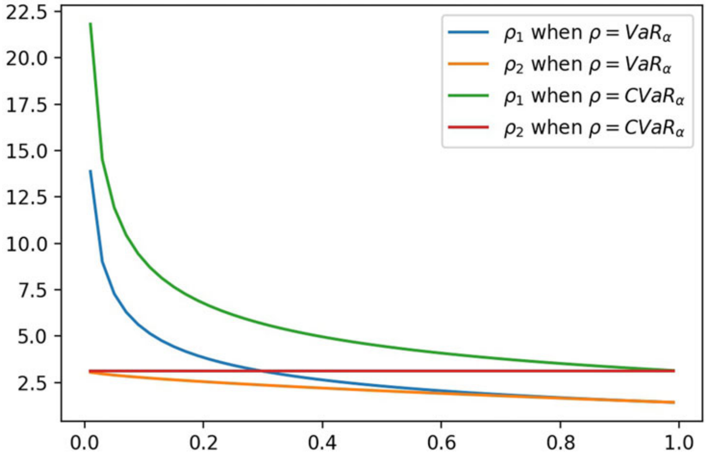

Figure 3. *For* $\lambda \in (0, 1]$, $\rho_1 = \rho^{h\wedge(1+\theta)g}(X_\lambda)$ *and* $\rho_2 = \bar{\rho}^{\mathcal{P}_\lambda}(\ell(X; I^*_{[\bar{\rho}^{\mathcal{P}_\lambda},\pi,X]}))$*, where* $\zeta = 3$, $\eta = 2$, $\theta = 4$*, and* $\alpha = 0.2$.

condition for the problem quantified by a distortion risk measure to have common optimal solutions with its robust counterpart. The robustness of the problem is discussed. In addition, we investigate the corresponding cooperative problem and determine the optimal indemnity function. Our results demonstrate that the value function in the cooperative model exceeds that of the non-cooperative model.

In addition, it is interesting to discuss the reinsurer's risk quantification in the optimization problems. In practice, the insurer and the reinsurer can have heterogeneous beliefs and, therefore, do not agree on the underlying probability measures. For the studies of standard optimal reinsurance design problems along this direction, we refer to Boonen and Ghossoub (2019), Chi (2019), Ghossoub (2019), and references therein. Furthermore, the reinsurer may raise her own uncertainty concern. For example, the reinsurer can determine the reinsurance premium in the worst-case scenario, say $\pi^{\uparrow}(I(X)) = \sup_{F\in\mathcal{S}} \pi(I(X^F))$, where $\mathcal{S}$ is the reinsurer's uncertainty set. Here, $\mathcal{S}$ is not necessarily align with the insurer's uncertainty set, and $\mathcal{S}$ can be induced by, for examples, ϕ-divergence, Wasserstein metrics, or moments constraints. In an alternative formulation, we can propose equilibrium problems for the insurer and the reinsurer when they adopt their own uncertainty sets. These challenging questions require future work.

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References

Asimit, A.V., Bignozzi, V., Cheung, K.C., Hu, J. and Kim, E.-S. (2017) Robust and pareto optimality of insurance contracts. *European Journal of Operational Research*, **262**(2), 720–732.

Bellini, F., Klar, B., Müller, A. and Gianin, E. (2014) Generalized quantiles as risk measures. *Insurance: Mathematics and Economics*, **54**, 41–48.

Bernard, C., Pesenti, S.M. and Vanduffel, S. (2024) Robust distortion risk measures. *Mathematical Finance*, **34**(3), 774–818.

Birghila, C. and Pflug, G.C. (2019) Optimal xl-insurance underwasserstein-type ambiguity. *Insurance: Mathematics and Economics*, **88**, 30–43.

Boonen, T.J. and Ghossoub, M. (2019) On the existence of a representative reinsurer under heterogeneous beliefs. *Insurance: Mathematics and Economics*, **88**, 209–225.

Boonen, T.J. and Jiang, W. (2024) Robust insurance design with distortion risk measures. *European Journal of Operational Research*, **316**(2), 694–706.

Boonen, T.J., Liu, F. and Wang, R. (2021) Competitive equilibria in a comonotone market. *Economic Theory*, **72**(4), 1217–1255.

Borch, K. (1960) An attempt to determine the optimum amount of stop loss reinsurance. *Transactions of the 16th International Congress of Actuaries*, I(1), pp. 597–610.

https://doi.org/10.1017/asb.2025.1 Published online by Cambridge University Press

Cai, J., Liu, F. and Yin, M. (2024) Worst-case risk measures of stop-loss and limited loss random variables under distribution uncertainty with applications to robust reinsurance. *European Journal of Operational Research*, **318**(1), 310–326.

Cai, J. and Weng, C. (2016) Optimal reinsurance with expectile. *Scandinavian Actuarial Journal*, **2016**(7), 624–645.

Chen, L., He, S. and Zhang, S. (2011) Tight bounds for some risk measures, with applications to robust portfolio selection. *Operations Research*, **59**(4), 847–865.

Cheung, K., Sung, K.C.J., Yam, S.C.P. and Yung, S. (2014) Optimal reinsurance under general law-invariant risk measures. *Scandinavian Actuarial Journal*, **2014**(1), 72–91.

Chi, Y. (2019) On the optimality of a straight deductible under belief heterogeneity. *ASTIN Bulletin*, **49**(1), 243–262.

Embrechts, P., Liu, H., Mao, T. and Wang, R. (2020) Quantile-based risk sharing with heterogeneous beliefs. *Mathematical Programming Series B*, **181**, 319–347.

Embrechts, P., Schied, A. and Wang, R. (2022) Robustness in the optimization of risk measures. *Operations Research*, **70**, 95–110.

Fissler, T., Liu, F., Wang, R. and Wei, L. (2024) Elicitability and identifiability of tail risk measures. arXiv:2404.14136.

Föllmer, H. and Schied, A. (2016) *Stochastic Finance: An Introduction in Discrete Time*, 4th edn. Berlin: Walter de Gruyter.

Ghaoui, L.E., Oks, M. and Oustry, F. (2003) Worst-case value-at-risk and robust portfolio optimization: A conic programming approach. *Operations Research*, **51**(4), 543–556.

Ghossoub, M. (2019) Budget-constrained optimal insurance with belief heterogeneity. *Insurance: Mathematics and Economics*, **89**, 79–91.

Hu, X., Yang, H. and Zhang, L. (2015) Optimal retention for a stop-loss reinsurance with incomplete information. *Insurance: Mathematics and Economics*, **65**, 15–21.

Li, J.Y.-M. (2018) Closed-form solutions for worst-case law invariant risk measures with application to robust portfolio optimization. *Operations Research*, **66**(6), 1533–1541.

Liu, F., Cai, J., Lemieux, C. and Ruodu, W. (2020) Convex risk functionals: Representation and applications. *Insurance: Mathematics and Economics*, **90**, 66–79.

Liu, F., Mao, T., Wang, R. and Wei, L. (2022) Inf-convolution, optimal allocations, and model uncertainty for tail risk measures. *Mathematics of Operations Research*, **47**(3), 2494–2519.

Liu, F. and Wang, R. (2021) A theory for measures of tail risk. *Mathematics of Operations Research*, **46**, 835–1234.

Liu, H. and Mao, T. (2022) Distributionally robust reinsurance with Value-at-Risk and Conditional Value-at-Risk. *Insurance: Mathematics and Economics*, **107**, 393–417.

Rüschendorf, L. and Vanduffel, S. (2020) On the construction of optimal payoffs. *Decisions in Economics and Finance*, **43**, 129–153.

Scarf, H.E., Arrow, K. and Karlin, S. (1957) *A min-max solution of an inventory problem*. Rand Corporation Santa Monica.

Ziegel, J.F. (2016) Coherence and elicitability. *Mathematical Finance*, **26**, 901–918.

A. Appendix

Proof of Theorem **3.1.** For any $I \in \mathcal{I}$, $\overline{\rho}^{\mathcal{P}_\lambda}(R_I(X)) = \rho(R_I(X)_\lambda)$ because $\overline{\rho}^{\mathcal{P}_\lambda}$ is a λ-tail risk measure generated by ρ. Under Assumption 1, the insurer's objective function in (3.2) can be expressed as:

$$\begin{aligned}\overline{\rho}^{\mathcal{P}_\lambda}(\ell(X;I)) &= \overline{\rho}^{\mathcal{P}_\lambda}(R_I(X)) + \pi(I(X)) = \rho(R_I(X)_\lambda) + \pi((I(X)-I(v_\lambda))_+ + I(X)\wedge I(v_\lambda)) \\ &= \rho(R_I(X)_\lambda - R_I(v_\lambda)) + \pi((I(X)-I(v_\lambda))_+) + R_I(v_\lambda) + \pi(I(X)\wedge I(v_\lambda)).\end{aligned} \tag{A1}$$

It is easy to see that

$$\begin{aligned}\min_{I\in\mathcal{I}} \overline{\rho}^{\mathcal{P}_\lambda}(\ell(X;I)) &= \min_{0\leqslant a\leqslant v_\lambda}\left\{\min_{I\in\mathcal{I},I(v_\lambda)=a} \overline{\rho}^{\mathcal{P}_\lambda}(\ell(X;I))\right\} \\ &\geqslant \min_{0\leqslant a\leqslant v_\lambda}\left\{\min_{I\in\mathcal{I},I(v_\lambda)=a}\{\rho(R_I(X)_\lambda - R_I(v_\lambda)) + \pi((I(X)-I(v_\lambda))_+)\}\right. \\ &\qquad \left. + \min_{I\in\mathcal{I},I(v_\lambda)=a}\{R_I(v_\lambda) + \pi(I(X)\wedge I(v_\lambda))\}\right\}.\end{aligned} \tag{A2}$$

For any $0 \leqslant a \leqslant v_\lambda$, define

$$\min_{I\in\mathcal{I},I(v_\lambda)=a}\{R_I(v_\lambda) + \pi(I(X)\wedge I(v_\lambda))\}, \tag{A3}$$

$$\min_{I\in\mathcal{I},I(v_\lambda)=a}\{\rho(R_I(X)_\lambda - R_I(v_\lambda)) + \pi((I(X)-I(v_\lambda))_+)\}. \tag{A4}$$

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If $I_a^* \in \mathcal{I}$ with $I_a^*(v_\lambda) = a$ is a solution to both problems (A3) and (A4), then I_a^* is a solution to $\min_{I\in\mathcal{I}, I(v_\lambda)=a} \overline{\rho}^{\mathcal{P}_\lambda}(\ell(X;I))$ in (A2), and the infinitely dimensional problem (3.2) is reduced to a 1-dimensional problem $\min_{0\leqslant a\leqslant v_\lambda} \overline{\rho}^{\mathcal{P}_\lambda}(\ell(X;I_a^*))$. Heuristically, the problem $\min_{I\in\mathcal{I}, I(v_\lambda)=a} \overline{\rho}^{\mathcal{P}_\lambda}(\ell(X;I))$ can be decomposed into the sub-optimization problem (A3) focusing on "small-loss part" of X (left-tail) and the sub-optimization problem (A4) focusing on "large-loss part" of X (right-tail).

We first consider the sub-problem (A4). Note that the function $(I(x) - I(v_\lambda))_+$ is non-decreasing in x. Therefore, $\mathrm{VaR}_t((I(X) - I(v_\lambda))_+) = (I(\mathrm{VaR}_t(X)) - I(v_\lambda))_+$ for any $t \in (0, 1)$. It follows that

$$
\begin{aligned}
\frac{1}{1+\theta}\pi\left((I(X) - I(v_\lambda))_+\right) &= \int_0^\infty g\left(\mathbb{P}((I(X) - I(v_\lambda))_+ > t)\right)\,\mathrm{d}t = \int_0^1 \mathrm{VaR}_t((I(X) - I(v_\lambda))_+) g'(t)\,\mathrm{d}t \\
&= \int_0^1 (I(\mathrm{VaR}_t(X)) - I(v_\lambda))_+\, g'(t)\,\mathrm{d}t = \int_0^\lambda (I(\mathrm{VaR}_t(X)) - I(v_\lambda))\, g'(t)\,\mathrm{d}t \\
&= \int_0^1 (I(\mathrm{VaR}_{\lambda u}(X)) - I(v_\lambda))\, g'(u\lambda)\lambda\,\mathrm{d}u = \int_0^1 (I(\mathrm{VaR}_u(X_\lambda)) - I(v_\lambda))\, g'(u\lambda)\lambda\,\mathrm{d}u,
\end{aligned}
$$

where the last equality comes from the fact that, when $p \in (0, 1)$, $\mathrm{VaR}_t(X_p) = \mathrm{VaR}_{pt}(X)$. If $g(\lambda) = 0$, then $g(p) = 0$ and $g'(p) = 0$ for all $p \in (0, \lambda)$. It follows that $\pi\left((I(X) - I(v_\lambda))_+\right) = 0$. If $g(\lambda) > 0$, by the continuity of g, $g'(p) > 0$ for p in a neighborhood of λ and then

$$
\frac{1}{1+\theta}\pi\left((I(X) - I(v_\lambda))_+\right) = g(\lambda)\int_0^1 (I(\mathrm{VaR}_u(X_\lambda)) - I(v_\lambda))\, \bar{g}'_\lambda(u)\,\mathrm{d}u = g(\lambda)\rho^{\bar{g}_\lambda}\left(I(X_\lambda) - I(v_\lambda)\right).
$$

Meanwhile, since $\mathrm{VaR}_t(X_p) = \mathrm{VaR}_{pt}(X)$, $t, p \in (0, 1)$, and R_I is a non-decreasing function, we have $\mathrm{VaR}_t(R(X)_p) = \mathrm{VaR}_{tp}(R(X)) = R(\mathrm{VaR}_{tp}(X)) = R(\mathrm{VaR}_t(X_p)) = \mathrm{VaR}_t(R(X_p))$ for $t, p \in (0, 1)$. Therefore, $R(X)_\lambda$ and $R(X_\lambda)$ have the same distribution. Using the above expression, we can rewrite the objective function in the problem (A4) as:

$$
\begin{aligned}
\rho(R_I(X)_\lambda - R_I(v_\lambda)) + \pi((I(X) - I(v_\lambda))_+) &= \rho\left(R_I(X_\lambda) - R_I(v_\lambda)\right) + (1+\theta)g(\lambda)\rho^{\bar{g}_\lambda}\left(I(X_\lambda) - I(v_\lambda)\right) \\
&= \rho\left(\bar{R}_I(X_{\lambda,0})\right) + (1+\theta)g(\lambda)\rho^{\bar{g}_\lambda}\left(\bar{I}(X_{\lambda,0})\right),
\end{aligned}
$$

where $\bar{R}_I(y) \triangleq R_I(y + v_\lambda) - R_I(v_\lambda)$ and $\bar{I}(y) \triangleq I(y + v_\lambda) - I(v_\lambda)$ for all $y \geqslant 0$. It is easy to see that, $\bar{I} \in \mathcal{I}$ for any $I \in \mathcal{I}$, and conversely, for any $\bar{I} \in \mathcal{I}$, there always exists $I \in \mathcal{I}$ such that $I(x) = \bar{I}(x - v_\lambda) + I(v_\lambda)$ for $x \geqslant v_\lambda$. Therefore, if I^* is a solution to the problem (A4) with $I^*(v_\lambda) = a$, then $\bar{I}^*(x) = I^*(x + v_\lambda) - a$ is a solution to the problem (3.5); if $\bar{I}^* = I^*_{[\rho,\bar{\pi}_\lambda,X_{\lambda,0}]}$ is a solution to the problem (3.5), then any $I \in \mathcal{I}$ satisfying $I(x) = \bar{I}^*(x - v_\lambda) + a$ for $x \geqslant x_\lambda$ and $I(v_\lambda) = a$ is a solution of the problem (A4).

Next, we consider the sub-problem (A3). Take $a \in [0, v_\lambda]$ and $I \in \mathcal{I}$ with $I(v_\lambda) = a$. Consider $f \in \mathcal{I}$ and its retained loss function given below, respectively,

$$
f(x) = \begin{cases} (x - R_I(v_\lambda))_+, & x \leqslant v_\lambda, \\ I(x), & x > v_\lambda, \end{cases} \qquad \text{and} \qquad R_f(x) = x - f(x) = \begin{cases} x \wedge R_I(v_\lambda), & x \leqslant v_\lambda, \\ R_I(x), & x > v_\lambda. \end{cases}
$$

It is easy to see that $R_f(v_\lambda) = R_I(v_\lambda) = v_\lambda - a$, and $f(x) \leqslant I(x)$ for all $x \geqslant 0$. Since π is comonotonic, we have $R_f(v_\lambda) + \pi(f(X) \wedge f(v_\lambda)) \leqslant R_I(v_\lambda) + \pi(I(X) \wedge I(v_\lambda))$. It says that I is suboptimal to f in the problem (A3), and thus, f is an optimal solution.

In short, for any $a \in [0, v_\lambda]$, the function

$$
I_a(x) \triangleq \begin{cases} (x - v_\lambda + a)_+, & x < v_\lambda, \\ I^*[\rho, \bar{\pi}_\lambda, X_{\lambda,0}](x - v_\lambda) + a, & x \geqslant v_\lambda, \end{cases} \tag{A5}
$$

satisfies $I_a(v_\lambda) = a$, $I_a(x) = I^*[\rho, \bar{\pi}_\lambda, X_{\lambda,0}](x - v_\lambda) + a$ for $x \geqslant v_\lambda$, and $I_a(x) \leqslant I(x)$ for all $x < v_\lambda$ and $I \in \mathcal{I}$ with $I(v_\lambda) = a$. Therefore, I_a is an optimal solution to problems (A3) and (A4) simultaneously. Furthermore, I_a is an optimal solution to the inner problem of (A2).

Finally, we determine the optimal a to the outer problem in (A2). For each $a \in [0, v_\lambda]$,

$$\begin{aligned}\overline{\rho}^{\mathcal{P}_\lambda}\ (\ell(X; I_a)) &= \rho(R_{I_a}(X)_\lambda - R_{I_a}(v_\lambda)) + R_{I_a}(v_\lambda) + \pi(I_a(X)) \\ &= \rho(X_{\lambda,0} - I^*_{[\rho,\bar{\pi}_\lambda,X_{\lambda,0}]}(X_{\lambda,0})) + v_\lambda - a + (1+\theta)\int_{v_\lambda - a}^{v_\lambda} g(S_X(x))\,\mathrm{d}x + C,\end{aligned}$$

where $\rho(X_{\lambda,0} - I^*[\rho, \bar{\pi}_\lambda, X_{\lambda,0}](X_{\lambda,0}))$ and constant $C = (1+\theta)\int_{v_\lambda}^{\infty} g(S_X(x)) I^{*'}[\rho, \bar{\pi}_\lambda, X_{\lambda,0}](x - v_\lambda)\,\mathrm{d}x$ do not dependent on the value of a. Minimizing (A2) is equivalent to the problem

$$\min_{0\leqslant a\leqslant v_\lambda}\left\{-a + (1+\theta)\int_{v_\lambda - a}^{v_\lambda} g(S_X(x))\,\mathrm{d}x\right\}, \tag{A6}$$

which has a minimizer at $a^* = \inf\{0 \leqslant a \leqslant v_\lambda : g(S_X(v_\lambda - a)) \geqslant \frac{1}{1+\theta}\}$. □

Proof of Proposition **3.2.** Take $F = F_X$ and $S = S_X$. The objective function in the problem (3.17) is $L_1 = \int_0^\infty [(1+\theta)S(x) - h(S(x))]\, I'(x)\,\mathrm{d}x + \rho^h(X)$. By the similar argument in Boonen *et al.* (2021), the solutions to the problem (3.17) are

$$I^*(x) \triangleq I^*[\rho, (1+\theta)\mathbb{E}, X](x) = \int_0^x \xi^*(y)\,\mathrm{d}y, \tag{A7}$$

where $\xi^*(y) = 1$ if $(1+\theta)S(y) < h(S(y))$, $\xi^*(y) = 0$ if $(1+\theta)S(y) > h(S(y))$, and $\xi^*(y) = \xi_1(y)$ can be any function within the range [0,1] if $(1+\theta)S(y) = h(S(y))$.

Since $\overline{\rho^h}^{\mathcal{P}_\lambda}$ is a λ-tail risk measure generated by ρ^h, we know that $\overline{\rho}^{\mathcal{P}_\lambda} = \rho^{\tilde{h}}$ is again a distortion risk measure with distortion function $\tilde{h}$. Similarly, the problem (3.18) has the objective function $L_2 = \int_0^\infty \left[(1+\theta)S(x) - \tilde{h}(S(x))\right] I'(x)\,\mathrm{d}x + \rho^{\tilde{h}}(X)$, and optimal solutions

$$\tilde{I}^*(x) \triangleq I^*[\overline{\rho}^{\mathcal{P}_\lambda}, (1+\theta)\mathbb{E}, X](x) = \int_0^x \tilde{\xi}^*(y)\,\mathrm{d}y, \tag{A8}$$

where $\tilde{\xi}^*(y) = 1$ if $(1+\theta)S(y) < \tilde{h}(S(y))$, $\tilde{\xi}^*(y) = 0$ if $(1+\theta)S(y) > \tilde{h}(S(y))$, and $\tilde{\xi}^*(y) = \xi_2(y)$ can be any function within the range [0,1] if $(1+\theta)S(y) = \tilde{h}(S(y))$.

Let $A \triangleq \{p \in (0,1) : (1+\theta)p < h(p)\}$, $E \triangleq \{p \in (0,1) : (1+\theta)p = h(p)\}$, $B \triangleq \{p \in (0, \frac{1}{1+\theta}) : (1+\theta)p > h(p)\}$, $\tilde{A} \triangleq \{p \in (0,1) : (1+\theta)p < \tilde{h}(p)\}$, $\tilde{E} \triangleq \{p \in (0,1) : (1+\theta)p = \tilde{h}(p)\}$, $\tilde{B} \triangleq \{p \in (0, \frac{1}{1+\theta}) : (1+\theta)p > \tilde{h}(p)\}$. Because $\tilde{h}(p) \geqslant h(p)$, we have $A \subset \tilde{A}$ and $\tilde{B} \subset B$.

We first prove that the problems (3.17) and (3.18) have at least one common optimal solution for a given $\lambda \in (0,1)$ if and only if $\tilde{A} \subset A \cup E$. If $\tilde{A} \subsetneqq A \cup E$, let $D = \tilde{A}/(A \cup E)$ and denote the indicator function as $\mathbb{1}_{\{\cdot\}}$. It then follows that $\tilde{I}^*(x) = \int_0^x \mathbb{1}_{\{S(y)\in\tilde{A}\}}\,\mathrm{d}y + \int_0^x \xi_2(y)\mathbb{1}_{\{S(y)\in\tilde{E}\}}\,\mathrm{d}y \geqslant \int_0^x \mathbb{1}_{\{S(y)\in\tilde{A}\}}\,\mathrm{d}y$. Because of the continuity of $\tilde{h}(p)$, there exists $0 < a < b$ such that $(S(b), S(a)) \subset D$ and $\tilde{I}^*(b) \geqslant \int_0^b \mathbb{1}_{\{S(y)\in D\}}\,\mathrm{d}y + \int_0^b \mathbb{1}_{\{S(y)\in A\cup E\}}\,\mathrm{d}y \geqslant b - a + I^*(b) > I^*(b)$ meaning that the problems (3.17) and (3.18) have no common optimal solutions when $\tilde{A} \subsetneqq A \cup E$, which leads to a contradiction. On the other hand, suppose that $\tilde{A} \subset A \cup E$, we can always equate (A7) and (A8) by adjusting the values of $\xi_1(y)$ and $\xi_2(y)$. Specifically, let $\xi_1(y) = 1$ when $S(y) \in \tilde{A}/A$, $\xi_2(y) = 0$ when $S(y) \in B/\tilde{B}$, and $\xi_1(y) = \xi_2(y)$ otherwise. Thus, $\tilde{A} \subset A \cup E$ is a necessary and sufficient condition for the problems (3.17) and (3.18) having at least one common optimal solution for a given $\lambda \in (0,1)$.

(a) "⇐" In this part, we will prove that condition (i) or (ii) implies $\tilde{A} \subset A \cup E$. (i) If $h(p) \geqslant (1+\theta)p$ for $p \in (0, p_0]$ and $\tilde{h}(p) \leqslant (1+\theta)p$ for $p \in [p_0, 1)$, where $p_0 \in (0, \frac{1}{1+\theta}]$, we have $\tilde{A} \subset (0, p_0] \subset A \cup E$. (ii) If $\tilde{h}(p) \leqslant (1+\theta)p$ for $p \in [0,1]$, then $\tilde{A} = \varnothing \subset A \cup E$. Thus, the problems (3.17) and (3.18) have at least one common optimal solution for a given $\lambda \in (0,1)$.
"⇒" The following part proves that $h(p)$ and $\tilde{h}(p)$ are classified into (i) or (ii) under condition $\tilde{A} \subset A \cup E$. If $\tilde{A} \neq \varnothing$, due to the continuity of $\tilde{h}(p)$, there exist $p_u, p_d \in [0, \frac{1}{1+\theta}]$ such that $\tilde{h}(p_u) = (1+\theta)p_u$, $\tilde{h}(p_d) = (1+\theta)p_d$, and $\tilde{h}(p) > (1+\theta)p$ for $p \in (p_d, p_u)$. We first assume

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$p_d > 0$. If $h(p_d) < \tilde{h}(p_d)$, the continuity of $h(p)$ and $\tilde{h}(p)$ implies that there exists $\delta > 0$ such that $(p_d, p_d + \delta) \subset \tilde{A}$ and $(p_d, p_d + \delta) \subsetneqq A \cup E$, which contradicts the premise $\tilde{A} \subset A \cup E$. Thus, we have $h(p_d) = \tilde{h}(p_d) = h(\frac{p_d}{\lambda})$. Because $h(p)$ is a continuous and increasing function, $h(p)$ should be a constant for $p \in [p_d, \frac{p_d}{\lambda}]$. Then there exists $\delta' > 0$ such that $(p_d, p_d + \delta') \subset \tilde{A}$ and $(p_d, p_d + \delta') \subsetneqq A \cup E$, which also leads to a contradiction. Hence, we have $p_d = 0$, which means that $\tilde{A}$ is a connected set with an infimum 0. Let $p_0 \triangleq \sup \tilde{A}$. Then $\tilde{A} = (0, p_0)$, $\tilde{h}(p_0) = (1+\theta)p_0$, and $p_0 \leqslant \frac{1}{1+\theta}$. Since $\tilde{A} \subset A \cup E$, we must have $h(p) \geqslant (1+\theta)p$ for $p \in (0, p_0]$ and $\tilde{h}(p) \leqslant (1+\theta)p$ for $p \in [p_0, 1)$, which is Case (i). If $\tilde{A} = \varnothing$ for $p \in [0, 1]$, it then follows that $\tilde{h}(p) \leqslant (1+\theta)p$, which is Case (ii).

(b) "$\Leftarrow$" If $h(p) \geqslant (1+\theta)p$ for $p \in (0, \frac{1}{1+\theta}]$, $\tilde{h}(p) \geqslant h(p)$ implies $\tilde{A} \subset A \cup E = (0, \frac{1}{1+\theta}]$ for $\lambda \in (0, 1)$. "$\Rightarrow$" Assume that the problems (3.17) and (3.18) have at least one common optimal solution for any $\lambda \in (0, 1)$. If $B \neq \emptyset$, there exist $p_a \in (0, \frac{1}{1+\theta})$ such that $h(p_a) < (1+\theta)p_a$. Let $\lambda < p_a$, it then follows that $h(p_a) < (1+\theta)p_a < \tilde{h}(\lambda) = 1$, which contradicts $\tilde{A} \subset A \cup E$. Thus, $B = \emptyset$, which means $h(p) \geqslant (1+\theta)p$ for $p \in (0, \frac{1}{1+\theta}]$. □

Proof of Lemma **4.1.** For simplicity, write $L_I(X) \triangleq \mathcal{E}_\alpha(R_I(X) + (1+\theta)\mathbb{E}[I(X)])$ and let $\mathcal{E}_I^R$ denote the α-expectile of $R(X) = X - I(X)$ for $I \in \mathcal{I}$. Define a two-layer function $h_{a,b,m}(x) = x - (x-a)_+ + (x-b)_+ - (x-m)_+$ with parameters $0 \leqslant a \leqslant b \leqslant m \leqslant \infty$ and $\mathcal{I}_1 = \{h_{a,b,m} \in \mathcal{I} : 0 \leqslant a \leqslant b = a + \mathcal{E}^R_{h_{a,b,m}} \leqslant m \leqslant \infty\}$. By the similar argument in Theorem 3.1 of Cai and Weng (2016), for $-1 \leqslant \theta \leqslant 0$, we have $\min_{I \in \mathcal{I}} L_I(X) = \min_{I \in \mathcal{I}_1} L_I(X)$.

Arbitrarily take and fix $h_{a,b,m} \in \mathcal{I}_1$. Define a one-layer function $I_{d,k}(x) = (x-d)_+ - (x-k)_+$ with $0 \leqslant d \leqslant \mathcal{E}^R_{h_{a,b,m}}$, $k = m + d - \mathcal{E}^R_{h_{a,b,m}}$, and $d \leqslant k$, such that $R_{I_{d,k}}(m) = R_{h_{a,b,m}}(m) = \mathcal{E}^R_{h_{a,b,m}}$ and $R_{I_{d,k}}(x) = R_{h_{a,b,m}}(x)$ when $x > m$. Therefore, for any pair (d,k), we have

$$\mathbb{E}\left[\left(R_{I_{d,k}}(X) - \mathcal{E}^R_{h_{a,b,m}}(X)\right)_+\right] = \mathbb{E}\left[\left(R_{h_{a,b,m}}(X) - \mathcal{E}^R_{h_{a,b,m}}(X)\right)_+\right]. \tag{A9}$$

Note that $R_{I_{d,k}}(x) \leqslant R_{h_{a,b,m}}(x)$ when $d = 0$ and $R_{I_{d,k}}(x) \geqslant R_{h_{a,b,m}}(x)$ when $d = \mathcal{E}^R_{h_{a,b,m}}$. Because $\mathbb{E}[R_{I_{d,k}}(X)]$ is a continuous function of d, there must exists $0 \leqslant \tilde{d} \leqslant \mathcal{E}^R_{h_{a,b,m}}$ such that $\mathbb{E}[R_{I_{\tilde{d},k}}(X)] = \mathbb{E}[R_{h_{a,b,m}}(X)]$. Thus, with $\pi = (1+\theta)\mathbb{E}$, we have $\pi(I_{\tilde{d},k}(X)) = \pi(h_{a,b,m}(X))$. In addition, since

$$\mathbb{E}[R_{I_{\tilde{d},k}}(X)] + \beta\mathbb{E}[(R_{I_{\tilde{d},k}}(X) - \mathcal{E}^R_{h_{a,b,m}}(X))_+] = \mathbb{E}[R_{h_{a,b,m}}(X)] + \beta\mathbb{E}[(R_{h_{a,b,m}}(X) - \mathcal{E}^R_{h_{a,b,m}}(X))_+] = \mathcal{E}^R_{h_{a,b,m}}(X),$$

by (4.1), we have $\mathcal{E}^R_{I_{\tilde{d},k}}(X) = \mathcal{E}^R_{h_{a,b,m}}(X)$. Therefore, $L_{I_{\tilde{d},k}}(X) = \mathcal{E}^R_{I_{\tilde{d},k}}(X) + \pi(I_{\tilde{d},k}(X)) = \mathcal{E}^R_{h_{a,b,m}}(X) + \pi(h_{a,b,m}(X)) = L_{I_{h_{a,b,m}}}(X)$. Because $h_{a,b,m}$ is arbitrarily taken, $\min_{I \in \mathcal{I}_1} L_I(X) = \min_{I \in \mathcal{I}_2} L_I(X)$.

For $I_{d,k} \in \mathcal{I}_2$, define m such that $R_{I_{d,k}}(m) = \mathcal{E}^R_{I_{d,k}}(X)$. Then we can get $m \geqslant k$ and $m = k - d + \mathcal{E}^R_{I_{d,k}}$. The parameter m can be viewed as an implicit function of the other two parameters d and k. Because $\mathcal{E}^R_{I_{d,k}}(X) = \mathbb{E}[R_{I_{d,k}}(X)] + \beta\mathbb{E}[(R_{I_{d,k}}(X) - \mathcal{E}^R_{I_{d,k}}(X))_+] = \mathbb{E}[R_{I_{d,k}}(X)] + \beta\mathbb{E}[(X-m)_+]$, we can get

$$\frac{\partial m}{\partial d} = -1 + \frac{\partial}{\partial d}\mathcal{E}^R_{I_{d,k}}(X) = -1 + \frac{\partial}{\partial d}\left\{\mathbb{E}[R_{I_{d,k}}(X)] + \beta\mathbb{E}[(X-m)_+]\right\} = -1 + S_X(d) - \beta S_X(m)\frac{\partial m}{\partial d}$$
$$\frac{\partial m}{\partial k} = 1 + \frac{\partial}{\partial k}\mathcal{E}^R_{I_{d,k}}(X) = 1 + \frac{\partial}{\partial k}\left\{\mathbb{E}[R_{I_{d,k}}(X)] + \beta\mathbb{E}[(X-m)_+]\right\} = 1 - S_X(k) - \beta S_X(m)\frac{\partial m}{\partial k}$$

which give $\frac{\partial m}{\partial d} = \frac{-F_X(d)}{1+\beta S_X(m)} < 0$ and $\frac{\partial m}{\partial k} = \frac{F_X(k)}{1+\beta S_X(m)} > 0$. Furthermore,

$$L_{I_{d,k}}(X) = \mathcal{E}^R_{I_{d,k}}(X) + (1+\theta)\mathbb{E}[I_{d,k}(X)] = m - k + d + (1+\theta)\int_d^k S_X(t)\,\mathrm{d}t.$$

Since $-1 \leqslant \theta \leqslant 0$, we have $\frac{\partial}{\partial d}L_{I_{d,k}}(X) = \frac{\partial m}{\partial d} + 1 - (1+\theta)S_X(d) = \frac{\beta S_X(m)F_X(d)}{1+\beta S_X(m)} - \theta S_X(d) \geqslant 0$, and $\frac{\partial}{\partial k}L_{I_{d,k}}(X) = \frac{\partial m}{\partial k} - 1 + (1+\theta)S_X(k) = \frac{-\beta S_X(m)F_X(k)}{1+\beta S_X(m)} + \theta S_X(k) \leqslant 0$, which means $L_{I_{d,k}}(X)$ increases with respect to d while decreases with respect to k. Then $L_{I_{d,m}}(X)$ reaches its minimum at $d = 0$ and $k = m$, indicating $\mathcal{E}^R_{I_{d,k}} = 0$ and $I^*_{[\mathcal{E}_\alpha,(1+\theta)\mathbb{E},X]}(x) = x$ for $x \geqslant 0$. This completes the proof. □