COMPLETELY INJECTIVE SEMIGROUPS WITH CENTRAL IDEMPOTENTS

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(Received 23 October, 1967; revised 26 April, 1968)

1. Introduction. A right [left] unitary S-system is a set M with right [left] operators in a semigroup S with 1, where $x1 = x$ [$1x = x$] for all $x \in M$. We define a semigroup S with 1 to be completely right [left] injective provided that every right [left] unitary S-system is injective. The main purpose of this paper is to determine a structure for completely right [left] injective semigroups whose idempotents are in the centre.

For a semigroup S whose idempotents are in the centre, we prove that S is completely right injective if and only if S has a zero element and each right ideal is generated by an idempotent. These semigroups are exactly of the type that are unions of disjoint groups whose identity elements are dually well-ordered, and where the multiplication is determined by homomorphisms between groups. This is a special case of Theorem 4.11 of [**2**, p. 128].

2. Structure theorems. Since we are concerned with right unitary S-systems in this paper, we shall use the term *S-system* to mean " right unitary S-system ". *Consequently all semigroups in this paper will contain an identity element* 1. An S-system M_S is *injective* if and only if, for every S-monomorphism $g: P_S \to R_S$ and for every S-homomorphism $h: P_S \to M_S$, there exists an S-homomorphism $h^*: R_S \to M_S$ such that $h^*g = h$. It follows that, if N_S is an injective subsystem of M_S, then there exists an idempotent S-epimorphism $f: M_S \to N_S$.

2.1. Definition. A semigroup S is called *completely right* [*left*] *injective*‡ if and only if S contains an identity element 1 and every right [left] unitary S-system is injective. S is called *completely injective* if and only if it is both completely right and left injective.

2.2. Theorem. *If S is completely right injective, then every right ideal of S is generated by an idempotent.*

Proof. If H is a right ideal of S, then H is an injective subsystem of S_S. Hence there exists an idempotent S-epimorphism $f: S \to H$. Thus $H = f(S) = f(1)S$ and from

$$f(1)f(1) = f(1 . f(1)) = f(1)$$

it follows that $f(1)$ is idempotent.

2.3. Corollary. *Every completely right injective semigroup is regular.*

Proof. This follows immediately from 2.2 and Lemma 1.13 of [**2**, p. 27].

† Research supported by the National Science Foundation under grant GP-6816 for the first author.

‡ In [3], this term was used in the case of S-systems with zero.

https://doi.org/10.1017/S0017089500000471 Published online by Cambridge University Press

2.4. Corollary. *Every completely right injective semigroup is semisimple.*

Proof. Let A be an ideal of a completely right injective semigroup S. By 2.2, $A = eS$, where e is idempotent. Then $A = e^2S \subseteq A^2$, and $A^2 = A$. Hence, from exercise 7 of [2, p. 76], S is semisimple.

It is known that the full transformation semigroup T_X on a finite set X of cardinal n is a semisimple semigroup. However, for $n > 2$, T_X is not completely right injective, for not every right ideal is principal. Using the terminology of [1], an S-system M is said to be *weakly injective* if and only if, for any right ideal A of S and S-homomorphism $f: A_S \to M_S$, there exists an element z in M such that $f(x) = zx$, for all x in A.

2.5. Lemma. *Let S be a semigroup for which every right ideal of S is generated by an idempotent. Then every S-system is weakly injective.*

Proof. Let M be an S-system, A a right ideal of S, and $f: A_S \to M_S$ an S-homomorphism. By hypothesis, $A = eS$, for some idempotent $e \in S$. Setting $z = f(e)$, we have

$$f(x) = f(ex) = f(e)x = zx,$$

for all x in A.

2.6. Theorem. *Let S be a semigroup whose idempotents are in the centre of S. Then S is completely right injective if and only if S has a zero and each right ideal of S is generated by an idempotent.*

Proof. Suppose that S is completely right injective. By 2.2, every right ideal of S is generated by an idempotent. Let S^0 denote the semigroup $S \cup 0$ defined as in [3, p. 4]. Clearly S^0 is an S-system containing S. By hypothesis, S is an injective S-system and hence there exists an S-homomorphism $f: S^0 \to S$ which extends the identity map, 1_S, on S. For each $a \in S$, we have $f(0)a = f(0a) = f(0)$. Thus $f(0)$ is a left zero element of S and hence is idempotent. Since $f(0)$ belongs to the centre of S, it follows that $f(0)$ is the zero element of S and $S = S^0$.

Conversely, assume that S is a semigroup with zero whose idempotents belong to the centre of S and that every right ideal of S is generated by an idempotent. Let M and $P \subseteq R$ be S-systems, and $f: P \to M$ be an S-homomorphism of P into M. We show that M is injective. Consider the set of all pairs (P', f') consisting of subsystems P' of R containing P and S-homomorphisms f' of P' into M which extend f. We partially order this set by the relation: $(P', f') \leqq (P'', f'')$ if and only if $P' \subseteq P''$ and f'' extends f'. Since any totally ordered subset has an upper bound in the set, the maximal principle applies to assure us of a maximal pair (P_0, f_0). We prove that $P_0 = R$.

Suppose that $P_0 \subset R$ and let $r \in R$ be such that $r \notin P_0$. Set $A = \{a \in S \mid ra \in P_0\}$. We need to consider the two cases: A non-empty or A empty. For each case we will be able to define an S-homomorphism h of rS into M which agrees with f_0 on $P_0 \cap rS$.

Suppose that A is non-empty; then A is a right ideal of S and, by hypothesis, $A = eS$, where e is idempotent. Since $r \notin P_0$, it follows that $A \subset S$, and hence $e \neq 1$. The map $g: A \to M$ defined by $g(a) = f_0(ra)$ $(a \in A)$ is an S-homomorphism of A into M. By 2.5, M is weakly injective. Thus, for some z in M, we have $f_0(ra) = za$ for all a in A.

Define $h: rS \to M$ by $h(rs) = zes$ for all $s \in S$. We assert that h is single-valued. Suppose that $rs_1 = rs_2$. Since e is central, $res_1 = rs_1e = rs_2e = res_2$, and consequently

$$h(rs_1) = zes_1 = f_0(res_1) = f_0(res_2) = zes_2 = h(rs_2).$$

Also $h = f_0$ on $P_0 \cap rS$. Suppose that $x \in P_0 \cap rS$. Then $x = ra \in P_0$, where $a \in A$. Since $ea = a$, we have $h(x) = h(ra) = zea = za = f_0(ra) = f_0(x)$. The map h is clearly an S-homomorphism of rS into M.

Suppose that A is empty. Define $h: rS \to M$ by $h(x) = m0$ for all $x \in rS$, where m is an arbitrary but fixed element of M. Then $P_0 \cap rS$ is empty and $h(x)s = (m0)s = m0 = h(xs)$ for all $x \in rS$ and $s \in S$. Hence h is also an S-homomorphism in this case.

Set $P^* = P_0 \cup rS$, and let $f^*: P^* \to M$ be the map defined by $f^*(x) = f_0(x)$ for $x \in P_0$, and $f^*(x) = h(x)$ for $x \in rS$. By the above, f^* is an S-homomorphism of P^* into M which extends f_0. Hence $(P^*, f^*) > (P_0, f_0)$, which contradicts the maximality of the pair (P_0, f_0). Thus $P_0 = R$, and M is injective.

In the following two lemmas we assume that S is a completely right injective semigroup such that every idempotent of S is in the centre of S.

2.7. Lemma. *The right ideals of S are totally ordered and satisfy the ascending chain condition (A.C.C.). Moreover, the set of all idempotents E of S, under the natural partial ordering, form a dually well-ordered set in the sense that every non-empty subset K of E contains a greatest element in K.*

Proof. If eS and fS are two right ideals of S, then $eS \cup fS$ is a principal right ideal by 2.2. Thus either $eS \cup fS = eS$ or $eS \cup fS = fS$, which implies that either $fS \subseteq eS$ or $eS \subseteq fS$.

If $\{e_\alpha S \mid \alpha \in I\}$ is a family of right ideals of S, then $\bigcup_{\alpha \in I} e_\alpha S = e_\beta S$ for some $\beta \in I$. Hence S satisfies the A.C.C. for right ideals.

The natural partial ordering on E is defined by $e \leqq f$ (e, f in E) if and only if $ef = fe = e$. Since E is commutative, $e \leqq f$ if and only if $eS \subseteq fS$. The well-ordering follows from the A.C.C.

2.8. Lemma. *The semigroup S is an inverse semigroup which is a union of disjoint groups.*

Proof. By 2.3, S is regular. Since the idempotents of S commute, then, by Theorem 1.17 of [2, p. 28], S is an inverse semigroup. Let $a \in S$, $e = aa^{-1}$ and $f = a^{-1}a$. Since e and f are central idempotents,

$$e = e^2 = aa^{-1}aa^{-1} = afa^{-1} = faa^{-1} = fe = (a^{-1}a)e = a^{-1}ea = a^{-1}(aa^{-1})a = a^{-1}a = f.$$

Therefore $a = ea = ae$ and $aa^{-1} = a^{-1}a = e$ and these together imply that $a \in H_e$, the $\mathscr{H}$-class containing e. From Theorem 2.16 of [2, p. 59], H_e is a group. Thus S is a union of groups.

We shall now establish the structure for our completely right injective semigroups, which is a special case of [2, p. 128]. We shall say that a set is *dually well-ordered* if and only if it is a semilattice in which every non-empty subset has a greatest element.

2.9. STRUCTURE THEOREM. *Let X be a dually well-ordered set containing a least element o such that, for each $\alpha \in X$, there corresponds in a one-to-one manner a group G_α with identity e_α, the group corresponding to o being the one-element group $\{e_0\}$. For each α, β of X with $\alpha > \beta$, let there correspond a homomorphism $f_{\beta,\alpha}$ of G_α into G_β, such that, if $\alpha > \beta > \gamma$, then $f_{\gamma,\beta} f_{\beta,\alpha} = f_{\gamma,\alpha}$. Let $f_{\alpha,\alpha}$ be the identity mapping of G_α and let S be the union of all the G_α ($\alpha \in X$). Define the product of a_α and a_β in S ($a_\alpha \in G_\alpha$, $a_\beta \in G_\beta$) by $a_\alpha a_\beta = f_{\gamma,\alpha}(a_\alpha) f_{\gamma,\beta}(a_\beta)$, where $\gamma = \alpha\beta = \alpha \wedge \beta$ in X.*

Then S is a completely right injective semigroup whose idempotents are in the centre of S. Conversely, every such semigroup is exactly of this form.

Proof. From Theorem 4.11 of [2, p. 128], S is an inverse semigroup which is a union of groups. Thus the idempotents of S are central. The element e_0 is the zero element of S for $\alpha \geqq o$, for all $\alpha \in X$, and $a_\alpha e_0 = f_{\alpha 0,\alpha}(a_\alpha) f_{\alpha 0,0}(a_0) = e_0 e_0 = e_0$. From the product defined in S, the map $\alpha \to e_\alpha$ is an order preserving isomorphism of the semilattice X onto the semilattice E of idempotents of S, where $e \leqq f$ ($e, f \in E$) if and only if $ef = fe = e$. Consequently $\alpha \geqq \beta$ (α, $\beta \in X$) if and only if $e_\alpha \geqq e_\beta$. Since X is dually well-ordered, E is dually well-ordered.

Let R be a right ideal of S. Since S is regular, R contains an idempotent. By the dual well-ordering of E, it follows that R contains a greatest idempotent e. Thus $eS \subseteq R$. If $a \in R$, then $aa^{-1} \in R$. Hence $eaa^{-1} = aa^{-1}$, and $ea = a \in eS$. Therefore $eS = R$. It follows from 2.6 that S is completely right injective.

The converse statement follows directly from 2.7, 2.8 and Lemmas 4.9 and 4.10 of [2, pp. 127–28].

2.10. PROPOSITION. *If S is a completely right injective semigroup whose idempotents are in the centre of S, then S is completely left injective.*

Proof. Let L be a left ideal of S. If $a \in LS$, then $a = us$, where $u \in L$ and $s \in S$. Since S is an inverse semigroup and the idempotents of S are central, it follows that

$$a = us = (uu^{-1}u)s = usu^{-1}u = (usu^{-1})u \in L.$$

Hence $LS \subseteq L$ and so L is a two-sided ideal of S. Thus every left ideal of S is generated by an idempotent. By the dual statement of 2.6 we have the proposition.

We now determine the explicit sets which form the maximal subgroups of 2.9.

2.11. PROPOSITION. *If S is a completely right injective semigroup whose idempotents are in the centre of S, then the $\mathcal{H}$-classes of S, which are the maximal subgroups of S, are precisely the sets $eS \backslash fS$, where $e, f \in E$ and fS is a maximal right ideal of S contained in eS. The element e is the identity of $eS \backslash fS$.*

Proof. By 2.8, S is an inverse semigroup which is a union of groups. This implies, as stated in [2, p. 127], that all the relations of Green are the same, and the equivalence classes are just the maximal subgroups of S. That is, if $a \in S$, then $H_a = R_a = L_a = D_a = J_a$, all being the maximal subgroup of S containing a.

Let fS be a maximal right ideal of S contained in eS, where e and f are idempotents. Then $f < e$ and no idempotent of S is between f and e. If $a \in eS \backslash fS$, then $a = ea$ and hence

https://doi.org/10.1017/S0017089500000471 Published online by Cambridge University Press

$aa^{-1} = eaa^{-1} \in eS$. Since $aa^{-1} \leqq e$ and the idempotents of S are totally ordered, it follows that $aa^{-1} = e$. Hence $a\mathscr{R}e$ and so $eS \backslash fS \subseteq R_e$. On the other hand, if $b \in R_e$, then $bS = eS \supseteq fS$ and so $b \notin fS$. Therefore $eS \backslash fS = R_e = H_e$.

Conversely, let H_a $(a \in S)$ be an $\mathscr{H}$-class of S. Let $e \in H_a$ be the identity element of the group H_a. By 2.7, the right ideals of S satisfy the A.C.C. Hence eS contains a maximal right ideal fS of S, where f is idempotent. By the above, $eS \backslash fS$ is the $\mathscr{H}$-class containing e and so $eS \backslash fS = H_e = H_a$.

For inverse semigroups, we have, from 2.2, 2.6, 2.8 and 4.8 of [**2**, p. 127],

2.12. PROPOSITION. *A semigroup S with zero is an inverse semigroup which is a union of groups, and whose right ideals are principal if and only if S is completely right injective with idempotents in the centre of S.*

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